Plane curves with 3 or 4 total inflection points

Filip Cools * and Marc Coppens †

July 4, 2007

Abstract. — In this article, we will study plane curves of a certain degree $d$ with 3 or 4 total inflection points. In particular, we will study their image in the moduli spaces. Also a result on curves with 5 total inflection points is included.

MSC. — 14H10, 14H50, 14N20.

0 Notation and introduction

We will first fix some notations. Let $\mathbb{P}^2$ be the projective plane over some algebraically closed field $k$ and let $\mathcal{P}^2$ be the incidence relation in $(\mathbb{P}^2)^* \times \mathbb{P}^2$, i.e. $\mathcal{P}^2 = \{(L, P) \mid P \in L\}$.

If $d$ and $e$ are nonzero natural numbers, we denote by $V_{d,e} \subset (\mathcal{P}^2)^e$ the set of elements $(\mathcal{L}, \mathcal{P}) = (((L_1, P_1), \ldots, (L_e, P_e))$ with $P_i \not\in L_j$ for all $i \neq j$ (hence also $L_i \neq L_j$ for $i \neq j$) and such that there exists a plane curve $\Gamma$ (not necessarily irreducible) of degree $d$, not containing one of the lines $L_i$, with intersection number $i(\Gamma.L_i, P_i) = d$. We say that in this case the pairs $(L_i, P_i)$ are total inflection points of $\Gamma$. We write $\overline{V_{d,e}}$ to denote the closure of $V_{d,e}$ in $(\mathcal{P}^2)^e$.

If $(\mathcal{L}, \mathcal{P}) \in V_{d,e}$, denote by $V(\mathcal{L}, \mathcal{P}) \subset \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ the set $\{s \!\!\mid \!\!\mid dP_1 + \ldots + dP_e \subset Z(s)\}$, whereby $dP_i$ is the subscheme of $\mathbb{P}^2$ corresponding to the divisor $dP_i$ on $L_i$. So the associated linear system $\mathbb{P}(V(\mathcal{L}, \mathcal{P}))$ consists of curves $\Gamma$ of degree $d$ having $(L_i, P_i)$ as total inflection point for all $i$. If $1 \leq f \leq e$, we will mostly write $(\mathcal{L}_f, \mathcal{P}_f)$ to denote the element $((L_1, P_1), \ldots, (L_f, P_f)) \in V_{d,f}$ (unless stated otherwise).


†Katholieke Hogeschool Kempen, Departement Industrieel Ingenieur en Biotechniek, Kleinhoefstraat 4, B-2440 Geel, Belgium, email: Marc.Coppens@khk.be; the author is affiliated with K.U.Leuven as Research Fellow.
We write \( V_{d,e} \) to denote the union of the spaces of curves \( \mathbb{P}(V(\mathcal{L}, \mathcal{P})) \) with \( (\mathcal{L}, \mathcal{P}) \in V_{d,e} \). We denote the set of points corresponding to smooth plane curves of \( V_{d,e} \) by \( V_{d,e}^\circ \). Let \( m_{d,e} : V_{d,e}^\circ \rightarrow M_{(d-1)(d-2)/2} \) be the moduli map and denote its image by \( M(V_{d,e}) \).

In [7], the case \( d = 4 \) (i.e. quartic curves) has been studied intensively. The main tool used there is the so-called \( \lambda \)-invariant, which is nothing else than a cross ratio of four points (see also [4]). In [2], the cases \( e = 1, 2 \) have been handled and also the cases \( e = 3, 4 \) for some special configurations of the lines \( L_i \) and the points \( P_i \). In [3], a few general results are proven on curves with total inflection points.

In Section 1, we will consider the case \( e = 3 \) (so three total inflection points). We will give a full description of the components of \( V_{d,3} \) for \( d \geq 2 \) and \( M(V_{d,3}) \) for \( d \geq 5 \), prove that they are rational and compute their dimensions (see Theorem 1.1 and Theorem 1.5). In Section 2, we consider the case \( e = 4 \). Again we find a full list of the components of \( V_{d,4} \) for \( d \geq 3 \) and \( M(V_{d,4}) \) for \( d \geq 6 \). We will prove that all components of \( V_{d,4} \) (Theorem 2.2) and almost all components of \( M(V_{d,4}) \) (Theorem 2.11) are rational. In Section 3, we will prove a result on the case \( e = 5 \) (Theorem 3.2).

We recall a proposition proven in [3], which we will use several times during this article.

**Proposition 0.1.** Let \( (\mathcal{L}, \mathcal{P}) \in V_{d,e} \).

(a) \( \dim(V(\mathcal{L}, \mathcal{P})) = \binom{d+e+2}{2} + 1 \), where \( \binom{n}{2} \) is defined to be 0 if \( n < 2 \).

(b) Let \( L \) be a line in \( \mathbb{P}^2 \) with \( P_i \notin L \) for \( 1 \leq i \leq e \). Let \( V_L(\mathcal{L}, \mathcal{P}) \) be the image of the restriction map \( V(\mathcal{L}, \mathcal{P}) \rightarrow \Gamma(L, \mathcal{O}_L(d)) \). Let \( P_{10} = L_1 \cap L \) for \( 1 \leq i \leq e \). If \( d \geq e \), \( \dim(V_L(\mathcal{L}, \mathcal{P})) = d - e + 2 \) and \( \mathbb{P}(V_L(\mathcal{L}, \mathcal{P})) \) is a linear system \( g_d^{d-e+1} \) on \( L \) containing \( P_{10} + \ldots + P_{e0} + g_{d-e}^{d-e} \). If \( d < e \), \( \dim(V_L(\mathcal{L}, \mathcal{P})) = 1 \).

(c) Under the assumptions of (b), for \( P \in L \) with \( P \neq P_{10} \) for all \( 1 \leq i \leq e \) one has \( ((\mathcal{L}, \mathcal{P}), (L, P)) \in V_{d,e+1} \) if and only if \( dP \in \mathbb{P}(V_L(\mathcal{L}, \mathcal{P})) \).

The following lemma is well-known. Since we cannot find a good reference, we include a proof for sake of completeness.

**Lemma 0.2.** Let \( C_1 \) and \( C_2 \) be smooth plane curves of degree \( d \geq 4 \). Then \( C_1 \) and \( C_2 \) are isomorphic if and only if they are projectively equivalent; i.e. there exists an automorphism \( \phi \) of \( \mathbb{P}^2 \) such that \( \phi(C_1) = C_2 \).

**Proof.** In case \( C_1 \) and \( C_2 \) are isomorphic curves, both are defined by a linear system \( g_d^2 \) on the same curve \( C \). Let \( g_i \) be the linear system on \( C \) defining \( C_i \)
(i ∈ \{0, 1\}). In order to prove the existence of φ, it is enough to prove that g_1 = g_2. Take D ∈ g_2 general, then D = P_1 + \ldots + P_d with P_i \neq P_j for i \neq j. We identify C with C_1, hence it is enough to prove that D = C_1L for a line L in \mathbb{P}^2. Let L be the line connecting P_1 and P_2 and assume P_d \notin L. From the Adjunction Formula, it follows that the canonical linear system on C_1 is defined by intersections of C_1 with plane curves of degree d − 3. Using d − 3 lines in \mathbb{P}^2, it is possible to find canonical divisors K_i on C such that K_i \cap D = P_1 + \ldots + P_{d−i} for 0 ≤ i ≤ d − 3. On the other hand, using L and d − 4 suited lines in \mathbb{P}^2, we can find a canonical divisor K_{d−2} containing P_1 + \ldots + P_{d−2} but not P_d. This proves that D imposes at least d − 1 conditions on K_C, hence h^0(K_C − D) ≤ h^0(K_C) − (d − 1) = g − d + 1. From Riemann-Roch, it follows that h^0(D) = deg(D) − g + 1 + h^0(K_C − D) ≤ (d − g + 1) + (g − d + 1) = 2. This contradicts D ∈ g_2. \qed

1 The case e = 3

Let (\mathcal{P}^2)^{3,0} \subset (\mathcal{P}^2)^3 be the set of points (\mathcal{L}, \mathcal{P}) = ((L_1, P_1), (L_2, P_2), (L_3, P_3)) with P_i \notin L_j for i \neq j and let (\mathcal{P}^2)^{3,0,2} \subset (\mathcal{P}^2)^3 be the subspace of (\mathcal{P}^2)^{3,0} of elements (\mathcal{L}, \mathcal{P}) such that the lines L_1, L_2 and L_3 have a common point S. Let (\mathcal{P}^2)^{3,0,1} = (\mathcal{P}^2)^{3,0} \setminus (\mathcal{P}^2)^{3,0,2}.

For (\mathcal{L}, \mathcal{P}) ∈ (\mathcal{P}^2)^{3,0}, we write g_{d−1} \mathcal{L} to denote the linear system \mathbb{P}(V_{L_3}(\mathcal{L}_2, \mathcal{P}_2)) on L_3. Let P_{1,2,3} = \langle P_1, P_2 \rangle \cap L_3. Since d(P_1, P_2) ∈ \mathbb{P}(V(\mathcal{L}_2, \mathcal{P}_2)), we find dP_{1,2,3} ∈ g_{d−1}^{−1}, hence

\[ g_{d−1}^{−1} = \langle P_{1,3} + P_{2,3} + g_{d−2}^{−2}, dP_{1,2,3} \rangle. \]

Moreover, (\mathcal{L}, \mathcal{P}) ∈ (\mathcal{P}^2)^{3,0,2} is equivalent to P_{1,3} = P_{2,3} and (\mathcal{L}, \mathcal{P}) ∈ V_{d,3} is equivalent to dP_3 ∈ g_{d−1}^{−1}.

We continue in case (\mathcal{L}, \mathcal{P}) ∈ (\mathcal{P}^2)^{3,0,1}. We can choose coordinates (X_1 : X_2 : X_3) on \mathbb{P}^2. Let R_1, R_2 and R_3 be the coordinate axes. There exists a coordinate transformation φ on \mathbb{P}^2 such that φ(L_i) = R_i : X_i = 0 for i ∈ \{1, 2, 3\}, φ(P_1) = Q_1 := (0 : 1 : 1) and φ(P_2) = Q_2 := (1 : 0 : 1), hence φ(P_3) = Q_3 := (1 : −t : 0) for some t ∈ k (note that P_3 is not contained in L_1), φ(P_{1,3}) = Q_{1,3} := (0 : 1 : 0), φ(P_{2,3}) = Q_{2,3} := (1 : 0 : 0) and φ(P_{1,2,3}) = Q_{1,2,3} := (1 : −1 : 0). On R_3 we can take (X_1 : X_2) as local coordinates, so on R_3 we have Q_{1,3} = (0 : 1), Q_{2,3} = (1 : 0), Q_{1,2,3} = (1 : −1) and Q_3 = (1 : −t). Identify g_d^3 on R_3 with the projective space \mathbb{P}(k[X_1, X_2]) of homogeneous forms of degree d and use homogeneous coordinates (a_d : a_{d−1} : \ldots : a_0) for \langle a_dX_1^d + a_{d−1}X_1^{d−1}X_2 + \ldots + a_0X_2^d \rangle. The subspace Q_{1,3} + Q_{2,3} + g_{d−2}^{−2} ⊂ g_d^3 is defined by the linear equations a_0 = a_d = 0, while dQ_{1,2,3} corresponds to \langle (X_1 + X_2)^d \rangle. It follows that g_{d−1}^{−1} = \langle Q_{1,3} + Q_{2,3} + g_{d−2}^{−2}, dQ_{1,2,3} \rangle ⊂ g_d^3 has as equation a_0 = a_d. The point Q_3 = (1 : −t) ∈ L_3 satisfies dQ_3 ∈ g_{d−1}^{−1} if and only if the form (tX_1 + X_2)^d satisfies the equation, i.e. if and only if t^d = 1.
The point \(Q_{1,2,3} = (1 : -1)\) on \(R_3\) is a solution of this equation. If \(\text{char}(k) = 0\) or \(\text{char}(k) = p > 0\) does not divide \(d\), there are exactly \(d\) solutions of this equation. If \(\text{char}(k) = p > 0\) divides \(d\), we can write \(d = d'p^e\) with \(d'\) not divisible by \(p\). In this case, the condition becomes \(t^{d'} = 1\), hence there are exactly \(d'\) solutions. In case \(\text{char}(k) = 0\) or \(\text{char}(k) = p > 0\) does not divide \(d\), we also write \(d' = d\).

Denote \(((R_1, Q_1), (R_2, Q_2), (R_3, Q_3))\) by \((\mathcal{R}, \mathcal{Q}(t))\) if \(Q_3 = (1 : -t : 0)\). By using 0.1, we see that \((\mathcal{R}, \mathcal{Q}(t)) \in V_{d,3}\) if and only if \(t^{d'} = 1\), hence if and only if \(t \in \mu_{d'} = \{1, \omega, \ldots, \omega^{d'-1}\}\) whereby \(\omega\) is a \(d'\)-th root of unity in \(k\) (for example, if \(k = \mathbb{C}\) we can take \(\omega = e^{2\pi i / d'}\)).

We get that if \((\mathcal{L}, \mathcal{P}) \in V_{d,3} \cap (\mathcal{P}^2)^{3,0,1}\), there exists a coordinate transformation \(\phi\) and an element \(t \in \mu_{d'}\) such that \(\phi(\mathcal{L}, \mathcal{P}) = (\mathcal{R}, \mathcal{Q}(t))\), hence \((\mathcal{L}, \mathcal{P}) = \phi^{-1}(\mathcal{R}, \mathcal{Q}(t))\). So we can conclude that

\[
V_{d,3} \cap (\mathcal{P}^2)^{3,0,1} = \bigcup_{t \in \mu_{d'}} \text{Aut}(\mathbb{P}^2). (\mathcal{R}, \mathcal{Q}(t))
\]

and thus it is the union of \(d'\) smooth components of dimension 8. For an element \(t \in \mu_{d'}\), we write \(V_t \subset V_{d,e}\) to denote the component containing \(\text{Aut}(\mathbb{P}^2). (\mathcal{R}, \mathcal{Q}(t))\).

Now we study the case where \((\mathcal{L}, \mathcal{P}) \in (\mathcal{P}^2)^{3,0,2}\). We can choose coordinates \((X_1 : X_2)\) on \(L_3\) such that \(P_{1,3} = P_{2,3} = (1 : 0)\) and \(P_{1,2,3} = (0 : 1)\). As before, we identify \(g_4^l\) on \(L_3\) with \(\mathbb{P}(k[1 : 1])\) and we use coordinates \(a_d : a_{d-1} : \ldots : a_0\) on \(g_4^l\). The linear subspace \(2P_{1,3} + g_4^{d-2} \subset g_4^d\) has linear equations \(a_d = a_{d-1} = 0\). Since \(dP_{1,2,3} \in g_4^d\) has coordinate \((1 : 0 : \ldots : 0)\), the equation of \(g_4^{d-1} = \langle 2P_{1,3} + g_4^{d-2}, dP_{1,2,3}\rangle \subset g_4^d\) is given by \(a_{d-1} = 0\). Since \(P_3 \neq P_{1,3} = P_{2,3}\) \((P_3\) is not contained in \(L_1\) or \(L_2)\), we may assume \(P_3 = (a : 1)\) on \(L_3\). Hence \(dP_3 \in g_4^{d-1}\) if and only if \((X_1 - aX_2)^d\) satisfies the equation, i.e. if and only if \(da = 0\). The solution \(a = 0\) corresponds to \(P_{1,2,3}\) and if \(\text{char}(k) = 0\) or \(\text{char}(k) = p > 0\) does not divide \(d\), there is no other solution. In this case, \((\mathcal{L}, \mathcal{P})\) belongs to the closure \(V_1\) of \(\text{Aut}(\mathbb{P}^2). (\mathcal{R}, \mathcal{Q}(1))\). However, if \(\text{char}(k) = p > 0\) divides \(d\), all points \(P_3\) on \(L_3\) satisfy \(dP_3 \in g_4^{d-1}\), so \((\mathcal{P}^2)^{3,0,2}\) is a component of \(V_{d,3}\). Notice that

\[
\text{Aut}(\mathbb{P}^2).((R_1', Q_1'), (R_2', Q_2'), (R_3', Q_3'))
\]

is a dense subset of \((\mathcal{P}^2)^{3,0,2}\), with \(R_1' : X_1 = 0, R_2' : X_2 = 0, R_3' : X_1 = X_2, Q_1' = (0 : 1 : 0), Q_2' = (1 : 0 : 0)\) and \(Q_3' = (1 : 1 : 1)\) (so \(S = (0 : 0 : 1)\)).

Since \(\text{Aut}(\mathbb{P}^2)\) is rational, we have proven the following theorem.

**Theorem 1.1.** In case \(\text{char}(k) = 0\), let \(d' = d\); in case \(\text{char}(k) = p > 0\), write \(d = pd'\) with \(c \geq 0\) and \(p \nmid d'\). There are exactly \(d'\) components of \(V_{d,3}\) intersecting \((\mathcal{P}^2)^{3,0,1}\). Each of them has dimension 8, the intersection with \((\mathcal{P}^2)^{3,0,1}\) is smooth and exactly one of them is not contained in \((\mathcal{P}^2)^{3,0,1}\). In case \(c \geq 1\), \((\mathcal{P}^2)^{3,0,2}\) is another 8-dimensional component of \(V_{d,3}\). Each component of \(V_{d,3}\) is rational.

We have the following generalization for the case where \(\text{char}(k) = p > 0\) is divisible by \(d\).
Proposition 1.2. Assume $\text{char}(k) = p > 0$ and $d = p^c d'$ with $c \geq 1$ and $p \nmid d'$. Let $e \leq p^c + 1$ and let $L_1, \ldots, L_e$ be lines through a common point $S$ and let $P_i \in L_i \setminus \{S\}$ for $1 \leq i \leq e$. Then $(\mathcal{L}, \mathcal{P}) \in V_{d,e}$.

Proof. We may assume $e \geq 3$. Let $L = \langle P_1, P_2 \rangle$ and assume $2 \leq x \leq e$ with $P_i \in L$ for $1 \leq i \leq x$ and $P_i \notin L$ otherwise. Write $f = e - x$. In case $f = 0$, we have $dL \in \mathbb{P}(V(\mathcal{L}, \mathcal{P}))$, hence $(\mathcal{L}, \mathcal{P}) \in V_{d,e}$. Let $f > 0$ and assume the claim holds for $f - 1$ (instead of $f$). Let $Q_{e-f+1} = L_{e-f+1} \cap L$. Let $(\mathcal{L}', \mathcal{P}') \in (\mathcal{P}_2)^{e,0}$ (resp. $(\mathcal{L}'', \mathcal{P}'') \in (\mathcal{P}_2)^{e-1,0}$) be obtained from $(\mathcal{L}, \mathcal{P})$ by replacing $(L_{e-f+1}, P_{e-f+1})$ by $(L_{e-f+1}, Q_{e-f+1})$ (resp. by omitting $(L_{e-f+1}, P_{e-f+1})$). Both $(\mathcal{L}', \mathcal{P}')$ and $(\mathcal{L}'', \mathcal{P}'')$ correspond to $f - 1$ instead of $f$, hence $(\mathcal{L}', \mathcal{P}') \in V_{d,e}$ and $(\mathcal{L}'', \mathcal{P}'') \in V_{d,e-1}$.

In order to prove the claim, it is enough to show $dP_{e-f+1} \in \mathbb{P}(V_{L_{e-f+1}}(\mathcal{L}'', \mathcal{P}''))$. There exists a $\Gamma \in V(\mathcal{L}', \mathcal{P}')$ such that $L_{e-f+1} \notin \Gamma$ and $L_{e-f+1} \cap \Gamma = L_{e-f+1}$. Since $V(\mathcal{L}', \mathcal{P}') \subset V(\mathcal{L}'', \mathcal{P}'')$ it follows that $dQ_{e-f+1} \in \mathbb{P}(V_{L_{e-f+1}}(\mathcal{L}'', \mathcal{P}''))$, hence

$$\mathbb{P}(V_{L_{e-f+1}}(\mathcal{L}'', \mathcal{P}'')) = ( (e-1)S + g_{d-e+1}^{d-e+1}, dQ_{e-f+1} ).$$

Choose coordinates $(x : y)$ on $L_{e-f+1}$ such that $S = (1 : 0)$ and $Q_{e-f+1} = (0 : 1)$. Use coordinates $(a_1 : \ldots : a_0)$ on $g_{d}^{d-e+1}$ on $L_{e-f+1}$ as before. The linear system $((e-1)S + g_{d-e+1}^{d-e+1}, dQ_{e-f+1})$ has equations $a_{d} = \ldots = a_{d-e+2} = 0$ and $dQ_{e-f+1} = (1 : 0 : \ldots : 0)$. Hence $\mathbb{P}(V_{L_{e-f+1}}(\mathcal{L}'', \mathcal{P}''))$ has equations $a_{d-1} = \ldots = a_{d-e+2} = 0$. For $P = (\alpha : \beta) \in L_{e-f+1}$ one has $dP \in \mathbb{P}(V_{L_{e-f+1}}(\mathcal{L}'', \mathcal{P}''))$ if and only if the form $(\beta x - \alpha y)^d$ satisfies those equations. This is equivalent to $\beta^{d-i} \alpha^i d = 0$ for $1 \leq i \leq e - 2$. In case $\alpha \beta \neq 0$, all those conditions are satisfied if and only if $e - 2 \leq p^c - 1$, hence $e \leq p^c + 1$.

Lemma 1.3. Assume $(\mathcal{L}, \mathcal{P}) \in V_{d,e}$ with $e \leq d$. Then a general element of $\mathbb{P}(V(\mathcal{L}, \mathcal{P}))$ is a smooth plane curve.

Proof. The claim follows immediately from [3, Prop. 2.1] by taking $z = 0$.

For each $t \in \mu_{d'}$, we write $V_t$ to denote the union of spaces $\mathbb{P}(V(\mathcal{L}, \mathcal{P}))$ for all $(\mathcal{L}, \mathcal{P}) \in V_t$. We denote the set of points corresponding to smooth plane curves of $V_t$ by $V_t^0$. Since $V_t^0 \subset V_{d,3}^0$, we can consider the image of $V_t^0$ under $m_{d,3} : V_{d,3}^0 \to M_{(d-1)(d-2)/2}$. We denote this image by $M(V_t)$.

Now we restrict to the case $\text{char}(k) = 0$.

Lemma 1.4. If $d \geq 5$ and $V_t$ is component of $V_{d,3}$, then a general element of $V_t^0$ has exactly 3 total inflection points.

Proof. Since $V_{d,3}$ has codimension 1 inside $(\mathcal{P}_2)^3$ (see Theorem 1.1 or [3, Ex. 3.4]), we find $V_{d,4}$ has codimension at least 2 inside $(\mathcal{P}_2)^4$, hence $\dim(V_{d,4}) \leq 10$ (in fact, we will show in Section 2 that equality holds). Indeed, fixing $(\mathcal{L}, \mathcal{P}) \in V_{d,3}$, the set

$$S = \{(L, P) \in \mathcal{P}_2 | ((\mathcal{L}, \mathcal{P}), (L, P)) \in V_{d,4}\}$$
We also need \( \phi \) holds if and only if \( \phi \) is dense open subset of \( M(V_i) \), where again the \( \sigma \) sign indicates the smooth curves. For a general curve \( C \in \mathbb{P}(V(R, Q(t)))^\sigma \), Lemma 0.2 implies that \( [C] = [C'] \in M_{(d-1)(d-2)/2} \) for some \( C' \in \mathbb{P}(V(R, Q(t')))^\sigma \) with \( t' \in \mu_d \) if and only if \( \phi(C) = C' \) for some \( \phi \in \text{Aut}(\mathbb{P}^2) \). Lemma 1.4 implies that the lines \( R_1, R_2 \) and \( R_3 \) are permuted by the map \( \phi \).

We write \( A = (a_{ij})_{i,j=1}^3 \) to denote a matrix corresponding to \( \phi \) and \( Q_3(t) = (1 : -t : 0) \) for some \( t \in \mu_d \).

First we consider the case where \( \phi((R_1, R_2, R_3)) = (R_2, R_1, R_3) \). In this case, the matrix becomes

\[
A = \begin{pmatrix}
0 & a_{12} & 0 \\
a_{21} & 0 & 0 \\
0 & 0 & a_{33}
\end{pmatrix}.
\]

We also need \( \phi(Q_1) = Q_2 \), hence \( a_{12} = a_{33} \), and \( \phi(Q_2) = Q_1 \), hence \( a_{21} = a_{33} \).

So \( \phi \) corresponds to the matrix

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

So \( \phi(Q_3(t)) = Q_3(t') \) implies \((t : 1 : 0) = (1 : -t' : 0)\), hence \( t' = 1/t \). Note that moreover the equality \( t = t' \) holds if and only if \( t = 1 \) or \( d \) is even and \( t = -1 \). We obtain \( M(V_i) = M(V_{1/t}) \).

In case \( \phi((R_1, R_2, R_3)) = (R_3, R_2, R_1) \), the matrix becomes

\[
\begin{pmatrix}
0 & 0 & a_{13} \\
0 & a_{22} & 0 \\
a_{31} & 0 & 0
\end{pmatrix}.
\]

We also need \( \phi(Q_1) = Q_2 \) and \( \phi(Q_3(t)) = Q_1 \), hence \( a_{13} = a_{31} = -t.a_{22} \). So \( \phi(Q_1) = Q_3(t') \) implies \((-t : 1 : 0) = (1 : -t' : 0)\) and thus again \( t' = 1/t \).

As above, the case \( \phi((R_1, R_2, R_3)) = (R_1, R_3, R_2) \) implies \( t' = 1/t \).

In case \( \phi((R_1, R_2, R_3)) = (R_2, R_3, R_1) \), the matrix becomes

\[
\begin{pmatrix}
0 & 0 & a_{13} \\
a_{21} & 0 & 0 \\
0 & a_{32} & 0
\end{pmatrix}.
\]

In this case we need \( \phi(Q_1) = Q_2 \) hence \( a_{13} = a_{32} \) and \( \phi(Q_3(t)) = Q_1 \) hence \( a_{21} = -t.a_{32} \). Since also \( \phi(Q_2) = Q_3(t') \), we get \((1 : -t : 0) = (1 : -t' : 0)\) and thus \( t = t' \). It follows that \( \phi \) acts on \( \mathbb{P}(V(R, Q(t)))^\sigma \).

In case \( \phi((R_1, R_2, R_3)) = (R_3, R_1, R_2) \), we find analogously as the above case that \( t = t' \) and that \( \phi \) acts on \( \mathbb{P}(V(R, Q(t)))^\sigma \).

In case \( t \neq 1 \) and \( t \neq -1 \) if \( d \) is even, we find a subgroup \( \mathbb{Z}/3\mathbb{Z} \subset \text{Aut}(\mathbb{P}^2) \) acting on \( \mathbb{P}(V(R, Q(t)))^\sigma \) such that \( \mathbb{P}(V(R, Q(t)))^\sigma / (\mathbb{Z}/3\mathbb{Z}) \) is birationally equivalent to \( M(V_i) \). Since \( \mathbb{P}(V(R, Q(t)))^\sigma \) is rational and \( \mathbb{Z}/3\mathbb{Z} \) is Abelian, it follows
that \( M(V_t) \) is also rational (using a result of E. Fischer, see [5]). This extends the result of Casnati and Del Centina (they consider the components \( M(V_1) \) and \( M(V_{-1}) \) in case \( d \) is even; see [2, Theorem B]) showing that all components of \( M(V_{d,3}) \) are rational. All the above results are summarized in the following theorem.

**Theorem 1.5.** If \( \text{char}(k) = 0 \) and \( d \geq 5 \), the set \( M(V_{d,3}) \) has \( 1 + \frac{d-1}{2} \) components if \( d \) is odd and \( 2 + \frac{d-2}{2} \) components if \( d \) is even. Moreover, each component is rational.

## 2 The case \( e = 4 \)

In this section we will for simplicity assume that \( k = \mathbb{C} \).

Assume that \( (\mathcal{L}, \mathcal{P}) \) is an element of \( V_{d,4} \), with no three of the lines \( L_1, L_2, L_3 \) and \( L_4 \) are concurrent. Since \( (\mathcal{L}, \mathcal{P}_3) = ((L_1, P_1), (L_2, P_2), (L_3, P_3)) \) is an element of \( V_{d,3} \cap (\mathcal{P}^2)_{3,0,1} \), there exists an element \( t \in \mu_d \) such that \( (\mathcal{L}, \mathcal{P}_3) \in V_t \). If we consider coordinates \( (X_1 : X_2 : X_3) \) on \( \mathbb{P}^2 \), there exists a coordinate transformation \( \phi \) such that \( \phi((\mathcal{L}, \mathcal{P}_3)) = (\mathcal{R}, Q(t)) \). Assume that \( R_4 := \phi(L_4) : X_3 = AX_1 + BX_2 \) (\( L_4 \) does not contain \( P_{1,2} \)). We can use \( (X_1 : X_2) \) as local coordinates on \( R_4 \). Identify the linear system \( g_{d}^{3} \) on \( R_4 \) with \( \mathbb{P}(k[X_1, X_2]_{d}) \) and use homogeneous coordinates \( (a_d : \ldots : a_0) \) for \( \langle a_dX_1^d + a_{d-1}X_1^{d-1}X_2 + \ldots + a_0X_2^d \rangle \).

Denote \( R_i \cap R_j \) by \( Q_{i,j} \) if \( 1 \leq i < j \leq 4 \).

Since \( Q_{1,4} = (0 : 1), Q_{2,4} = (1 : 0) \) and \( Q_{3,4} = (-B : A) \) on \( R_4 \), the linear system \( Q_{1,4} + Q_{2,4} + Q_{3,4} + g_{d-3}^{3} \subset g_{d}^{3} \) is defined by the following equations:

\[
\begin{align*}
    l_1(a_d, \ldots, a_0) := a_0 &= 0, \\
    l_2(a_d, \ldots, a_0) := a_d &= 0, \\
    l_3(a_d, \ldots, a_0) := a_d(-B)^d + a_{d-1}(-B)^{d-1}A + \ldots + a_0A^d &= 0.
\end{align*}
\]

If we define \( f_t \in k[X_1, X_2, X_3]_{d} \) to be

\[
(X_3 - X_1)^d + (X_3 - X_2)^d + (-1)^d(tX_1 + X_2)^d - (-1)^dX_1^d - (-1)^dX_2^d - X_3^d,
\]

it is easy to see that the curve \( \mathcal{C} \) in \( \mathbb{P}^2 \) with equation \( f_t(X_1 : X_2 : X_3) = 0 \) is contained in \( \mathbb{P}(V(\mathcal{R}, Q(t))) \), hence the divisor \( \mathcal{C} \cap L_4 \) is an element of \( g_{d}^{d-2} = \mathbb{P}(V_{R_d}(\mathcal{R}, Q(t))) \). The divisor \( \mathcal{C} \cap R_4 \) is defined by

\[
\tilde{a}_dX_1^d + \ldots + \tilde{a}_0X_2^d := f_t(X_1, X_2, AX_1 + BX_2) = 0,
\]

so we have the following equalities

\[
\begin{align*}
    l_1(\tilde{a}_d, \ldots, \tilde{a}_0) = \tilde{a}_0 &= (B - 1)^d, \\
    l_2(\tilde{a}_d, \ldots, \tilde{a}_0) = \tilde{a}_d &= (A - 1)^d, \\
    l_3(\tilde{a}_d, \ldots, \tilde{a}_0) = f_t(-B, A, 0) &= (-1)^d(A - tB)^d.
\end{align*}
\]
Since $g_{d}^{d-2} = \langle Q_{1,4} + Q_{2,4} + Q_{3,4} + g_{d-3}^{d-3}, C \cap R_{4} \rangle$, we find that $g_{d}^{d-2}$ is defined by
\[
\begin{align*}
&\left\{ \begin{array}{l}
l_{2}(\overline{a}_{d}, \ldots, \overline{a}_{0}), l_{1}(a_{d}, \ldots, a_{0}) = l_{1}(\overline{a}_{d}, \ldots, \overline{a}_{0}), l_{2}(a_{d}, \ldots, a_{0}), \\
l_{3}(\overline{a}_{d}, \ldots, \overline{a}_{0}), l_{2}(a_{d}, \ldots, a_{0}) = l_{2}(\overline{a}_{d}, \ldots, \overline{a}_{0}), l_{3}(a_{d}, \ldots, a_{0}).
\end{array} \right.
\end{align*}
\]

Assume that the local coordinates of $Q_{4}$ are $(\alpha : \beta)$. Proposition 0.1 implies that $dQ_{4} = \langle (\beta X_{1} - \alpha X_{2})^{d} \rangle \in g_{d}^{d-2}$, hence
\[
\begin{align*}
&\left\{ \begin{array}{l}
(A - 1)^{d}(-\alpha)^{d} = (B - 1)^{d}\beta^{d} \\
(-1)^{d}(A - tB)^{d} \beta^{d} = (A - 1)^{d}(-\beta B - A \alpha)^{d}
\end{array} \right.
\end{align*}
\]
so there exist $t'$ and $t''$ in $\mu_{d}$ such that
\[
\begin{align*}
&\left\{ \begin{array}{l}
-t'(A - 1) \alpha = (B - 1) \beta \\
t''(A - tB) \beta = t(A - 1)(A \alpha + B \beta)
\end{array} \right.
\end{align*}
\]
hence
\[
t(t' - 1)AB = (t't'' - t)A + tt'(1 - t'')B.
\]

**Proposition 2.1.** Under the above assumptions, we have:

i) $((L_{1}, P_{1}), (L_{2}, P_{2}), (L_{3}, P_{3})) \in V_{t}$

ii) $((L_{1}, P_{1}), (L_{2}, P_{2}), (L_{4}, P_{4})) \in V_{t'}$

iii) $((L_{2}, P_{2}), (L_{3}, P_{3}), (L_{4}, P_{4})) \in V_{t''}$

iv) $((L_{1}, P_{1}), (L_{3}, P_{3}), (L_{4}, P_{4})) \in V_{t't''/t}$

**Proof.** For i, we don’t have to prove anything. We will only prove statement iii (ii and iv are analogously). It is enough to prove that

\[
((R_{2}, Q_{2}), (R_{3}, Q_{3}), (R_{4}, Q_{4})) \in V_{t''}.
\]

Consider a coordinate transformation $\varphi$ on $\mathbb{P}^{2}$ such that $\varphi(R_{2}) : X_{1} = 0$, $\varphi(R_{3}) : X_{2} = 0$, $\varphi(R_{4}) : X_{3} = 0$, $\varphi(Q_{2}) = (0 : 1 : 1)$ and $\varphi(Q_{3}) = (1 : 0 : 1)$, hence
\[
\begin{pmatrix}
0 & -\frac{Bt + A}{t} & 0 \\
0 & 0 & 1 - A \\
-A & -B & 1
\end{pmatrix}
\]
is a matrix corresponding to $\varphi$. So the image of $Q_{4} = (\alpha, \beta, A \alpha + B \beta)$ under $\varphi$ is equal to $(\frac{\beta - Bt + A}{t} : (1 - A)(A \alpha + B \beta) : 0) = (1 : -t'' : 0)$. \[\square\]
We can rewrite the equation $t(t' - 1)AB = (t't'' - t)A + tt'(1 - t'')B$ as

$$(t' - 1)AB = (\frac{tt''}{t} - 1)A + t'(1 - t'')B.$$ 

If $t', t''$ and $t't''/t$ are fixed and not equal to 1, we have a smooth conic of lines $L_4$ and for each line $L_4$ on this conic, there is only one point $P_4$ such that $(\mathcal{L}, \mathcal{P}) \in V_{d,4}$. So we get a 9-dimensional component of $V_{d,4}$.

If $t \neq 1$ and exactly one of the numbers $t', t''$ or $t't''/t$ is equal to 1, it is easy to see that $L_4$ moves in a pencil of lines through a fixed point ($A \neq 0$ and $B \neq 0$ since otherwise $L_4$ contains $P_{1,2}$ respectively $P_{1,3}$). Once $L_4$ is fixed, we have only one choice for $P_4$ so that $(\mathcal{L}, \mathcal{P}) \in V_{d,4}$. Again, this gives us a 9-dimensional component of $V_{d,4}$.

If at least two of the numbers $t, t', t''$ or $t't''/t$ are equal to 1, the points $P_1, P_2, P_3, P_4$ are collinear and so $t = t' = t'' = t't''/t = 1$. In this case, we get no condition on the line $L_4$ and for each line $L_4$ we have one point $P_4$ such that $(\mathcal{L}, \mathcal{P}) \in V_{d,4}$, in particular $P_4$ is the point on $L_4$ collinear with $P_1, P_2$ and $P_3$. Hence this case gives rise to a 10-dimensional component of $V_{d,4}$.

Let $\nu_d$ be the set of elements $(t, t', t'')$ with $t, t', t'' \in \mu_d$ and no 2 or 3 of the elements $t, t', t''$ and $t't''/t$ equal to 1. It is easy to see that $(t, t', t'') \in \nu_d$ if $(t, t', t'') \in (\mu_d)^3$ is not of the form $(a, 1, 1), (1, a, 1), (1, 1, a), (a, a, 1), (a, 1, a)$ or $(1, a, 1/a)$ with $a \neq 1$, hence $|\nu_d| = d^3 - 6(d - 1)$. We denote for $(t, t', t'') \in \nu_d$, the corresponding component of $V_{d,4}$ by $V_{t,t',t''}$. Notice that $V_{t,t',t''}$ is 9-dimensional for each $(t, t', t'') \in \nu_d \setminus \{(1, 1, 1)\}$ and that $V_{1,1,1}$ is 10-dimensional. All these components are rational, since they are birationally equivalent to $\text{Aut}(\mathbb{P}^2) \times \mathbb{P}^1$ or $\text{Aut}(\mathbb{P}^2) \times (\mathbb{P}^1)^2$.

Now assume that $(\mathcal{L}, \mathcal{P})$ is an element of $V_{d,4}$, with

$$(\mathcal{L}_3, \mathcal{P}_3) = ((L_1, P_1), (L_2, P_2), (L_3, P_3)) \in (\mathcal{P}^2)^{3,0,2},$$

hence $L_1, L_2$ and $L_3$ are concurrent. From the case $e = 3$, it follows that $P_1, P_2$ and $P_3$ are collinear (since $(\mathcal{L}_3, \mathcal{P}_3) \in V_{d,3} \cap (\mathcal{P}^2)^{3,0,2}$) and $(\mathcal{L}_3, \mathcal{P}_3) \in V_1$.

If $L_4$ contains the intersection point $S = L_1 \cap L_2 \cap L_3$, we have that all lines $L_1, L_2, L_3$ and $L_4$ are concurrent and the points $P_1, P_2, P_3$ and $P_4$ have to be collinear, hence $(\mathcal{L}, \mathcal{P}) \in V_{1,1,1,1}$.

If $L_4$ does not contain the point $S$, there exist elements $t'$ and $t''$ in $\mu_d$ such that $((L_1, P_1), (L_2, P_2), (L_4, P_4)) \in V_{t'}$ and $((L_2, P_2), (L_3, P_3), (L_4, P_4)) \in V_{t''}$, hence $(\mathcal{L}, \mathcal{P}) \in V_{1,t',t''}$.

The above results are summarized in the following theorem.

**Theorem 2.2.** For $k = \mathbb{C}$, the set $V_{d,4}$ has $d^3 - 6(d - 1) - 1$ components of dimension 9 and one component of dimension 10. Each component is rational.
For all \((t, t', t'')\) \(\in \nu_d\), denote by \(V_{t,t',t''}\) the union of the spaces of curves \(\mathbb{P}(V(L, P))\) with \((L, P) \in V_{t,t',t''}\) and by \(V_{t,t',t''}^\circ\) the subset of points corresponding to smooth curves. Since \(V_{t,t',t''}^\circ \subset V_{t,t',t''}\), we can consider the image of \(V_{t,t',t''}^\circ\) under the moduli map \(m_{d,4}: V_{t,t',t''}^\circ \rightarrow M_{(d-1)(d-2)/2}\). Denote this image by \(M(V_{t,t',t''})\).

If \((\mathcal{L}_3, \mathcal{P}_3) \in V_t\) general, denote by \(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3) \subset V_{t,t',t''}\) the union of spaces of curves \(\mathbb{P}(V((\mathcal{L}_3, \mathcal{P}_3), (L_4, P_4)))\) with \((\mathcal{L}_3, \mathcal{P}_3), (L_4, P_4)\) \(\in V_{t,t',t''}\) and by \(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3)^\circ\) the points corresponding to smooth curves. Let \(M(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3))\) be the image of \(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3)^\circ\) under the moduli map \(m_{d,4}\). Since for a general element \((\mathcal{L}, \mathcal{P}) \in V_{t,t',t''}\), there exists a \(\phi \in \text{Aut}(\mathbb{P}^2)\) and an element \((L_4, P_4) \in \mathcal{P}\) such that \((\mathcal{L}, \mathcal{P}) = \phi((\mathcal{L}_3, \mathcal{P}_3), (L_4, P_4))\), we have that \(M(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3))\) is an open dense subset of \(M(V_{t,t',t''})\). So if we want to prove that \(M(V_{t,t',t''})\) is rational, it is enough to show that \(M(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3))\) is rational for a general element \((\mathcal{L}_3, \mathcal{P}_3) \in V_t\).

**Proposition 2.3.** If \(d \geq 6\) and \((t, t', t'') \in \nu_d \setminus \{(1, 1, 1)\}\), then a general curve in \(V_{t,t',t''}^\circ\) has exactly 4 total inflection points. In particular, if \((\mathcal{L}, \mathcal{P}) \in V_{t,t',t''}\) general, a general curve of \(\mathbb{P}(V(\mathcal{L}, \mathcal{P}))^\circ\) has exactly 4 total inflection points.

**Proof.** Suppose \(d \geq 6\) and that a general curve in \(V_{t,t',t''}^\circ\) has more than 4 inflection points. Then there exists a closed subset \(W \subset V_{d,5}\) such that the map \(p: W \rightarrow V_{d,4}: (\mathcal{L}, \mathcal{P}) \mapsto (\mathcal{L}_4, \mathcal{P}_4)\) has as image \(V_{t,t',t''}\) and such that the union \(W\) of spaces of curves \(\mathbb{P}(V(\mathcal{L}, \mathcal{P}))\) with \((\mathcal{L}, \mathcal{P}) \in W\) is equal to \(V_{t,t',t''}\). We know that \(\dim W = \dim V + \binom{d-3}{2}\) and \(\dim V_{t,t',t''} = 9 + \binom{d-2}{2}\), hence \(\dim W = 9 + \binom{d-2}{2} - \binom{d-3}{2} = 6 + d \geq 12\). Since \(\dim(V_{t,t',t''}) = 9\), the dimension of a general fibre of \(p\) is equal to \(d - 3 \geq 3\). On the other hand, such a general fiber can be seen as a subset of \(\mathcal{P}_2\) and hence its dimension is at most 3. So it follows that \(d = 6\) and a general fiber of \(p\) is equal to \(\mathcal{P}_2\). If we define the map \(q\) to be the projection \(W \rightarrow V_{d,3}: (\mathcal{L}, \mathcal{P}) \mapsto ((L_1, P_1), (L_2, P_2), (L_3, P_3))\), we get that \(p(W) = (\mathcal{P}_2)^3\), hence \(V_{d,3} = (\mathcal{P}_2)^3\), a contradiction. \(\square\)

**Proposition 2.4.** Assume \(d \geq 6\) and \((t, t', t'') \in \nu_d \setminus \{(1, 1, 1)\}\). If \((\mathcal{L}_3, \mathcal{P}_3)\) is a general element of \(V_t\), the space \(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3)^\circ\) is rational.

**Proof.** There exists a smooth rational curve \(\Gamma \subset (\mathbb{P}^2)^*\) (if \(1 \not\in \{t', t'', t't''/t\}\), \(\Gamma\) is a conic; if \(t \neq 1\) and only one of the numbers \(t', t''\) and \(t't''/t\) is equal to 1, \(\Gamma\) is a line), such that for \(L_4 \in \Gamma\) general one finds a unique point \(P_4 \in L_4\) such that \(((\mathcal{L}_3, \mathcal{P}_3), (L_4, P_4)) \in V_{t,t',t''}\).

This defines a curve \(\tilde{\Gamma} \subset \mathbb{P}^2 \times (\mathbb{P}^2)^*\) together with a projection \(\tilde{\Gamma} \rightarrow \Gamma: (L_4, P_4) \mapsto L_4\) that is generically injective, hence \(\tilde{\Gamma} \rightarrow \Gamma\) is a birational equivalence. Since \(\Gamma\) is smooth, the normalization map of \(\tilde{\Gamma}\) defines an inverse morphism \(\Gamma \rightarrow \tilde{\Gamma}\), so \(\tilde{\Gamma}\) is isomorphic to \(\Gamma\) and \(\tilde{\Gamma}\) is rational and smooth.

On \(\tilde{\Gamma} \times \mathbb{P}^2\), consider the closed subscheme \(\mathcal{D}\) flat over \(\tilde{\Gamma}\) with the fiber over \((L_4, P_4)\) being \(dP_4 \subset L_4\).
Let $p_1 : \tilde{\Gamma} \times \mathbb{P}^2 \to \tilde{\Gamma}$ and $p_2 : \tilde{\Gamma} \times \mathbb{P}^2 \to \mathbb{P}^2$ be the projections and consider

$$\Phi : (p_2^*(\mathcal{O}_{\mathbb{P}^2}(d))) \to p_2^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \mathcal{O}_D$$

and

$$p_1*(\Phi) : p_1*(p_2^*(\mathcal{O}_{\mathbb{P}^2}(d))) = \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \times \tilde{\Gamma} \to p_1*(p_2^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \mathcal{O}_D) = E_D$$

whereby $E_D$ is locally free of rank $d$. Let $\tilde{\Phi}$ be the map $p_1*(\Phi)$ restricted to $V(\mathcal{L}_3, \mathcal{P}_3) \times \tilde{\Gamma}$ and consider the short exact sequence

$$0 \to \text{Ker}(\tilde{\Phi}) \to V(\mathcal{L}_3, \mathcal{P}_3) \times \tilde{\Gamma} \to \text{Im}(\tilde{\Phi}) \to 0.$$ 

Ker($\tilde{\Phi}$) and Im($\tilde{\Phi}$) are vector bundles over $\tilde{\Gamma}$ since they are torsion free and $\tilde{\Gamma}$ is a smooth curve. For $\tilde{x} \in \tilde{\Gamma}$, one has $\text{Tor}_1(\text{Im}(\tilde{\Phi}), k(\tilde{x})) = 0$, hence we have an exact sequence

$$0 \to \text{Ker}(\tilde{\Phi}) \otimes k(\tilde{x}) \to V(\mathcal{L}_3, \mathcal{P}_3) \to \text{Im}(\tilde{\Phi}) \otimes k(\tilde{x}) \to 0$$

with $\text{dim}[\text{Im}(\tilde{\Phi}) \otimes k(\tilde{x})] = d - 2$ by construction. From Im($\tilde{\Phi}$) $\subset E_D$, we obtain the following commutative diagram

$$\begin{array}{ccc}
V(\mathcal{L}_3, \mathcal{P}_3) & \to & \text{Im}(\tilde{\Phi}) \otimes k(\tilde{x}) \\
\downarrow \Phi(\tilde{x}) & & \downarrow u \\
E_D \otimes k(\tilde{x}) & \end{array}$$

By definition of $\tilde{\Gamma}$ one has $\text{rank}(\Phi(\tilde{x})) = d - 2$, hence $u$ is injective. This shows Ker($\tilde{\Phi}$) $\otimes k(\tilde{x})$ = Ker($\tilde{\Phi}$), hence Ker($\tilde{\Phi}$) $\otimes k(\tilde{x}) = V_d((\mathcal{L}_3, \mathcal{P}_3), (L_4, P_4))$. Consider the projection of $\mathbb{P}(\text{Ker}(\tilde{\Phi})) \subset \mathbb{P}(V(\mathcal{L}_3, \mathcal{P}_3)) \times \tilde{\Gamma}$ on $\mathbb{P}(V(\mathcal{L}_3, \mathcal{P}_3))$. Since $\mathbb{P}(\text{Ker}(\tilde{\Phi}))$ is rational and this projection is generically injective (see Prop. 2.3), we find that the image of the projection is rational.

Now let $S_4$ be the symmetric group of order 4, where we denote $\sigma \in S_4$ by $(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$. Define $\theta : S_4 \times \nu_d \to \nu_d$ as the map that maps $(\sigma, (t, t', t''))$ to $(t_0, t_1, t_2)$ if the component consisting of elements

$$(\mathcal{L}_\sigma, \mathcal{P}_\sigma) := ((L_{\sigma(1)}, P_{\sigma(1)}), \ldots, (L_{\sigma(4)}, P_{\sigma(1)})) \in V_{d, 4}$$

with $(\mathcal{L}, \mathcal{P}) \in V_{t, t', t''}$ is equal to $V_{t_0, t_1, t_2}$.

**Proposition 2.5.** If $d \geq 6$ and $(t, t', t''), (t_0, t_1, t_2) \in \nu_d \setminus \{(1, 1, 1)\}$, we have $M(V_{t, t', t''}) = M(V_{t_0, t_1, t_2})$ if and only if $\theta(\sigma, (t, t', t'')) = (t_0, t_1, t_2)$ for some element $\sigma \in S_4$.
Proof. The image of a curve contained in $\mathcal{V}_{t,t',t''}^o$ (resp. $\mathcal{V}_{t_0,t_1,t_2}^o$) under an automorphism $\phi \in \text{Aut}(\mathbb{P}^2)$ remains in $\mathcal{V}_{t,t',t''}^o$ (resp. $\mathcal{V}_{t_0,t_1,t_2}^o$), hence $M(\mathcal{V}_{t,t',t''}) = M(\mathcal{V}_{t_0,t_1,t_2})$ if and only if $\mathcal{V}_{t,t',t''}^o = \mathcal{V}_{t_0,t_1,t_2}^o$. Since $d \geq 6$, a general curve contained in $\mathcal{V}_{t,t',t''}^o$ or $\mathcal{V}_{t_0,t_1,t_2}^o$ has exactly 4 total inflection points. Thus if $\mathcal{V}_{t,t',t''}^o = \mathcal{V}_{t_0,t_1,t_2}^o$, the total inflection points just are ordered in a different way and so there has to be a permutation $\sigma \in S_4$ such that $V_{t_0,t_1,t_2} = \{(L_\sigma, P_\sigma) \mid (L, P) \in V_{t,t',t''}\}$. \hfill $\square$

Write $O[t,t',t'']$ to denote $\theta(S_4 \times \{(t,t',t'')\})$ and $S[t,t',t'']$ to denote

$$\{\sigma \in S_4 \mid \theta(\sigma, (t,t',t'')) = (t,t',t'')\}$$

for $(t,t',t'') \in \nu_d$.

**Proposition 2.6.** For each $(t,t',t'') \in \nu_d$, we have that $S[t,t',t'']$ is a subgroup of $S_4$ and $|S[t,t',t'']| |O[t,t',t'']| = 24 = |S_4|$. Moreover, the sets $O[t,t',t'']$ with $(t,t',t'') \in \nu_d$ form a partition of $\nu_d$.

Proof. Note that $\theta$ is a left group action of $S_4$ on $\nu_d$, $S[t,t',t'']$ is the stabilizer of $(t,t',t'')$ and $O[t,t',t'']$ is the orbit of $(t,t',t'')$. The statement of the proposition now follows from classical theorems on group actions. \hfill $\square$

Consider the following table.

<table>
<thead>
<tr>
<th>$\sigma \in S_4$</th>
<th>$\theta(\sigma, (t,t',t''))$</th>
<th>$T \in \nu_d$ with $\theta(\sigma, T) = T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1,2,3,4)$</td>
<td>$(t,t',t'')$</td>
<td>$T = (t,t',t'') \in \nu_d$</td>
</tr>
<tr>
<td>$(1,3,2,4)$</td>
<td>$(\frac{t''}{t}, \frac{t'}{t'}, \frac{1}{t''})$</td>
<td>$T = (1,1,1)$ or $T = (-1, a, -1)$ with $a \in \mu_d \setminus {1}$ and $d$ even</td>
</tr>
<tr>
<td>$(1,2,4,3)$</td>
<td>$(t',t, \frac{1}{t'})$</td>
<td>$T = (1,1,1)$ or $T = (a, a, -1)$ with $a \in \mu_d \setminus {1}$ and $d$ even</td>
</tr>
<tr>
<td>$(1,4,2,3)$</td>
<td>$(\frac{1}{t''}, \frac{t'}{t''}, t'')$</td>
<td>$T = (a, \frac{1}{a}, a^3)$ with $a \in \mu_d$</td>
</tr>
<tr>
<td>$(1,3,4,2)$</td>
<td>$(\frac{t'}{t''}, \frac{1}{t'}, \frac{1}{t''})$</td>
<td>$T = (a, \frac{1}{a}, a^3)$ with $a \in \mu_d$</td>
</tr>
<tr>
<td>$(1,4,3,2)$</td>
<td>$(\frac{t'}{t''}, \frac{1}{t'}, \frac{1}{t''})$</td>
<td>$T = (1,1,1)$ or $T = (a, -1, -1)$ with $a \in \mu_d \setminus {1}$ and $d$ even</td>
</tr>
</tbody>
</table>

*continued on the next page*
In the third columns, all $T \in \nu_d$ are listed such that $\theta(\sigma, T) = T$. These
Proposition 2.7. Let \((t, t', t'') \in \nu_d\). The set \(O[t, t', t'']\) has less than 24 elements if and only if \((t, t', t'')\) is of the form

\[
(a^3, a, a), (a, \frac{1}{a}, a^3), (a, a^3, \frac{1}{a}), (a, a, a), (a, a, a), (a, \frac{1}{a}, a) \text{ or } (a, a, \frac{1}{a})
\]

or \(d\) is even and \((t, t', t'')\) is of the form

\[
(-1, -1, a), (-1, a, -1), (a, -1, -1), (-1, a, \frac{1}{a}), (a, -1, a), (a, a, -1), (-a, a, a), (a, \frac{1}{a}, -a) \text{ or } (a, -a, \frac{1}{a}).
\]

Moreover, then \(O[t, t', t'']\) is of the form \(\{(1, 1, 1), (-1, -1, -1)\} \text{ (d even)}, \{(i, -i, i), (i, -i, -i)\} \text{ (d \(\in\) 4\(\mathbb{Z}\)},

\[
A(a) := \left\{(a^3, a, a), (a, \frac{1}{a}, a^3), (a, a^3, \frac{1}{a}), (\frac{1}{a}, a, a), \right. \\
\left. (\frac{1}{a^3}, \frac{1}{a}, \frac{1}{a^3}), (\frac{1}{a}, a, \frac{1}{a^3}), (\frac{1}{a^3}, a, \frac{1}{a}) \right\}
\]

for some \(a \in \{\omega^j \mid 0 < j < d/2\} \setminus \{i\},

\[
B(a) := \left\{(a, a, a), (a, \frac{1}{a}, a), (a, a, \frac{1}{a}), (\frac{1}{a}, \frac{1}{a}, \frac{1}{a}), (\frac{1}{a}, a, \frac{1}{a}), (\frac{1}{a}, \frac{1}{a}, a) \right\}
\]

for some \(a \in \{\omega^j \mid 0 < j < d/2\},

\[
C(a) := \left\{(-1, -1, a), (-1, a, -1), (a, -1, -1), (-1, a, \frac{1}{a}), (a, -1, a), (a, a, -1), \\
(-1, -1, \frac{1}{a}), (-1, \frac{1}{a}, -1), (\frac{1}{a}, -1, -1), (-1, \frac{1}{a}, a), (\frac{1}{a}, -1, \frac{1}{a}), (\frac{1}{a}, \frac{1}{a}, -1) \right\}
\]

for some \(a \in \{\omega^j \mid 0 < j < d/2\} \text{ (d even)}\) or

\[
D(a) := \left\{(-a, a, a), (a, \frac{1}{a}, -a), (a, -a, \frac{1}{a}), (a, -a, -a), (-a, -a, \frac{1}{a}), (-a, a, -\frac{1}{a}), \right. \\
\left. (-\frac{1}{a}, \frac{1}{a}, \frac{1}{a}), (\frac{1}{a}, a, -\frac{1}{a}), (\frac{1}{a}, -a, \frac{1}{a}), (\frac{1}{a}, -a, -\frac{1}{a}), (-\frac{1}{a}, -a, \frac{1}{a}), (-\frac{1}{a}, \frac{1}{a}, -a) \right\}
\]

for some \(a \in \{\omega^j \mid 0 < j < d/4\} \text{ (d even).}\)
Proposition 2.8. Let \((t, t', t'')\) for all \((t, t', t'')\). If and only if \(\{Id\} \not\subseteq S[t, t', t'']\), hence if and only if there exists a \(\alpha \in S_4 \setminus \{Id\}\) such that \(\theta(\alpha, (t, t', t'')) = (t, t', t'')\). These \((t, t', t'')\) can be found in the third column of the table above. For such \((t, t', t'')\) \(\in \nu_d\) (with \(|O[t, t', t'']| < 24\), we can compute \(O[t, t', t'']\) by using the second column of the table. We use hereby the fact that the sets \(O[t, t', t'']\) form a partition of \(\nu_d\).

If \((t, t', t'')\) \(\in \nu_d\) does not appear in the list given in Proposition 2.7, we have \(|O[t, t', t'']| = 24\). Hence, in case \(d\) is odd, the number of components of \(M(V_{d,4})\) is equal to

\[
1 + 2 \cdot \frac{d - 1}{2} + \frac{d^3 - 6(d - 1) - 1 - (8 + 6) \cdot \frac{d - 1}{2}}{24} = \frac{d^3 + 11d + 12}{24}.
\]

Analogously, we can see that if \(d\) is odd, \(M(V_{d,4})\) has \(\frac{d^3 + 20d}{24}\) components.

The following proposition gives us a full list of the subgroups \(S[t, t', t''] \subset S_4\) for all \((t, t', t'')\) \(\in \nu_d\) (up to isomorphism).

Proposition 2.8. Let \((t, t', t'')\) \(\in \nu_d\).

- if \((t, t', t'') = (1, 1, 1)\) or \((-1, -1, -1)\) (\(d\) even): \(S[t, t', t''] = S_4\),
- if \(d \in 4\mathbb{Z}\), \((t, t', t'') = (-i, i, i)\) or \((i, -i, -i)\): \(S[t, t', t''] = A_4\),
- if \((t, t', t'')\) \(\in A(a)\) for some \(a \in \{\omega^j | 0 < j < d/2\} \setminus \{i\}\): \(S[t, t', t''] \cong \mathbb{Z}_5\),
- if \((t, t', t'')\) \(\in B(a)\) for some \(a \in \{\omega^j | 0 < j < d/2\}\): \(S[t, t', t''] \cong \mathbb{Z}_4\),
- if \((t, t', t'')\) \(\in C(a)\) for some \(a \in \{\omega^j | 0 < j < d/2\}\): \(S[t, t', t''] \cong \mathbb{Z}_2\),
- if \((t, t', t'')\) \(\in D(a)\) for some \(a \in \{\omega^j | 0 < j < d/4\}\): \(S[t, t', t''] \cong \mathbb{Z}_2\),
- in the other cases: \(S(t, t', t'') = \{Id\} \subset S_4\).

Proof. To compute \(S[t, t', t'']\) for an element \((t, t', t'')\) \(\in \nu_d\), we only have to write down all the elements \(\sigma \in S_4\) with \(\theta(\sigma, (t, t', t'')) = (t, t', t'')\) (this can be done using the table). The number of elements \(|S[t, t', t'']|\) is equal to \(24/|O[t, t', t'']|\).

Proposition 2.9. If \(d \geq 6\) and

\[(t, t', t'') \in \nu_d \setminus \{(1, 1, 1), (-1, -1, -1), (-i, i, i), (i, -i, -i)\},\]

the component \(M(V_{t,t',t''}) \subset M(V_{d,4})\) is rational.
Proof. Let \((\mathcal{L}, \mathcal{P}_3) \in V_t\) be general. It is enough to prove that \(M(V_{t',t''}(\mathcal{L}, \mathcal{P}_3))\) is rational. We will first prove that

\[
M(V_{t',t''}(\mathcal{L}, \mathcal{P}_3)) \cong \frac{V_{t',t''}(\mathcal{L}, \mathcal{P}_3)^o}{S[t, t', t'']},
\]

Assume that \(C\) and \(C'\) are general smooth curves in \(V_{t',t''}(\mathcal{L}, \mathcal{P}_3)^o\) with \(C \in \mathbb{P}(V((\mathcal{L}, \mathcal{P})))\) and \(C' \in \mathbb{P}(V(\mathcal{L}', \mathcal{P}'))\). We have that \([C] = [C'] \in M(V_{t',t''}(\mathcal{L}, \mathcal{P}_3))\) if and only if there exists an automorphism \(\phi\) of \(\mathbb{P}^2\) such that \(\phi(C) = C'\). If the latter happens, since \(C\) and \(C'\) have exactly 4 total inflection points, the automorphism \(\phi\) changes the order of the total inflection points, hence there exists a permutation \(\sigma \in S_4\) such that \(\phi(\mathcal{L}, \mathcal{P}) = (\mathcal{L}_\sigma, \mathcal{P}_\sigma')\). Thus we get that \((\mathcal{L}', \mathcal{P}')\) and \((\mathcal{L}_\sigma', \mathcal{P}_\sigma')\) are contained in \(V_{t',t''}\), so \(\sigma \in S[t, t', t'']\). On the other side, if \(C \in \mathbb{P}(V(\mathcal{L}, \mathcal{P})) \subset V_{t',t''}(\mathcal{L}, \mathcal{P}_3)^o\) and \(\sigma \in S[t, t', t'']\), the curve \(C\) also belongs to \(\mathbb{P}(V(\mathcal{L}_\sigma, \mathcal{P}_\sigma))\). Since there exists an automorphism \(\varphi\) of \(\mathbb{P}^2\) such that \(\varphi(\mathcal{L}_\sigma, \mathcal{P}_\sigma)\) is of the form \(((\mathcal{L}_3, \mathcal{P}_3), (\mathcal{L}_3', P_3'))\), we have \(C' = \varphi(C) \in V_{t',t''}(\mathcal{L}, \mathcal{P}_3)^o\).

By Proposition 2.8, we have that \(S[t, t', t''] \subset S_4\) is Abelian. By using a result due to E. Fischer (see [5]) and Proposition 2.4, we conclude that \(M(V_{t',t''}(\mathcal{L}, \mathcal{P}_3))\) is rational.

\(\square\)

Remark 2.10. From [2, Theorem C] follows that \(M(V_{1,1,1})\) is rational. We cannot use the above arguments in order to prove that \(M(V_{t',t''})\) is rational for \((t, t', t'') \in \{(1, 1, 1), (-1, -1, -1), (-i, i, i), (i, -i, -i)\}\), since in each of those cases, the group \(S[t, t', t'']\) is not Abelian.

The results on the components of \(M(V_{d,4})\) are summarized in the following theorem.

**Theorem 2.11.** Assume \(k = \mathbb{C}\) and \(d \geq 6\). If \(d\) is odd, \(M(V_{d,4})\) has \(\frac{d^3+11d+12}{24}\) components and each of these components is rational. If \(d\) is even, \(M(V_{d,4})\) has \(\frac{d^3+20d}{24}\) components and at most \(\tau\) of them are not rational, whereby \(\tau = 3\) if \(4|d\) and \(\tau = 1\) if \(d \equiv 2 \mod 4\).

### 3 A result for the case \(e = 5\)

We will first give a new proof of the following result of A.M. Vermeulen (see [7, Prop. 2.12]).

**Proposition 3.1.** Let \((\mathcal{L}, \mathcal{P})\) be an element of \((\mathbb{P}^2)^e\) such that no 3 lines are concurrent. If for all \(2 \leq i < j \leq e\), we have \(((L_1, P_i), (L_i, P_i), (L_j, P_j)) \in V_{d,3}\), then \((\mathcal{L}, \mathcal{P}) \in V_{d,e}\).
Proof. It is easy to see that it is enough to prove the following claim: assume
\((\mathcal{L}, \mathcal{P}) = ((L_1, P_1), \ldots, (L_e, P_e)) \in V_{d,e}\) and \((L, P) \in \mathcal{P}^2\) such no 3 of the lines
\(L_1, \ldots, L_e, L\) are concurrent. If moreover \(((L_1, P_1), (L_i, P_i), (L, P)) \in V_{d,3}\) for all
\(i \in \{2, \ldots, e\}\), we have \(((\mathcal{L}, \mathcal{P}), (L, P)) \in V_{d,e+1}\).

We will first consider the case where \(e \leq d + 1\). Let \(\Gamma \in \mathbb{P}(V(\mathcal{L}, \mathcal{P}))\) be a
curve not containing one of the lines \(L_i\). We have
\[
\mathbb{P}(V_L(\mathcal{L}, \mathcal{P})) = \langle P_{1,0} + \ldots + P_{e,0} + g_{d-e}^{d-e}, \Gamma.L \rangle
\]
\((\{\Gamma.L\})\) in case \(e = d + 1\) with \(P_{i,0} = L_i \cap L\). Since \(\Gamma \in \mathbb{P}(V((L_1, P_1), (L_i, P_i)))\)
and \(((L_1, P_1), (L_i, P_i), (L, P)) \in V_{d,3}\) for all \(i \in \{2, \ldots, e\}\), one has
\[
dP \in \mathbb{P}(V((L_1, P_1), (P_i, L_i))) = \langle P_{1,0} + P_{i,0} + g_{d-2}^{d-2}, \Gamma.L \rangle.
\]
Of course, we have
\[
\mathbb{P}(V_L(\mathcal{L}, \mathcal{P})) \subset \bigcap_{i=2}^{e} \mathbb{P}(V_L((L_1, P_1), (L_i, P_i)))
\]
and \(\dim(\mathbb{P}(V_L(\mathcal{L}, \mathcal{P}))) = d - e + 1\).

For \(2 \leq i \leq e\), we have \(P_{1,0} + \ldots + P_{i,0} + g_{d-e}^{d-e} \subset \bigcap_{j=2}^{i} \mathbb{P}(V_L((L_1, P_1), (L_j, P_j)))\).

Take \(F \in g_{d-i}^{d-i}\) with \(P_{i+1,0} \not\in F\), then \(P_{1,0} + \ldots + P_{i,0} + F \not\in P_{1,0} + P_{i+1,0} + g_{d-2}^{d-2}\). If
\(P_{1,0} + \ldots + P_{i,0} + F \in \mathbb{P}(V_L((L_1, P_1), (L_{i+1}, P_{i+1})))\), then for some \(G \in g_{d-2}^{d-2}\), we
have \(P_{1,0} + \ldots + P_{i,0} + F \in \langle P_{1,0} + P_{i+1,0} + G, \Gamma.L \rangle\), hence
\[
\Gamma.L \in \langle P_{1,0} + P_{i+1,0} + G, P_{1,0} + \ldots + P_{i,0} + F \rangle,
\]
so \(P_{1,0} \in \Gamma \cap L\). This implies \(L_1 \subset \Gamma\) since \(P_{1,0} \in \Gamma \cap L_1\), so we get a contradiction
and \(P_{1,0} + \ldots + P_{i,0} + F \not\in \mathbb{P}(V_L((L_1, P_1), (L_{i+1}, P_{i+1})))\) and so
\[
\dim \left\{ \bigcap_{j=2}^{i+1} \mathbb{P}(V_L((L_1, P_1), (L_j, P_j))) \right\} < \dim \left\{ \bigcap_{j=2}^{i} \mathbb{P}(V_L((L_1, P_1), (L_j, P_j))) \right\}.
\]
This proves \(\dim \left\{ \bigcap_{j=2}^{e} \mathbb{P}(V_L((L_1, P_1), (L_j, P_j))) \right\} = d - e + 1\), hence
\[
\mathbb{P}(V_L(\mathcal{L}, \mathcal{P})) = \bigcap_{j=2}^{e} \mathbb{P}(V_L((L_1, P_1), (L_j, P_j)))
\]
and so \(dP \in \mathbb{P}(V_L(\mathcal{L}, \mathcal{P}))\), hence \(((\mathcal{L}, \mathcal{P}), (L, P)) \in V_{d,e+1}\).

Now assume \(e \geq d + 2\). We have \(\mathbb{P}(V((L_1, P_1), \ldots, (L_{d+1}, P_{d+1}))) = \{\Gamma\}\). Since
\((\mathcal{L}, \mathcal{P}) \in V_{d,e}\), we have \(dP_i = \Gamma.L_i\) for all \(i \in \{1, \ldots, e\}\). The previous part
of this proof implies \(((L_1, P_1), \ldots, (L_{d+1}, P_{d+1}), (L, P)) \in V_{d,d+2}\), hence \(dP \in \Gamma.L\).
So we find \(\Gamma \in \mathbb{P}(V((\mathcal{L}, \mathcal{P}), (L, P)))\), i.e. \(((\mathcal{L}, \mathcal{P}), (L, P)) \in V_{d,e+1}\).
Theorem 3.2. Assume that $V \subset V_{d,5}$ a component is of dimension 10 such that for a general $(\mathcal{L}, \mathcal{P}) \in V$, no three of the points $P_1, \ldots, P_5$ are collinear. Then $d$ is even and $(\mathcal{L}, \mathcal{P}) \in V_{d,5}$.

Proof. Assume that

$$((L_1, P_1), (L_2, P_2), (L_3, P_3), (L_4, P_4)) \in V_{t_1, t_2, t_3}$$

and

$$((L_1, P_1), (L_2, P_2), (L_3, P_3), (L_5, P_5)) \in V_{t_1, t'_2, t''_3}.$$ 

Note that $\dim(V_{t_1, t'_2, t''_3}) = \dim(V_{t_1, t_2, t_3}) = 9$ (we even have that none of the numbers $t_1, t'_2, t''_3$ or $t_2'$ is equal to 1) and that $t_1 = t_2$. Since $\dim(V_{t_1}) = 8$ and $\dim(V) = 10$, for general elements $((\mathcal{L}_3, \mathcal{P}_3), (\mathcal{L}_4, \mathcal{P}_4)) \in V_{t_1, t'_2, t''_3}$ and $((\mathcal{L}_3, \mathcal{P}_3), (\mathcal{L}_5, \mathcal{P}_5)) \in V_{t_1, t_2, t_3}$ we have $((\mathcal{L}_3, \mathcal{P}_3), (\mathcal{L}_4, \mathcal{P}_4), (\mathcal{L}_5, \mathcal{P}_5)) \in V$.

One needs $((L_1, P_1), (L_2, P_2), (L_4, P_4), (L_5, P_5)) \in V_{t_1, t'_2, t''_3}$ general for some $t_3, t'_3, t''_3 \in \mu_d$ (we omitted the accents in $(L'_4, P'_4)$ and $(L_5', P_5')$ for notational reasons). Using the base $\{(L_1, P_1), (L_2, P_2), (L_3, P_3)\}$, let $(X_1 : X_2 : X_3)$ be the coordinates of $\mathbb{P}^2$, and by using $\{(L_1, P_1), (L_2, P_2), (L_4, P_4)\}$, let them be $(X'_1 : X'_2 : X_3')$. Assume $L_4$ has as equation $X_3 = A X_1 + B X_2$. It is easy to see that

$$\begin{pmatrix} 1 - A & 0 & 0 \\ 0 & 1 - B & 0 \\ -A & -B & 1 \end{pmatrix}$$

is a matrix corresponding to the coordinate transformation from the coordinates $(X_1 : X_2 : X_3)$ to $(X'_1 : X'_2 : X_3')$. The equation $X'_3 = A' X'_1 + B' X'_2$ of $L_5$ becomes


Hence $(A, B) = (A'(1 - A) + A, B'(1 - B) + B)$ is a general solution of

$$(t'_2 - 1) A + (t''_2(t''_2 - 1) A + t'_2(1 - t''_2) B. \tag{1}$$

This equation holds for general $(A', B')$ satisfying

$$(t'_3 - 1) A' B' = (t''_3(t''_3 - 1) A' + t'_3(1 - t''_3) B'. \tag{2}$$

Since (2) is an equation without constant term, the constant term in (1) has to be equal to zero, so we get

$$(t'_2 - 1) A B = (t''_2(t''_2 - 1) A + t'_2(1 - t''_2) B. \tag{3}$$

This equation should be satisfied for general $(A, B)$ satisfying

$$(t'_1 - 1) A B = (t''_1(t''_1 - 1) A + t'_1(1 - t''_1) B, \tag{4}$$

18
hence (3) and (4) have to define the same curve. If we take $A = 1$ we get that

$$B = \frac{t_1 t''_2}{t_1 t'_1} - 1 = \frac{t_2 t''_2}{t_2 t'_2} - 1.$$ 

Since $t_1 = t_2$, we get $t'_1 t''_1 = t'_2 t''_2$, hence the coefficients of $A$ in (3) and (4) are equal. So we get that the coefficients of $AB$ in (3) and (4) are also equal, so $t'_1 = t'_2$ and thus $t'_2 = t''_2$. We can conclude that $((L_1, P_1), (L_2, P_2), (L_3, P_3), (L_4, P_4))$ and $((L_1, P_1), (L_2, P_2), (L_3, P_3), (L_5, P_5))$ belong to the same component of $V_{d,4}$.

Now let $((\mathcal{L}^i), (\mathcal{P}^j)) = ((\mathcal{L}_3, \mathcal{P}_3), (L_j, P_j))$ be general elements of $V_{1, t_1, t''_1}$ for each $j = 4, \ldots, m$. We see that $((\mathcal{L}_3, \mathcal{P}_3), (L_i, P_i), (L_j, P_j)) \in V$ for each $i, j \in \{4, \ldots, m\}$ with $i \neq j$. Hence we get that $((L_1, P_1), (L_i, P_i), (L_j, P_j)) \in V_{d,3}$, so $((\mathcal{L}, \mathcal{P})) = ((L_1, P_1), (L_2, P_2), \ldots, (L_m, P_m)) \in V_{d,m}$. Let $\Gamma \in \mathbb{P}(V(\mathcal{L}, \mathcal{P}))$ with $L_i \not\subset \Gamma$ for all $i$ and let $n_1 \Gamma_1 + \ldots + n_s \Gamma_s$ be its decomposition into irreducible curves. Write $d_i$ to denote the degree of $\Gamma_i$.

Assume $s \geq 2$. Since

$$d = i(\Gamma, L_j, P_j) = \sum_{i=1}^{s} n_i i(\Gamma_i, L_j, P_j) \leq \sum_{i=1}^{s} n_i d_i = d,$$

we get that $(L_j, P_j)$ is also a total inflection point of $\Gamma_i$ for all $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, m\}$. A fortiori, the points $P_1, \ldots, P_m$ are contained in $\bigcap_{i=1}^{s} \Gamma_i$, thus $m$ is bounded. We get a contradiction, so $s = 1$ and $\Gamma = n_1 \Gamma_1$.

If $d_1 \geq 3$, then the number of total inflection points of $\Gamma_1$ is bounded by $d$, hence $d_1 \leq 2$. Since $d_1 = 1$ is excluded, we find $d_1 = 2$ and $(\mathcal{L}, \mathcal{P}) \in V_{2,5}$. $\square$

**Remark 3.3.** A point $(\mathcal{L}, \mathcal{P}) \in (\mathcal{P}^2)^5$ is contained in $V_{2,5}$ if and only if there exists a smooth conic $C \subset \mathbb{P}^2$ through the five points $P_1, \ldots, P_5$ such that $L_i$ is the tangent line to $C$ at $P_i$ for all $i$. It is clear that $V_{2,5}$ is 10-dimensional (see also [3, Ex. 3.5]).

**Acknowledgement**

Both authors are partially supported by the Fund of Scientific Research - Flanders (G.0318.06).

**References**


