On $G_{k-1,k}$-defectivity of smooth surfaces and threefolds

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Abstract. — In this paper, we prove a rough characterization for $G_{k-1,k}$-defective $n$-dimensional non-degenerate varieties $X \subset \mathbb{P}^N$ if $k \geq n$. In the case of smooth surfaces or threefolds, we give a fine classification.

Keywords. — Grassmann-defectivity; secant varieties; surfaces; threefolds.

1 Introduction

Let $X$ be an irreducible non-degenerate projective variety of dimension $n$ in $\mathbb{P}^N$ and let $h$ and $k$ be integers such that $0 \leq h \leq k \leq N$. Then $G_{h,k}(X)$ is the closure in $G(h,N)$ of the set of $h$-dimensional linear subspaces contained in the span of $k+1$ different points of $X$ and is called the $h$-Grassmannian of $(k+1)$-secant $k$-planes of $X$. We say that $X$ is $G_{h,k}$-defective if the dimension of $G_{h,k}(X)$ is smaller then the expected dimension, which is the minimum between $(h+1)(N-h)$ and $(k+1)n + (k-h)(h+1)$.

In case $h = 0$, the variety $G_{0,k}(X)$ is just the $k$-th secant variety $S_k(X)$ of $X$. A variety $X$ is called $k$-defective if it is $G_{0,k}$-defective. Such varieties are intensively studied in [16].

In case $h > 0$, little is known. The most important reason for this is the lack of a so-called Terracini lemma, which in case $h = 0$ gives a description for the tangent space on $S_k(X)$ in a general point. Nevertheless, for example in [4] is shown that irreducible curves are not $G_{h,k}$-defective and in [5] there is given a classification of surfaces with $G_{1,2}$-defect. There is also a rough classification for varieties having $G_{n-1,n}$-defect together with a fine classification for $G_{2,3}$-defective smooth threefolds (see [7]).

Beside the intrinsic importance of $G_{h,k}$-defective varieties, defective varieties

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are also important for some extrinsic reasons. For example, varieties with $G_{h,k}$-defect have a strange behaviour under projections. Waring’s problem for forms (see [2, 6, 9]) gives us another extrinsic reason for studying defective varieties. This problem is in connection with the $G_{h,k}$-behaviour of Veronese embeddings of projective spaces.

In this paper we will classify the smooth surfaces $X$ in $\mathbb{P}^N$ with $G_{k-1,k}$-defect for $k > 2$.

**Theorem 1.1.** Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate surface and $k > 2$. Then $X$ is $G_{k-1,k}$-defective if and only if $N = k + 3$ and $X$ is of minimal degree $k + 2$.

We will also give a full classification of smooth threefolds $X \subset \mathbb{P}^N$ with $G_{k-1,k}$-defect for $k > 3$.

**Theorem 1.2.** Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate threefold and $k > 3$. Then $X$ is $G_{k-1,k}$-defective if and only if $X$ is one of the following varieties:

1. $X$ is a threefold of minimal degree $k + 2$ in $\mathbb{P}^{k+1}$;
2. $X$ is a threefold of minimal degree $k + 3$ in $\mathbb{P}^{k+5}$;
3. $X$ is the projection in $\mathbb{P}^{k+4}$ of a threefold of minimal degree $k + 3$ in $\mathbb{P}^{k+5}$;
4. $k = 4$ and $X$ is the (linearly normal) embedding in $\mathbb{P}^8$ of the blowing-up of $\mathbb{P}^3$ at a point.
5. $k = 5$ and $X$ is the image of the 2-uple embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$.

Compared with the classification of smooth $G_{2,3}$-defective varieties with $N \geq 7$ (see [7]), the first three cases are totally analogous.

Before proving Theorem 1.1 and Theorem 1.2 we will first give a rough characterization for $G_{k-1,k}$-defective $n$-dimensional varieties with $k \geq n$. Here we don’t require that $X$ needs to be smooth.

**Proposition 1.3.** Let $X$ be an $n$-dimensional variety in $\mathbb{P}^N$ and let $k \geq n$ be an integer. Then $X$ is $G_{k-1,k}$-defective if and only if $N \geq n + k + 1$ and one of the following properties hold for $k + 1$ general points $P_0, \ldots, P_k$ on $X$:

1. For each $i \in \{0, \ldots, k\}$, there exists a line $L_i$ on $X$ containing $P_i$ such that the linear span of the lines has dimension $k + 1$.
2. There exists a rational normal curve $\Gamma$ of degree $k + 1$ on $X$ containing $P_0, \ldots, P_k$. 

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We can see that both properties are enough for $G_{k-1,k}$-defectivity. In case $n$ is equal to 2 or 3, we will prove that the first property is the same as saying that $X$ is a cone (see Section 4). If $X$ satisfies the second property, we will prove that $X$ has sectional genus at most $n - 2$ (see Section 5).

2 Some conventions and generalities

2.1. Conventions. We denote the $N$-dimensional projective space over the field $\mathbb{C}$ of the complex numbers by $\mathbb{P}^N$. We write $\mathbb{G}(h, N)$ to denote the Grassmannian of $h$-dimensional linear subspaces of $\mathbb{P}^N$.

An $n$-dimensional variety $X$ in $\mathbb{P}^N$ is an irreducible reduced $n$-dimensional Zariski-closed subset of $\mathbb{P}^N$. We say that a variety $X \subset \mathbb{P}^N$ is non-degenerate if $X$ is not contained in a hyperplane of $\mathbb{P}^N$.

Let $X$ be a non-degenerate $n$-dimensional variety in $\mathbb{P}^N$. We say that a closed subscheme $Y \subset X$ is a $m$-dimensional section of $X$ if $Y$ is the scheme-theoretical intersection of $X$ with a linear subspace $\mathbb{P}^{N-n+m}$ of $\mathbb{P}^N$ such that all irreducible components have dimension $m$. We will often use the notions of curve section, surface section and hyperplane section in case $m$ is equal to respectively 1, 2 and $n - 1$.

The linear span $\langle Y \rangle$ of a closed subscheme $Y$ of $\mathbb{P}^N$ is the intersection of all hyperplanes $H \subset \mathbb{P}^N$ containing $Y$ as a closed subscheme. This linear span is always a linear subspace of $\mathbb{P}^N$. If $P_0, \ldots, P_r$ are different points of $\mathbb{P}^N$, we write $\langle P_0, \ldots, P_r \rangle$ to denote the linear span of the reduced subscheme of $\mathbb{P}^N$ supported by those points.

Let $Y$ be a closed subscheme of $\mathbb{P}^N$ and let $P \in Y$. We can take a hyperplane $H \subset \mathbb{P}^N$ such that $P \not\in H$ and identify $\mathbb{P}^N\setminus H$ with the affine space $\mathbb{A}^N$ and $Y\setminus(Y \cap H)$ with a closed subscheme of $\mathbb{A}^N$ (containing $P$). We can define the Zariski-tangent space $T_P(Y\setminus(Y \cap H)) \subset \mathbb{A}^N$ by using the equations of the subscheme $Y\setminus(Y \cap H)$. Its closure in $\mathbb{P}^N$ is called the embedded tangent space $T_P(Y)$ in $\mathbb{P}^N$ of $Y$ at $P$.

If $D_1$ and $D_2$ are divisors on a smooth surface $S$, we will write $D_1.D_2$ to denote the intersection number of those divisors. If $D$ is an effective divisor on $S$, then saying $D$ is irreducible means $D$ is integral (i.e. also reduced) by convention.

2.2. Definition of $G_{k-1,k}(X)$. Let $X \subset \mathbb{P}^N$ be a non-degenerate $n$-dimensional variety and let $k \leq N$ be an integer. The set of points $(P_0, \ldots, P_k)$ in $X^{k+1}$ with $\dim(\langle P_0, \ldots, P_k \rangle) = k$ is non-empty and open; so we have a rational map $\omega : X^{k+1} \dashrightarrow \mathbb{G}(k, N)$. An element of the image of $\omega$ is called a $(k + 1)$-secant
Another important example can be given by taking an abstract projective variety of the twisting sheaf of Serre $O_{2.5}$. Examples. If $S$ is an abstract projective variety and $G$ a polarized variety is a pair $(\alpha, \beta)$ implies that $X$ the uniform position lemma for curves (see [1] and [3, Proposition 2.6] for the argument) implies that $X$ points are contained in a general curve section of $k$. Let $2.4$. From Proposition 1.1 in [5] it follows that $G_{\kappa, k}$ is equal to $\expdim(G_{\kappa, k}(X)) = \min\{(k + 1)n + k, k(N - n)\}$.

If $\dim(G_{\kappa, k}(X))$ is smaller then this expected dimension, we say that $X$ has $G_{\kappa, k}$-defect. It is easy to see that in case $k \geq n$ the expected dimension of $G_{\kappa, k}(X)$ is equal to $(k + 1)n + k$ if and only if $N \geq n + k$.

If $\dim(G_{\kappa, k}(X)) = (k + 1)n + k - a$ and $N \geq n + k + 1$, for a general element $H \in G_{\kappa, k}(X)$ the set of $(k + 1)$-secant $k$-planes of $X$ containing $H$ has dimension $a$.

2.3. Let $X$ be a non-degenerate variety in $\mathbb{P}^N$ and let $k \leq N$ be an integer. From Proposition 1.1 in [5] it follows that $G_{\kappa, k}(X) := \text{im}(\omega)$ is equal to $\mathbb{G}(k, N)$ if $N \leq n + k$. Hence, $X$ is not $G_{\kappa, k}$-defective if $N \leq n + k$ since in this case $G_{\kappa, k}(X) := \alpha(\beta^{-1}(\text{im}(\omega))) = \mathbb{G}(k - 1, N)$. If $k > n$, this also follows from [9].

2.4. Let $X \subset \mathbb{P}^N$ be a non-degenerate $n$-dimensional variety with $N \geq n + k + 1$ for some integer $k$ and let $P_0, \ldots, P_k$ be general points on $X$. Then these $k + 1$ points are contained in a general curve section of $X$ in some $\mathbb{P}^{N-1}$. So the uniform position lemma for curves (see [1] and [3, Proposition 2.6] for the argument) implies that $X \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_k\}$ as a scheme. This implies that $\omega : X^{k+1} \rightarrow \mathbb{G}(k, N)$ is generically injective.

2.5. Polarized varieties. A polarized variety is a pair $(V, \mathcal{S})$ such that $V$ is an abstract projective variety and $\mathcal{S}$ is an ample invertible sheaf on $V$.

2.5.1. Examples. If $X \subset \mathbb{P}^N$ is a variety and $O_X(1)$ is the restriction to $X$ of the twisting sheaf of Serre $O_{\mathbb{P}^N}(1)$, the pair $(X, O_X(1))$ is a polarized variety. Another important example can be given by taking an abstract projective variety $V$ and a locally free sheaf $\mathcal{E}$ on $V$. Let $\mathbb{P}(\mathcal{E})$ be the projective bundle associated to $\mathcal{E}$ and let $O_{\mathbb{P}(\mathcal{E})}(1)$ be the associated tautological sheaf (see [12, p. 162]). If this

\[
\begin{tikzpicture}
    \node (G) at (0,0) {$\mathbb{G}(k - 1, N)$};
    \node (G2) at (2,0) {$\mathbb{G}(k, N)$};
    \node (I) at (1,1) {$I$};
    \draw[->] (I) -- (G) node[midway, left] {$\alpha$};
    \draw[->] (I) -- (G2) node[midway, right] {$\beta$};
\end{tikzpicture}
\]
sheaf is ample then \((\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))\) is a polarized variety and is called a scroll on \(V\).

### 2.5.2. Sectional genus.

For a polarized variety we can define the notion of sectional genus (for a general definition, see [10]). If \(S\) is very ample on \(V\) and \(V \subset \mathbb{P}^N\) is the embedding of \(V\) using the global sections of \(S\), then the sectional genus of \((V, S)\) is defined as being the arithmetic genus of a general curve section of \(V \subset \mathbb{P}^N\).

The classification of smooth polarized varieties \((V, S)\) of sectional genus at most one is given in [10, Section 12]. We only consider the case where \(V = X \subset \mathbb{P}^N\) and \(S = \mathcal{O}_X(1)\) with \(n = \dim(X) = 3\) and \(N \geq 8\).

If the sectional genus is 0 we only have scrolls of vectorbundles on \(\mathbb{P}^1\) as possibilities. Moreover, if \(X\) is embedded using the complete linear system then \(X\) is of minimal degree, so \(\deg(X) = N - 2\). We can obtain all smooth threefolds \(X \subset \mathbb{P}^N\) of minimal degree in this way.

If the sectional genus is equal to 1, the only possibilities are scrolls of vectorbundles on elliptic curves and Del Pezzo varieties. In our situation a Del Pezzo variety is one of the following possibilities (see [10, Section 8]):

i. \(\deg(X) = 7\); \(X\) is isomorphic to the blowing-up \(\sigma : \text{Bl}_Q(\mathbb{P}^3) \rightarrow \mathbb{P}^3\) at one point \(Q\) and \(\mathcal{O}_X(1) \cong \sigma^*(\mathcal{O}_{\mathbb{P}^3}(2)) \otimes \mathcal{O}_{\text{Bl}_Q(\mathbb{P}^3)}(-E)\) where \(E\) is the exceptional divisor.

ii. \(\deg(X) = 8\) and \((X, \mathcal{O}_X(1)) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))\).

### 2.6. Theorems of Bertini.

Let \(\mathcal{L}\) be a linear system on a smooth projective variety \(V\) without fixed components. Then, for a general element \(D \in \mathcal{L}\) the singular locus \(\text{Sing}(D)\) is contained in the locus of fixed points of \(\mathcal{L}\) on \(V\) and \(D\) is irreducible unless \(\mathcal{L}\) is composed with a pencil. For the proofs of this properties, see [13, 17, 18].

A linear system \(\mathcal{L}\) is composed by a pencil if and only if there exists a morphism \(f : W \rightarrow C\) with \(\sigma : W \rightarrow V\) the blowing-up of \(V\) at the fixed points of \(\mathcal{L}\) and \(C\) a curve such that the following holds. There is a linear system \(\mathcal{L}'\) on \(C\) with \(\dim(\mathcal{L}) = \dim(\mathcal{L}')\) such that for all \(D \in \mathcal{L}\) there exists a \(D' \in \mathcal{L}'\) such that \(D = \sigma(f^{-1}(D'))\). Using a Stein factorization and a desingularization for \(W\), one can see that we can assume that the general fibre of \(f\) is irreducible.

### 2.7. If \(D_1\) is an irreducible reduced divisor on a smooth projective variety \(V\) and \(D_2\) is an effective divisor on \(V\) linear equivalent to \(D_1\), then \(D_2\) is connected. For an argument, see Section 2.6 in [7].
3 A rough characterization

Proof of Proposition 1.3: Let $X \subset \mathbb{P}^N$ be an $n$-dimensional variety with $G_{k-1,k}$-defect for some $k \geq n$. From Sec. 2.3 it follows that $N \geq n + k + 1$, hence $\dim(G_{k-1,k}(X)) < (k + 1)n + k$ (Sec. 2.2).

Take $H \in G_{k-1,k}(X)$ general and consider the closure in $X^{k+1}$ of the set of points $(P_0, \ldots, P_k)$ with $P_i \neq P_j$ for all $i \neq j$ and $H \subset \langle P_0, \ldots, P_k \rangle$. Let $a$ be its dimension and let $\Omega_{H,k}$ be an $a$-dimensional component of that set. We know that $a \geq 1$. Take a general element $(P_0, \ldots, P_k)$ of $\Omega_{H,k}$. Since we have chosen $H \in G_{k-1,k}(X)$ generally, $(P_0, \ldots, P_k)$ is a general element of $X^{k+1}$. In particular, $(P_0, \ldots, P_k) \cap X = \{P_0, \ldots, P_k\}$ as a scheme. Now let $(Q_0, \ldots, Q_k)$ be another general element of $\Omega_{H,k}$.

Claim 1. For each $i \in \{0, \ldots, k\}$ one has $Q_i \not\in \{P_0, \ldots, P_k\}$.

Proof Claim 1: analogous to the proof of Claim 3.1 in [7]. □

Write $L = \langle P_0, \ldots, P_k \rangle$ and $M = \langle Q_0, \ldots, Q_k \rangle$. Since $L \neq M$, $\dim(L) = \dim(M)$ and $H \subset L \cap M$; one has $H = L \cap M$ and $\dim(\langle L \cup M \rangle) = k + 1$. Write $\mathbb{P}^{k+1} = \langle L \cup M \rangle$.

Claim 2. $\mathbb{P}^{k+1} \cap X$ is not finite.

Proof Claim 2: Assume $\mathbb{P}^{k+1} \cap X$ is finite.

Subclaim 2.1. A general linear subspace of $\mathbb{P}^N$ of dimension $N - n + 1$ containing $\mathbb{P}^{k+1} \cap X$ gives rise to an irreducible curve section of $X$ smooth at $P_0, \ldots, P_k$.

Proof Subclaim 2.1: analogous to the proof of Subclaim 3.3 in [7]. □

Denote by $\Psi'_0$ the closure of the set of elements $(P_0, \ldots, P_k; Q_0, \ldots, Q_k)$ in $X^{k+1} \times X^{k+1}$ such that $\dim(\langle P_0, \ldots, P_k \rangle) = k$, $P_i \neq P_j$ and $Q_i \neq Q_j$ for $i \neq j$, \{P_0, \ldots, P_k\} \neq \{Q_0, \ldots, Q_k\}$ and $H \subset \langle Q_0, \ldots, Q_k \rangle$ for some $(k - 1)$-dimensional linear subspace $H$ of $\langle P_0, \ldots, P_k \rangle$.

Subclaim 2.2. There exists an irreducible component $\Psi_0$ of $\Psi'_0$ of dimension $(k + 1)n + k + a$ dominating the first factor $X^{k+1}$.

Proof Subclaim 2.2: analogous to the proof of Subclaim 3.4 in [7]. □

Now consider the closure $\Psi_1 \subset \Psi_0 \times \mathbb{G}(N - n + 1, N)$ of the set of pairs $(P_0, \ldots, P_k; Q_0, \ldots, Q_k; G)$ with the dimension of $\langle P_0, \ldots, P_k, Q_0, \ldots, Q_k \rangle$ equal to $k + 1$ and $\langle P_0, \ldots, P_k, Q_0, \ldots, Q_k \rangle \subset G$. The dimension of a general fibre of
the projection $\Psi_1 \to \Psi_0$ is $(N - n - k)(n - 1)$, hence $\dim(\Psi_1) = (k + 1)n + k + a + (N - n - k)(n - 1)$. This implies that a general non-empty fiber of $\tau : \Psi_1 \to \mathbb{G}(N - n + 1, N)$ has dimension at least $(k + 1)n + k + a + (N - n - k)(n - 1) - (N - n + 2)(n - 1) = 2k - n + 2 + a$.

For $G \in \tau(\Psi_1)$ general we have by Subclaim 2.1 that $G \cap X$ is an irreducible curve $C \subset \mathbb{P}^{N-n+1}$ spanning $\mathbb{P}^{N-n+1}$. So we find a subset $S \subset C^{2k+2}$ of dimension $2k - n + 2 + a \geq k + 3$ such that for $(P_0, \ldots, P_k, Q_0, \ldots, Q_k) \in S$ the points impose at most $k + 2$ conditions on hyperplanes. Since we can choose $k + 3$ of those points general on $C$, we conclude that $k + 3$ general points of $C$ do not impose independent conditions on hyperplanes. Hence, $N - n + 1 \leq k + 1$ and so $N \leq n + k$. This gives us a contradiction. $\square$

Now we know that $\dim(\mathbb{P}^{k+1} \cap X) \geq 1$. Since $\dim(L \cap X) = 0$ and $L$ is a hyperplane in $\mathbb{P}^{k+1}$, we find $\dim(\mathbb{P}^{k+1} \cap X) = 1$. Denote by $\Gamma$ an irreducible curve in $\mathbb{P}^{k+1} \cap X$.

Claim 3. Either $\Gamma \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_k\}$ or $\Gamma \cap \{P_0, \ldots, P_k\}$ is only one point. In the second case $\mathbb{P}^{k+1} \cap X$ contains a line $L_i$ with $L_i \cap \{P_0, \ldots, P_k\} \neq \emptyset$ for each $i \in \{0, \ldots, k\}$.

Proof Claim 3: Assume that $\Gamma \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_l\}$ for some $0 \leq l < k$. Let $m$ be an integer such that $l < m \leq k$. We will now prove using a monodromy argument that there exists another component $\Gamma' \subset \mathbb{P}^{k+1} \cap X$ such that $\Gamma' \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_{l-1}, P_m\}$.

Let $\Theta_1 \subset X^{k+1} \times \mathbb{G}(k-1, N)$ be the closure of the set of points $((P_0, \ldots, P_k), H)$ such that $P_i \neq P_j$ for $i \neq j$, $\dim ((P_0, \ldots, P_k)) = k$ and $H \subset \langle P_0, \ldots, P_k \rangle$.

Consider the projections $p_{1,1} : \Theta_1 \to X^{k+1}$ and $p_{1,2} : \Theta_1 \to \mathbb{G}(k-1, N)$. Since $p_{1,1}$ is surjective with irreducible general fibers of dimension $k$, we see that $\Theta_1$ also is irreducible and of dimension $(k + 1)n + k$. The fibers of $p_{1,2}$ have dimension at least $a$. Denote $\Theta_1 \times_{\mathbb{G}(k-1, N)} \Theta_1$ by $\Theta_2$ and consider the projections $p_{2,i} : \Theta_2 \to \Theta_1$ onto the $i$-th factor for $i \in \{1, 2\}$. Let $\Delta$ be the diagonal of $\Theta_1$ in $\Theta_2$. If $((P_0, \ldots, P_k), H)$ is a general element of $\Theta_1$ then $p_{2,2}(p_{2,1}^{-1}((P_0, \ldots, P_k), H))$ contains $\Omega_{H,k}$ as an irreducible component; more precisely, $\Omega_{H,k}$ corresponds to the irreducible component of $p_{2,1}^{-1}((P_0, \ldots, P_k), H)$ intersecting $\Delta$. It follows that $\Delta$ is contained in a unique irreducible component $\Theta$ of $\Theta_2$. If $p_1 : \Theta \to \Theta_1$ denotes the restriction of the projection $p_{2,1}$ to $\Theta$, we obtain $p_1^{-1}((P_0, \ldots, P_k), H) = \Omega_{H,k}$. Consider $\Theta \subset X^{k+1} \times X^{k+1} \times \mathbb{G}(k-1, N)$ and let $\Theta_3 \subset \Theta \times X$ be the set of elements $(((P_0, \ldots, P_k), (Q_0, \ldots, Q_k), H), R)$ with $R \in \langle P_0, \ldots, P_k, Q_0, \ldots, Q_k \rangle$. By assumption, there is a curve $\Gamma$ in the fibre of $p_3 : \Theta_3 \to \Theta$ with $\Gamma \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_l\}$. Let $\Theta_4$ be the irreducible component of the Hilbert scheme parameterizing curves in fibres of
the projection $p_3$ containing the point that parameterizes $\Gamma$. Let $q : \Theta_4 \rightarrow \Theta$ be the natural morphism. Let $\Xi \subset \Theta_4 \times X$ be the universal curve and let $q' : \Theta_4 \times X \rightarrow \Theta_4$ be the projection. Consider the sections $S_i : \Theta_4 \rightarrow \Theta_4 \times X$ with $S_i(z) = (z, P_i)$ if $q(z) = ((P_0, \ldots, P_k), (Q_0, \ldots, Q_k), H)$. For a general point $z$ of $\Theta_4$ we have $S_i(z) \subseteq \Xi$ if and only if $i \in \{0, \ldots, l\}$. By construction and assumption, $\Theta_4$ is irreducible and $q$ is surjective. Let $z' \in \Theta_4$ with $q(z') = ((P_0, \ldots, P_{l-1}, P_{l+1}, P_l, \ldots, P_k), (Q_0, \ldots, Q_k), H)$. The point $q(z')$ belongs to $\Theta$ because $\Omega_{H,k}$ is determined by $H$ and $\{P_0, \ldots, P_k\}$, thus independent of the order of the points $P_0, \ldots, P_k$. Hence, $z' \in \Theta_4$ corresponds to a curve $\Gamma' \subset \mathbb{P}^{k+1} \cap X$ with $P_0, \ldots, P_{l-1}, P_{l+1} \in \Gamma'$. So, we have proved the statement above for $m = l + 1$; analogous we can prove the statement for other values of $m$.

When we take $l = 0$ we immediately get the second part of the statement of the Claim. If $l > 0$, $P_0 \in \Gamma \cap \Gamma' \subset \mathbb{P}^{k+1} \cap X$ hence $\dim(T_{P_0}(\mathbb{P}^{k+1} \cap X)) \geq 2$. Thus we get a contradiction because $\dim(T_{P_0}(L \cap X)) = 0$. So we proved also the first part of the statement of the Claim. \(\square\)

If $\Gamma \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_k\}$, we find $\Gamma \cap L = \{P_0, \ldots, P_k\}$ as a scheme because $X \cap L = \{P_0, \ldots, P_k\}$ as a scheme and $\Gamma \subset X$. Hence $\deg(\Gamma) = k + 1 = \text{codim}_{\mathbb{P}^{k+1}}(\Gamma) + 1$ and so $\Gamma$ is a rational normal curve. In this case, we find that $k + 1$ general points on $X$ are contained in a rational normal curve of degree $k + 1$ on $X$. \(\blacksquare\)

## 4 The first case of the characterization

Here we will study the first case occurring in the Proposition: for general points $P_0, \ldots, P_k \in X$ there exist lines $L_i$ on $X$ containing $P_i$ for each $i \in \{0, \ldots, k\}$ such that $\dim(L_0, \ldots, L_k) = k + 1$. Remember that a generally chosen element $(P_0, \ldots, P_k, Q_0, \ldots, Q_k, H) \in \Theta$ determines $L_0, \ldots, L_k$ uniquely. By monodromy on $\Theta$, a property that holds for some subset of $\{L_0, \ldots, L_k\}$ holds for each subset of the same cardinality.

**Claim 4.1.** If $k$ lines of $\{L_0, \ldots, L_k\}$ span a linear subspace of dimension $k$, then $X$ is a cone.

**Proof:** analogous to the proof of Claim 3.6 in [7]. \(\square\)

Assume that $X$ is not a cone. From Claim 4.1, we know $\dim(\langle L_1, \ldots, L_k \rangle) \neq k$, hence $\langle L_1, \ldots, L_k \rangle = \mathbb{P}^{k+1}$. Notice that $\dim(\langle L_i, P_1, \ldots, P_k \rangle) = k$ for all $i \in \{1, \ldots, k\}$ because $L_i \not\subset \langle P_1, \ldots, P_k \rangle$.

Now, let $1 \leq i < j \leq k$. If $\dim(\langle L_i, L_j, P_1, \ldots, P_k \rangle) = k$, then it follows that $L_j \subset \langle L_i, P_1, \ldots, P_k \rangle$ and thus $L_l \subset \langle L_i, P_1, \ldots, P_k \rangle$ for each $l \in \{1, \ldots, k\}$ by mo-
nodromy. Hence, \( \dim(\langle L_1, \ldots, L_k \rangle) = k \), a contradiction. So \( \langle L_i, L_j, P_1, \ldots, P_k \rangle = \mathbb{P}^{k+1} \).

Now fix \( P_1, \ldots, P_k \) on \( X \) and let \( P_0(t) \) be a 1-parameter family on \( X \) with \( P_0(0) = P_0 \). Consider also a 1-parameter family \( H(t) \subset \langle P_0(t), P_1, \ldots, P_k \rangle \) of linear subspaces of dimension \( k-1 \) with \( H(0) = H \) and 1-parameter families \( Q_0(t), \ldots, Q_k(t) \) on \( X \) with \( Q_i(0) = Q_i \) for each \( i \) and \( H(t) \subset \langle Q_0(t), \ldots, Q_k(t) \rangle \). Those families imply the existence of 1-parameter families \( L_0(t), \ldots, L_k(t) \) of lines on \( X \) with \( L_i(0) = L_i \) for \( i \in \{0, \ldots, k\} \), \( P_i \in L_i(t) \) for all \( i \in \{1, \ldots, k\} \), \( P_0(t) \in L_0(t) \) and \( \dim(\langle L_0(t), \ldots, L_k(t) \rangle) = k+1 \) for each value of the parameter \( t \). We may assume that \( P_0(t) \not\in \mathbb{P}^{k+1} \) for general values of \( t \). If \( L_i(t) = L_i \) for all \( i \in \{1, \ldots, k\} \) and for a general value of \( t \), then \( P_0(t) \in \langle P_0(t), P_1, \ldots, P_k \rangle \subset \langle L_1(t), \ldots, L_k(t) \rangle = \mathbb{P}^{k+1} \), a contradiction. By monodromy we can assume that \( L_i(t) \neq L_i \) for all \( i \in \{0, \ldots, k\} \).

So there is a family of lines on \( X \) through each general point of \( X \).

Remark 4.2. If \( X \) is a surface, one can easily see that this situation cannot occur.

Proposition 4.3. Let \( X \subset \mathbb{P}^N \) \( (N \geq k + 4, k \geq 3) \) be a threefold such that for \( k+1 \) general points \( P_0, \ldots, P_k \) on \( X \) there exist lines \( L_0, \ldots, L_k \) on \( X \) such that \( P_i \in L_i \) for \( i \in \{0, \ldots, k\} \) and \( \dim(\langle L_0, \ldots, L_k \rangle) = k+1 \), then \( X \) is a cone.

Proof: Assume that \( X \) is not a cone. For a general point \( P \) on \( X \) there exists a 1-dimensional family of lines on \( X \) through \( P \). Hence, \( X \) contains a 3-dimensional family of lines. By [14] or [15], \( X \) is embedded in \( \mathbb{P}^N \) as a \( \mathbb{P}^2 \)-bundle over a curve \( K \). Let \( K_P \) be the 2-dimensional component of the union of all lines on \( X \) through \( P \). We know that \( K_P \) is a plane. Using a 1-parameter family \( P_0(t) \) we find 1-parameter families \( L_1(t) \) and \( L_2(t) \) in respectively \( K_{P_1} \) and \( K_{P_2} \). We have

\[
\langle P_0(t), P_1, \ldots, P_k \rangle \subset \langle L_1(t), L_2(t), P_3, \ldots, P_k \rangle \subset \langle K_{P_1}, K_{P_2}, P_3, \ldots, P_k \rangle.
\]

Since \( \dim(\langle K_{P_1}, K_{P_2}, P_3, \ldots, P_k \rangle) \leq k+3 \) and thus \( X \not\subset \langle K_{P_1}, K_{P_2}, P_3, \ldots, P_k \rangle \), we can choose the parameter family \( P_0(t) \) such that \( P_0(t) \not\in \langle K_{P_1}, K_{P_2}, P_3, \ldots, P_k \rangle \) for general values of the parameter \( t \). This gives us a contradiction and finishes the proof. ■

5 The second case of the characterization

Proposition 5.1. Let \( X \subset \mathbb{P}^N \) \( (N \geq n + k + 1, k \geq n) \) be an \( n \)-dimensional variety such that for \( k+1 \) general points \( P_0, \ldots, P_k \) on \( X \) there exists a rational normal curve \( \Gamma \) on \( X \) of degree \( k+1 \) containing \( P_0, \ldots, P_k \). Then, the geometric genus of a general curve section of \( X \) is at most \( n - 2 \).
Proof: Denote the family of rational normal curves of degree $k+1$ on $X$ by $\{\Gamma\}$. By assumption, $\dim(\{\Gamma\}) \geq (k+1)n - (k+1) = (n-1)(k+1)$.

Because $k \leq N - n + 1$, $k+1$ general points on $X$ are contained in a curve section of $X$. So, taking $k+1$ general points $P_0, \ldots, P_k$ on $X$ can be done by first taking a general curve section $C'$ of $X$ and then considering $k+1$ general points on $C'$. Bertini’s theorems imply that $C'$ is irreducible and smooth at $P_0, \ldots, P_k$. Write $C' = X \cap G_0'$ with $G_0'$ a linear subspace of $\mathbb{P}^N$ of dimension $N - n + 1$. Consider a general linear subspace $H \subset L = \langle P_0, \ldots, P_k \rangle$ of dimension $k-1$ and let $(Q_0, \ldots, Q_k)$ be a general element of $\Omega_{H,k}$. Hence, $G' = \langle G_0' \cup \{Q_0\} \rangle \subset \mathbb{P}^N$ is a linear subspace of dimension $N - n + 2$. Consider $S' = X \cap G'$. Since $C'$ is an irreducible curve and $G_0'$ is a hyperplane of $G'$, we find that $S'$ is an irreducible surface. Since $C'$ is smooth at $P_0, \ldots, P_k$ we see that $S'$ is smooth at $P_0, \ldots, P_k$.

Let $I' \subset \{\Gamma\} \times \mathbb{G}(N - n + 2,N)$ be the inclusion relation. The dimension of a general fibre of $I' \rightarrow \{\Gamma\}$ is $(N - n - k + 1)(n-2)$. Hence, we obtain an irreducible component $I$ of $I'$ containing $(\Gamma, G')$ of dimension greater than or equal to $(N-n-k+1)(n-2) + (k+1)(n-1)$, with $\Gamma$ the rational normal curve contained in $X \cap \langle P_0, \ldots, P_k, Q_0, \ldots, Q_k \rangle$. Consider the projection $\nu : I \rightarrow \mathbb{G}(N - n + 2,N)$. The dimension of a general non-empty fibre of $\nu$ is at least

$$(N - n - k + 1)(n-2) + (k+1)(n-1) - (N-n+3)(n-2) = k - n + 3.$$ 

If we consider the fibre above $G'$, we find that $S'$ contains a subfamily of $\{\Gamma\}$ of dimension at least $k - n + 3$. Let $S$ be the minimal resolution of singularities of $S'$. We become a family $\{\gamma\}$ of rational curves on $S$ of dimension at least $k - n + 3$ by considering the strict transforms of the curves in $\{\Gamma\}$ on $S'$. Denote the strict transforms on $S$ of $\Gamma$ and $C'$ by resp. $\gamma$ and $C''$. Any two points of $S$ can be connected by means of a rational curve in $\{\gamma\}$. This implies $h^1(S,\mathcal{O}_S) = 0$, so the family $\{\gamma\}$ is contained in a linear system $\{\gamma\}$ of dimension at least $k - n + 3$. This linear system induces a linear system $|g|$ on the normalization $C$ of $C''$. Since $S'$ is smooth at $P_0, \ldots, P_k$, we find that $S$ and $S'$ are isomorphic above neighborhoods of those points. Since $\dim(|C'' - \gamma|) \geq 1$ ($C''$ is a divisor corresponding to the morphism $S \rightarrow G' \cong \mathbb{P}^{N-n+2}$ and $\gamma$ corresponds to $\Gamma$ with $\dim(\{\Gamma\}) = k+1$), no curve of $|\gamma|$ contains $C''$, hence $\dim(|g|) \geq k - n + 3$. Since $\Gamma \cap C'' = \{P_0, \ldots, P_k\}$ as a scheme, we find $\gamma \in |g|$ gives rise to $P_0 + \ldots + P_k \in |g|$. Since $P_0, \ldots, P_k$ are general points of $C$, we see that $|g|$ is non-special and $\dim(|g|) = \deg(g) - g(C) = k+1 - g(C)$. Thus, $k + 1 - g(C) \geq k - n + 3$, so $g(C) \leq n - 2$. ■

6 Some examples

Proposition 6.1. Let $X \subset \mathbb{P}^N$ be an $n$-dimensional smooth variety of minimal degree. If $k \geq n$ and $n + k + 1 < N \leq 2n + k - 1$ then $X$ has $G_{k-1,k}$ defect.
Proof: Notice that \( n \geq 3 \) because \( n + k + 1 < 2n + k - 1 \).

Take \( k + 1 \) general points \( P_0, \ldots, P_k \) on \( X \) and choose a linear subspace \( \mathbb{P}^{N-k-1} \subset \mathbb{P}^N \) disjoint with \( \langle P_0, \ldots, P_k \rangle \). Consider the projection of \( X \) on \( \mathbb{P}^{N-k-1} \) with center \( \langle P_0, \ldots, P_k \rangle \) and let \( Y \) be the closure of the image of that projection. Then \( Y \) is also an \( n \)-dimensional variety of minimal degree.

From the classification of varieties of minimal degree (see [8]) follows that \( X \) is a smooth rational normal scroll. In particular \( X \) has a bundle structure \( \pi : X \to \mathbb{P}^1 \) such that \( L(P) := \pi^{-1}(P) \subset X \subset \mathbb{P}^N \) is a linear subspace of dimension \( n - 1 \). For \( P \in \mathbb{P}^1 \) general \( L(P) \cap \langle P_0, \ldots, P_k \rangle = \emptyset \) because \( X \cap \langle P_0, \ldots, P_k \rangle = \{ P_0, \ldots, P_k \} \). Hence, on \( Y \) the image of \( L(P) \) is again a linear subspace of dimension \( n - 1 \) of \( \mathbb{P}^{N-k-1} \). So \( Y \) cannot be a cone over a Veronese surface. If \( N = n + k + 2 \) it follows that \( Y \) is a quadric in \( \mathbb{P}^{n+1} \). This quadric contains linear subspaces of dimension \( n - 1 \), so \( Y \) is singular ([11, Chapter 6, Section 1]). Let \( s \) be a general point of the singular locus of \( Y \), which is a linear subspace of \( \mathbb{P}^{n+1} \). The image of \( L(P) \) on \( Y \) contains \( s \), for \( P \in \mathbb{P}^1 \) general. Let \( G = \langle P_0, \ldots, P_k, s \rangle \) then \( \dim(G) = k + 1 \) and \( \langle P_0, \ldots, P_k \rangle \) is a hyperplane in \( G \). Since \( L(P) \cap G \neq \emptyset \) for \( P \in \mathbb{P}^1 \) general, \( \dim(X \cap G) \geq 1 \). Let \( \Gamma \) be a curve in \( X \cap G \) intersecting \( L(P) \) for general \( P \in \mathbb{P}^1 \). Since \( \langle P_0, \ldots, P_k \rangle \) is a hyperplane in \( G \) and \( X \cap \langle P_0, \ldots, P_k \rangle = \{ P_0, \ldots, P_k \} \), we find two possibilities by similar monodromy arguments as in the proof of Claim 3 of Section 3. If \( \Gamma \) is a rational normal curve of degree \( k + 1 \) through \( P_0, \ldots, P_k \); the proof is finished. The second possibility is that \( \Gamma \) is a line. Then there exist lines \( \Gamma_0, \ldots, \Gamma_k \) on \( X \) such that \( P_i \in \Gamma_i \) for all \( i \in \{ 0, \ldots, k \} \). If we denote \( \pi(P_i) \) by \( P_i' \), then \( P_i' \in L(P_i') \). The line \( \Gamma_0 \) intersects \( L(P_i') \) at a point \( P_i'' \) different from \( P_i \). We have \( \langle P_i, P_i' \rangle \cup \Gamma_1 \subset X \cap G \) and \( \langle P_i, P_i'' \rangle \neq \Gamma_1 \) (\( \Gamma_1 \) is not contained in \( L(P_i') \)). This contradicts \( \dim(T_{P_i}(X \cap G)) \leq 1 \). Hence the second possibility cannot occur.

If \( n + k + 2 < N \leq 2n + k - 1 \), it follows that \( Y \) is a scroll with \( \dim(\text{Sing}(Y)) \geq 2n + k - 1 - N \geq 0 \). So we can finish this proposition by taking the same arguments as in the case \( N = n + k + 2 \). 

Remark 6.2. If \( n = 3 \) this proposition says that minimal threefold \( X \subset \mathbb{P}^{k+5} \) is \( G_{k-1,k} \)-defective for \( k \geq 3 \). Let \( \tilde{X} \subset \mathbb{P}^{k+4} \) be the image of \( X \subset \mathbb{P}^{k+5} \) under the projection with center \( P \in \mathbb{P}^{k+5} \setminus X \). The curve \( \Gamma \) of the proof of the proposition above gives rise to a rational normal curve \( \tilde{\Gamma} \subset \tilde{X} \) of degree \( k + 1 \) containing \( k + 1 \) general points on \( \tilde{X} \). So, \( \tilde{X} \) is also \( G_{k-1,k} \)-defective.

Proposition 6.3. Let \( X \subset \mathbb{P}^{n+k+1} \) be an \( n \)-dimensional smooth variety of minimal degree \( k + 2 \), not being the Veronese surface in \( \mathbb{P}^5 \). If \( k \geq n \) then \( X \) has \( G_{k-1,k} \)-defect.

Proof: Consider a general surface section \( S \subset \mathbb{P}^{k+3} \) of \( X \). Then \( S \) is smooth and of minimal degree \( k + 2 \). Since \( S \) is not the Veronese surface (\( X \) is smooth), it is
a smooth rational normal scroll surface.

We will use some results on smooth rational normal scroll surfaces. We know that they are isomorphic to a Hirzebruch surface $\mathbb{F}_r = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}_{21}(r))$ for some $r \in \mathbb{N}$. If $r \geq 1$, those surfaces contain a curve $B$ with negative self-intersection $B^2 = -r$ and have a 1-dimensional linear system of curves $F$ with $F^2 = 0$ and $F.B = 1$. In case $r = 0$, $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and we can take $B = \mathbb{P}^1 \times \{0\}$ ($B^2 = 0$) and $F = \{p\} \times \mathbb{P}^1$ for $p \in \mathbb{P}^1$.

Let $b$ (respectively $f$) be the element of $\text{Pic}(\mathbb{F}_r)$ corresponding to the curve $B$ (respectively $F$). Write $h = b + rf$. It is well-known that $\text{Pic}(\mathbb{F}_r) = \mathbb{Z}h \oplus \mathbb{Z}f$.

For each $l > 0$, the linear system $|h + lf|$ is very ample on $\mathbb{F}_r$. For $l \geq 0$, one has $\dim(|h + lf|) = r + 2l + 1$ and $(h + lf)^2 = r + 2l$. Hence for $l \geq 1$ the linear system $|h + lf|$ gives rise to a surface $S \subset \mathbb{P}^{r+2l+1}$ of minimal degree. Those surfaces are the smooth rational normal scroll surfaces.

Let $\Gamma$ be an element of $|h + (l-1)f|$ for $l \geq 1$. We have $\dim(|(h + lf) - \Gamma|) = \dim(|f|) = 1$ hence $\dim(|\Gamma|) = r + 2l - 1$ for $\Gamma \subset S \subset \mathbb{P}^{r+2l+1}$. On the other hand $\deg(\Gamma) = (h + (l - 1)f).h + 2l - 1$, hence $\Gamma \subset S \subset \mathbb{P}^{r+2l+1}$ is a rational normal curve of degree $r + 2l - 1$. Since $\dim(|h + (l-1)f|) = r + 2l - 1$, any $r + 2l - 1$ general points on $S$ contain such a curve.

Now take $X$ as above and take $k + 1$ general points $P_0, \ldots, P_k$ on $X$. The points $P_0, \ldots, P_k$ can be considered as $k + 1$ general points on a general surface section $S \subset \mathbb{P}^{k+3}$ of $X$. Since $S$ is a smooth rational normal scroll surface, the points $P_0, \ldots, P_k$ are contained in a rational normal curve $\Gamma \subset S \subset \mathbb{P}^{k+3}$ of degree $k + 1$. This implies that $X$ is $G_{k-1,k}$-defective. ■

**Proposition 6.4.** Let $X$ be the 2-uple embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$. Then $X$ is $G_{4,5}$-defective.

**Proof:** Denote the 2-uple embedding $\mathbb{P}^3 \to X \subset \mathbb{P}^9$ by $\nu_2$. Let $P_0, \ldots, P_5$ be six general points on $X$ and denote their inverse images in $\mathbb{P}^3$ under $\nu_2$ by $Q_0, \ldots, Q_5$. These points are contained in a rational normal curve $\tilde{\Gamma} \subset \mathbb{P}^3$ of degree 3 (see [11, p. 530]). The image of $\tilde{\Gamma}$ under $\nu_2$ is a rational normal curve $\Gamma$ of degree 6 in $\mathbb{P}^9$ through $P_0, \ldots, P_5$ that is contained in $X$ since $\tilde{\Gamma}$ is cut out by quadrics in $\mathbb{P}^3$ (see again [11, p. 530]), so $X$ is $G_{4,5}$-defective. ■

**Proposition 6.5.** Let $X$ be the blowing-up of $\mathbb{P}^3$ in a point $Q$ linearly normal embedded in $\mathbb{P}^8$. Then $X$ is $G_{3,4}$-defective.

**Proof:** Let $P_0, \ldots, P_4$ be five general points of $X$. We may assume that non of those points is contained in the exceptional divisor $E \subset X$. We can consider $X$ as a subset of $\mathbb{P}^3 \times \mathbb{P}^2 \subset \mathbb{P}^{11}$ (with $\mathbb{P}^8 \subset \mathbb{P}^{11}$). Let $p : X \to \mathbb{P}^3$ be the projection to the first factor and let $Q_0, \ldots, Q_4$ be the images under $p$ of respectively $P_0, \ldots, P_4$. Hence there exists a rational normal curve $\tilde{\Gamma}$ in $\mathbb{P}^3$ containing $Q, Q_0, \ldots, Q_4$. The inverse image of $\tilde{\Gamma}$ under $p$ contains a rational normal curve $\Gamma$ in $X$ of degree 5.
containing \( P_0, \ldots, P_4 \), so \( X \) is \( G_{3,4} \)-defective. ■

7 What for smooth surfaces?

**Proof of Theorem 1.1:** We have already proved that smooth surfaces \( X \subset \mathbb{P}^{k+3} \) of minimal degree are \( G_{k-1,k} \)-defective (see Prop. 6.3).

So let \( X \) be a smooth \( G_{k-1,k} \)-defective surface in \( P^N \). Now we can use Proposition 1.3. It follows that \( N \geq k+3 \) and (since \( X \) is smooth) that for \( k+1 \) general points of \( X \) there exists a rational normal curve of degree \( k+1 \) on \( X \) through those points. Take \( k+1 \) general points \( P_0, \ldots, P_k \) on \( X \). One can assume that \( P_0, \ldots, P_k \) are general points on a general (smooth) curve section \( C \) of \( X \). Write \( \Gamma \subset X \) to denote the rational normal curve of degree \( k+1 \) through \( P_0, \ldots, P_k \). Since \( \dim(\langle \Gamma \rangle) = N - 1 \geq k + 2 \), we find \( \dim(|C - \Gamma|) \geq 1 \). Let \( C'' \) be a general element of \( |C - \Gamma| \). The linear system \( |C'| = |C - \Gamma| \) has no fixed component because \( \Gamma \) is the only curve in \( X \cap \langle \Gamma \rangle \) and \( X \cap \langle \Gamma \rangle \) is smooth in a general point of \( \Gamma \). Either \( C' \) is irreducible or it is the sum of irreducible curves in a pencil on \( X \). So, if \( C' \) would contain a curve \( \Gamma \), then \( C' \sim (\alpha - 1)\Gamma \) for some \( \alpha \geq 2 \) and so \( C \sim \alpha \Gamma \). So from \( \Gamma.C = k + 1 \) it would follow that \( \alpha(\Gamma.C) = k + 1 \). But this would contradict \( \alpha \geq 2 \), \( k > 2 \) and \( \Gamma.C \geq k \) (\( \dim|\Gamma| \geq k + 1 \)). Since \( \Gamma \cap C' \) is connected, we get \( \Gamma.C' \geq 1 \). Hence \( \Gamma.C = \Gamma.C' = k + 1 \) implies \( \Gamma.C = k \) and \( \Gamma.C' = 1 \). Since \( \dim|\Gamma| \geq k + 1 \) we find \( |\Gamma - C''| \neq 0 \). So we can write \( \Gamma \sim \beta.C' + C'' \) for some \( \beta \geq 1 \) and \( C'' \geq 0 \) with \( |C'' - C'| = 0 \).

If \( C'' = 0 \), then \( \beta(C'.C'') = \Gamma.C' = 1 \) implies \( \beta = 1 \) and \( C'.C' = 1 \). Since \( \beta^2(C'.C') = \Gamma.C' = k \), this gives us a contradiction with \( k > 2 \), so \( C'' \neq 0 \). Since \( C' \cap C'' \) is connected, we find \( C'.C'' \geq 1 \). From \( 1 = \Gamma.C' = \beta(C'.C') + C'.C'' \) it follows that \( C'.C' = 0 \) and \( C'.C'' \geq 1 \) because \( C'.C' \geq 0 \) (\( |C'| \) is 1-dimensional and has no fixed components). Thus,

\[
\deg(X) = C.C = (\Gamma + C').(\Gamma + C') = \Gamma.\Gamma + 2(\Gamma.C'') + C'.C' = k + 2.
\]

Since \( \text{codim}(X) + 1 = N - 1 \geq k + 2 \) it follows that \( N = k + 3 \) and that \( X \) is of minimal degree. ■

8 What for smooth threefolds?

**Proof of Theorem 1.2:** We have already proved that the threefolds of the statement are \( G_{k-1,k} \)-defective (see Sec. 6), so we only have to prove that there are no other threefolds with \( G_{k-1,k} \)-defect. Let \( X \subset \mathbb{P}^N \) be a smooth non-degenerate threefold with \( G_{k-1,k} \)-defect. From Proposition 1.3 and Section 4, it follows that \( N \geq n + k + 1 \) and that any \( k+1 \) general points on \( X \) are contained in a rational normal curve of degree \( k+1 \) on \( X \). Now fix \( k+1 \) general points \( P_0, \ldots, P_k \) on \( X \). We may assume that \( P_0, \ldots, P_k \) are contained in a general curve section \( C' \)
of $X$. Using the notations of the proof of Proposition 5.1, since $X$ is smooth and $\dim(X) = 3$ we have $C = C' = X \cap G'_0$ for some linear subspace $G'_0 \subset \mathbb{P}^N$ of dimension $N - 2$ and $S' = X \cap G'$ for some hyperplane $G' \subset \mathbb{P}^N$ containing $G'_0$. There is a 1-dimensional family of hyperplanes of $\mathbb{P}^N$ containing $G'_0$ and we distinguish two possibilities:

(a) The hyperplane $G'$ is a general element in this family; i.e. the projection morphism $\nu$ in the proof of Proposition 5.1 is surjective. In this case $S'$ is smooth since $X$ is smooth and $S'$ is a general surface section of $X$ (Sec. 2.6). The surface $S'$ contains a subfamily of $\{\Gamma\}$ of dimension at least $k$. 

(b) The hyperplane $G'$ is a special element in this family; i.e. the projection morphism $\nu$ in the proof of Proposition 5.1 is not surjective. In this case $S'$ contains a subfamily of $\{\Gamma\}$ of dimension at least $k + 1$. In particular the linear system $|g|$ on $C$ has degree $k + 1$ and dimension at least $k + 1$. Hence $S'$ has sectional genus 0, but $S'$ does not need to be smooth.

Case (a).

Write $\mathcal{L}$ to denote the linear system defining $S \subset \mathbb{P}^{N-1 \geq k+3}$. If $\mathcal{L}(-\Gamma)$ is defined as being $\{D \in \mathcal{L} \mid D - \Gamma \geq 0\}$, then $\dim(\mathcal{L}(-\Gamma)) \geq 1$ since $\dim(\langle \Gamma \rangle) = k + 1$. Notice that $\mathcal{L} - \Gamma = \{D - \Gamma \mid D \in \mathcal{L}(-\Gamma)\}$ does not have fixed components because $\Gamma$ is the only curve in $X \cap (\Gamma)$ and $X \cap (\Gamma)$ smooth in a general point of $\Gamma$. Let $C'$ be a general element of $\mathcal{L} - \Gamma$, then $\Gamma.(\Gamma + C') = k + 1$. Since $\Gamma \cup C'$ is connected we have $\Gamma.C' \geq 1$. On the other hand, since $S'$ contains a subfamily of $\{\Gamma\}$ of dimension at least $k$ we find $\Gamma.\Gamma \geq k - 1$. So we obtain two possibilities: $\Gamma.C' = 1$ and $\Gamma.\Gamma = k$ or $\Gamma.C' = 2$ and $\Gamma.\Gamma = k - 1$.

Case $\Gamma.C' = 2$ and $\Gamma.\Gamma = k - 1$.

First assume that $\mathcal{L} - \Gamma$ is composed with a pencil, so there is a morphism $f : \tilde{S} \rightarrow T$ with $T$ a curve and $\tilde{S}$ a blowing-up of $S$ at the fixed points of $\mathcal{L} - \Gamma$ such that $C' = f^{-1}(c_1) + f^{-1}(c_2)$ for $c_1 + c_2$ moving in a linear system on $T$. Indeed, $C'$ cannot be contained in a fibre of $f$ and each fibre of $f$ intersects $\Gamma$ otherwise $\Gamma.C'$ would be 0. Since $\Gamma$ dominates $T$, we find $T \cong \mathbb{P}^1$. So the fibres of $f$ form a linear system on $S$. Thus $C' \in |2C_0|$ for an irreducible curve $C_0$ with $\dim(C_0) = 1$ and $\Gamma.C_0 = 1$. Because $\dim|\Gamma| \geq k$, there are curves in $|\Gamma|$ that contain $C_0$. Suppose that $\Gamma \sim \alpha C_0 + C''$ for some $\alpha \geq 1$ and $C'' \geq 0$ with $|C'' - C_0| = 0$. If $C'' = 0$, it would follow $\Gamma \sim \alpha C_0$, hence $\alpha^2(C_0.C_0) = \Gamma.\Gamma = k - 1$ and $2\alpha(C_0.C_0) = \Gamma.C' = 2$, a contradiction (with $k > 3$).

So $C'' \neq 0$. Since $\alpha C_0 + C''$ is connected (Sec. 2.7) and $C_0$ irreducible, we find $C_0.C'' \geq 1$. We know that $2 = \Gamma.C' = \alpha(C_0.C') + C''.C' = 2\alpha(C_0.C_0) + 2(C''.C_0)$. Hence $C_0.C_0 = 0$ and $C''.C_0 = 1$, since $C_0.C_0 \geq 0 \ (\dim|C_0| = 1$ and $|C_0|$ has no
fixed components). This implies that $C'.C' = 0$ and so
\[ \deg(X) = \deg(S) = C.C = (\Gamma + C').(\Gamma + C') = k + 3. \]

Hence $N \in \{k + 4, k + 5\}$, because $\text{codim}(X) + 1 = N - 2 \geq k + 2$. Since $g(C) \leq 1$ and $C \sim \Gamma + C' + C''$ for $C'_0$ and $C''_0$ general on $S$, we find $p_a(\Gamma + C'_0 + C''_0) \leq 1$ and since $g(C'_0) = g(C''_0)$ it follows $g(C) = p_a(\Gamma + C'_0 + C''_0) = 0$. So the sectional genus of $X$ is 0. Now it follows from Theorema 12.1 in [10] that the polarized variety $(X, L)$ has $\Delta$-genus equal to 0. From the classification theory of polarized varieties (Section 2.5.2) it follows that $(X, L) = (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$. A linearly normal embedding of $(X, L)$ gives rise to a threefold $\bar{X} \subset \mathbb{P}^{k+5}$ of minimal degree $k + 3$. So $X = \bar{X}$ or $X$ is the projection of $\bar{X}$ in $\mathbb{P}^{k+4}$ with center $P \in \mathbb{P}^{k+5} \setminus X$. This gives rise to possibilities 2 and 3.

Assume now that $\mathcal{L} - \Gamma$ is not composed with a pencil. Hence in general $C'$ is irreducible (Sec. 2.6). Since $\Gamma.C' = 2$ we have
\[ g(C) = p_a(\Gamma + C') = 1 + \frac{1}{2} (\Gamma + C'.C). (\Gamma + C' + K) = p_a(C') + p_a(\Gamma) + 1 \leq 1. \]

Since $g(\Gamma) = 0$, we find $C' \cong \mathbb{P}^1$ and $X$ has sectional genus equal to 1. From $\dim(\Gamma) \geq k$, it follows $|\Gamma - C'| \neq \emptyset$. Now write $\Gamma \sim \alpha C' + C''$ for some $\alpha \geq 1$ and $C'' \geq 0$ with $|C'' - C'| = \emptyset$.

If $C'' = 0$, we have $\Gamma \sim \alpha C'$ and so
\[ k - 1 = \Gamma.C = \alpha^2(C'.C') = \alpha(\Gamma.C') = 2\alpha. \]

Hence $\alpha = \frac{k - 1}{2}$ and so $C'.C' = \frac{1}{2} = \frac{4}{k-1}$. Since $k > 3$ it follows $k = 5$, $\alpha = 2$, $\Gamma.C' = 4$, $C'.C' = 1$ and $\Gamma.C' = 2$; so $\deg(X) = C.C = 9(C'.C') = 9$. From the classification of polarized varieties $(X, L)$ with sectional genus 1 (Sec. 2.5.2) follows that $X$ has to be a scroll over an elliptic curve. This gives us a contradiction because $k + 1$ general points on $X$ are contained in a rational normal curve on $X$.

So we find $C'' \neq 0$. We have $\Gamma.C' \geq 0$ and $\Gamma.C'' \geq 0$ since $\Gamma$ has no fixed component. On the other hand, $C'.C' \geq 0$ since $\dim(|C'|) \geq 1$ and $C'$ has no fixed component. We also have
\[ k - 1 = \Gamma.C = \alpha(\Gamma.C') + \Gamma.C'' = 2\alpha + \Gamma.C'' \]
and
\[ \deg(X) = C.C = (\Gamma + C').(\Gamma + C') = k + 3 + C'.C'. 

First consider the case $k = 4$. Then $2\alpha + \Gamma.C'' = 3$ and so $\alpha = 1$ and $\Gamma.C'' = 1$. Since $2 = \Gamma.C' = C'.C' + C''.C'$ and $C'.C'' \geq 1$ ($C' \cup C''$ connected) we have two
Consider the first possibility. It follows \( \deg(X) = C.C = 7 \) and \( C''.C'' = -1 \)
(since \( \Gamma, \Gamma = 3 \)). So \( (X, L) \) is a smooth 3-dimensional variety with sectional genus 1 of degree 7. From the classification of polarized varieties with sectional genus 1 (see Sec. 2.5.2) follows that \( (X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \) using the classification of polarized varieties with sectional genus 1 (see Sec. 2.5.2). This implies that \( S \) needs to be a smooth quadric in \( \mathbb{P}^3 \) embedded by \( |2C' + C''| = |C| = |\mathcal{O}_S(2, 2)|. \)
This gives us a contradiction since \( C'.C' = 1 = C''.C'' \) and \( C''.C'' = 0 \).

Now let \( k = 5 \), thus \( 2\alpha + C''.C'' = 4 \). Hence we again have two possibilities: \( \alpha = 2 \) and \( \Gamma, C''.C'' = 0 \) or \( \alpha = 1 \) and \( \Gamma, C''.C'' = 2 \).

We start with the first possibility. Since \( \Gamma \sim 2C' + C'' \), we have \( 2 = \Gamma, C' = 2(C'.C') + C''.C'' \), hence \( C'.C' = 0 \) and \( C'.C'' = 2 \). It follows that \( \deg(X) = C.C = 8 \) and \( C''.C'' = -4 \). From Section 2.5.2 we see that \( (X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \).

Now we take a look at the second possibility. Since \( \Gamma \sim C' + C'' \), we have \( 2 = \Gamma, C' = C'.C' + C'.C'' \). Notice that \( C'.C' \leq 0 \) since there are no 3-dimensional smooth Del Pezzo varieties \( \bar{X} \) with \( \deg(\bar{X}) > 8 \). It follows \( C'.C' = 0 \), \( C''.C'' = 2 \), \( \deg(X) = C.C = 8 \) and \( C''.C'' = 0 \). From Section 2.5.2 we see that \( (X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \).

So, in both cases we end up with \( (X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \). This gives rise to the 2-uple embedding of \( \mathbb{P}^3 \) in \( \mathbb{P}^9 \), which is case 5 of the Theorem.

If \( k > 5 \) it follows \( \deg(X) = k + 3 + C'.C' > 8 \) since \( C'.C' \geq 0 \). This immediately gives us a contradiction since there are no 3-dimensional smooth Del Pezzo varieties \( \bar{X} \) with \( \deg(\bar{X}) > 8 \) (see Sec. 2.5.2).

Case \( \Gamma, C' = 1 \) and \( \Gamma, \Gamma = k \).

In particular, since \( |C'| \) has no fixed components, \( |C''| \) cannot be composed by a pencil and it follows that in general \( C'' \) is irreducible (Bertini’s theorem, see Sec. 2.6). Since \( \dim|\Gamma| \geq k \) and \( \Gamma, C' = 1 \), we can write \( \Gamma \sim \alpha C' + C'' \) for some \( \alpha \geq 1 \) and \( C'' \geq 0 \) with \( |C'' - C'| = 0 \). If \( C'' = 0 \) it follows \( \Gamma \sim \alpha C' \) and thus \( \alpha^2(C'.C') = \Gamma, \Gamma = k \) and \( \alpha(C'.C') = \Gamma, C' = 1 \), a contradiction with \( k > 3 \). Hence \( C'' \neq 0 \). We have

\[
\alpha(C'.C') + C'.C'' = (\alpha C' + C'').C' = \Gamma, C' = 1.
\]
Since $C'.C' \geq 0$ and $C'.C'' \geq 1$ we obtain $C'.C' = 0$ and so $\text{deg}(X) = C.C = k+2$. Because $\text{codim}(X) + 1 = N - 2 \geq k + 2$ we find that $X$ is a smooth threefold in $\mathbb{P}^{k+4}$ of minimal degree $k + 2$. From Proposition 6.3, it follows that such a threefold $X$ has $G_{k-1,k}$-defect. This gives rise to case 1 of the Theorem.

**Case (b).**

Because $C$ is a smooth hyperplane section of $S'$, $S'$ is smooth along $C$, hence $\text{Sing}(S') \cap C = \emptyset$. It follows that $\text{Sing}(S')$ is a finite set and so $S'$ is irreducible.

**Claim.** If $s \in \text{Sing}(S')$ and $\Gamma$ is a general curve in the set of curves $\{\Gamma\}$ in $S'$, then $s \notin \langle \Gamma \rangle$.

**Proof Claim:** First we are going to prove that $s \notin \Gamma$. Assume $s \in \Gamma$. Since $\text{Sing}(S')$ is finite, $s \in \Gamma$ for all curves $\Gamma$ on $S'$. So a general curve $\Gamma$ on $S'$ is completely determined by $k + 1$ points $P_0, \ldots, P_k$ on $C$ as being the only 1-dimensional component of $X \cap \langle P_0, \ldots, P_k, s \rangle$. The uniqueness follows from $X \cap \langle P_0, \ldots, P_k \rangle = \{P_0, \ldots, P_k\}$ as a scheme. Now take $k + 2$ general points $P_0, \ldots, P_{k-1}, Q, Q'$ on $C$ and let $\Gamma$ (respectively $\Gamma'$) be the curve in the family corresponding with $P_0, \ldots, P_{k-1}, Q$ (respectively $P_0, \ldots, P_{k-1}, Q'$). Because $\dim(\langle P_0, \ldots, P_{k-1}, Q, Q' \rangle) = k + 1$, we can consider a deformation of $C$ on $S'$ to another curve $C''$ containing $P_0, \ldots, P_{k-1}, Q, Q'$. Since $\Gamma$ and $\Gamma'$ are contained in $\langle C' \cup \{s\} \rangle$, the surface $S'$ is deformed into $S'' = X \cap \langle C' \cup \{s\} \rangle$. Because $\Gamma \cap \Gamma'$ is finite it follows $s \in \text{Sing}(S'')$. So for a general hyperplane $\mathbb{P}^{N-1} \subset \mathbb{P}^N$ with $\langle P_0, \ldots, P_{k-1}, Q, Q', s \rangle \subset \mathbb{P}^{N-1}$ we find $T_s(X) \subset \mathbb{P}^{N-1}$, hence $T_s(X) \subset \langle P_0, \ldots, P_{k-1}, Q, Q', s \rangle$. Since $s \notin C = X \cap \langle C \rangle$ and $\langle P_0, \ldots, P_{k-1}, Q, Q' \rangle \subset \langle C \rangle$, we have $\dim(T_s) = n - 1 = 2$ with $T = T_s(X) \cap \langle P_0, \ldots, P_{k-1}, Q, Q' \rangle$. If $s \in \langle C \rangle$ then $s \in C = \langle C \rangle \cap X$ and thus $s \notin \text{Sing}(S')$, a contradiction. So we have

$$T = T_s(X) \cap \langle P_0, \ldots, P_{k-1}, Q, Q' \rangle \subset T_s(X) \cap \langle C \rangle \subset T_s(X),$$

hence $T = T_s(X) \cap \langle C \rangle$ since $\dim(T) = 2$. This implies

$$T = T_s(X) \cap \langle C \rangle \subset \langle P_0, \ldots, P_{k-1}, Q, Q' \rangle \subset \langle C \rangle.$$

Since $P_0, \ldots, P_{k-1}, Q, Q'$ are generally chosen on $C$ and $k + 1 < N - 2$, we may assume that those points are contained in a general hyperplane of $\langle C \rangle$ (not containing $T$), a contradiction.

If $s \in \langle \Gamma \rangle \backslash \Gamma$ then $s$ is one of the finitely many points in $\langle \Gamma \rangle \cap X$ not on $\Gamma$. So a general curve $\Gamma$ is again completely determined by $k + 1$ points $P_0, \ldots, P_k$ on $C$. Take a deformation of $C$ on $X$ to another curve $C''$ containing $P_0, \ldots, P_k$. Since $\Gamma$ is contained in $\langle C' \cup \{s\} \rangle$ and $s \in \langle \Gamma \rangle$, the surface $S'$ deforms to $S'' = \langle C' \cup \{s\} \rangle \cap X$ with $s \in \text{Sing}(S'')$. As before we find $T_s(X) \subset \langle P_0, \ldots, P_k, s \rangle$ and thus $\dim(T_s(X) \cap \langle P_0, \ldots, P_k \rangle) \geq 2$ for general points $P_0, \ldots, P_k$ on $C$. Since
s \not\in \langle P_0, \ldots, P_k \rangle \subset \langle C \rangle \) (otherwise \( s \in C = X \cap \langle C \rangle \) and so \( s \not\in \text{Sing}(S') \)) we obtain \( T := T_s(X) \cap \langle C \rangle = T_s(X) \cap \langle P_0, \ldots, P_k \rangle \) and \( \dim(T) = 2 \). On the other hand, we may assume that \( P_0, \ldots, P_k \) are contained in a general hyperplane of \( \langle C \rangle \) since \( k < N - 2 \). So we get a contradiction. \( \square \)

Now take a minimal resolution of singularities \( \chi : S \to S' \). General curves \( C \) and \( \Gamma \) can be considered as curves on \( S \) and \( \Gamma \) is contained in a linear system on \( S \) of dimension at least \( k + 1 \). Since \( \Gamma.C = k + 1 \) and \( |\Gamma - C| = \emptyset \) the linear system of curves \( \Gamma \) is complete and induces a \( g_{k+1}^{k+1} \) on \( C \), so \( C \) is rational. We have \( \dim(|C - \Gamma|) \geq 1 \), since \( \dim(|C'|) = N - 2 \geq k + 2 \) and \( \dim(|\Gamma'|) = k + 1 \). Let \( C' \) be a general element of \( |C - \Gamma| \). The linear system \( |C'| = |C - \Gamma| \) has no fixed component since \( \Gamma \) is the only curve contained in \( X \cap \langle \Gamma \rangle \) and \( \text{Sing}(S') \cap \langle \Gamma \rangle = \emptyset \). So \( C' \) is irreducible or it is the sum of irreducible curves in a pencil. Hence, if \( C' \) would contain a curve \( \Gamma \), then \( C' \sim (\alpha - 1)\Gamma \) and \( C \sim \alpha \Gamma \) for some \( \alpha \geq 2 \). This would imply that \( k + 1 = \Gamma.C = \alpha(\Gamma.G) \), but \( \Gamma.G \geq k \) since \( \dim(|\Gamma|) \geq k + 1 \), a contradiction. So \( C' \) is irreducible. Since \( \Gamma \cup C' \) is connected, \( \Gamma.C' \geq 1 \). From \( k + 1 = \Gamma.C = \Gamma.G + \Gamma.C' \) then follows \( \Gamma.G = k \) and \( \Gamma.C' = 1 \). Since \( \dim(|\Gamma|) \geq k + 1 \) this also implies \( |\Gamma - C'| \neq \emptyset \).

We can write \( \Gamma \sim \beta C' + C'' \) for some \( \beta \geq 1 \) and \( C'' \geq 0 \) with \( |C'' - C'| = \emptyset \). If \( C'' = 0 \) then \( \Gamma \sim \beta C' \), hence \( \beta(C'.C') = \Gamma.C' = 1 \) and so \( \beta = 1 \) and \( C'.C' = 1 \). This would imply \( k = \Gamma.G = \beta^2(C'.C') = 1 \), a contradiction. So \( C'' \neq 0 \). We know \( C'.C' \geq 0 (|C'| \) has dimension at least \( 1 \)) and \( C'.C'' \geq 1 (C' \cup C'' \) connected), so \( \beta(C'.C') + C'.C'' = \Gamma.C' = 1 \) implies \( C'.C' = 0 \) and \( C'.C'' = 1 \). Hence

\[
\text{deg}(X) = C.C = (\Gamma + C').(\Gamma + C') = k + 2.
\]

Since \( \text{codim}(X) + 1 = N - 2 \geq k + 2 \) this implies \( N = k + 4 \) and \( X \) is a smooth threefold in \( \mathbb{P}^N \) with minimal degree \( k + 2 \). This case corresponds to case 1 of the Theorem. \( \blacksquare \)

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**References**


