MAM3000W PROJECT
Maximal Surfaces in Lorentz-Minkowski 3-Space $\mathbf{L}^3$

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Abstract

A *maximal surface* is a surface of zero mean curvature in *Lorentz-Minkowski space*. In this project, various characteristic properties of maximal surfaces in the 3-dimensional Minkowski space $\mathbf{L}^3$ will be studied.

First, an overview of general concepts in differential geometry used in later parts of the project is given. Then, different results regarding maximal surfaces are stated and proven. At the end of the project, a proof of the *Calabi-Bernstein Theorem* in $\mathbf{L}^3$, which gives important information about entire maximal surfaces, is given.

Contents

1 Introduction

Soap bubbles are for many people a fond child-hood memory. But besides the joys they brought us at an earlier age, soap bubbles have interesting mathematical properties too. In fact, it turns out that a thin soap film naturally always forms itself into a *minimal surface*.[1] In this project, several properties of *maximal surfaces*, the counterpart of the minimal surface in *Minkowski space* will be studied.

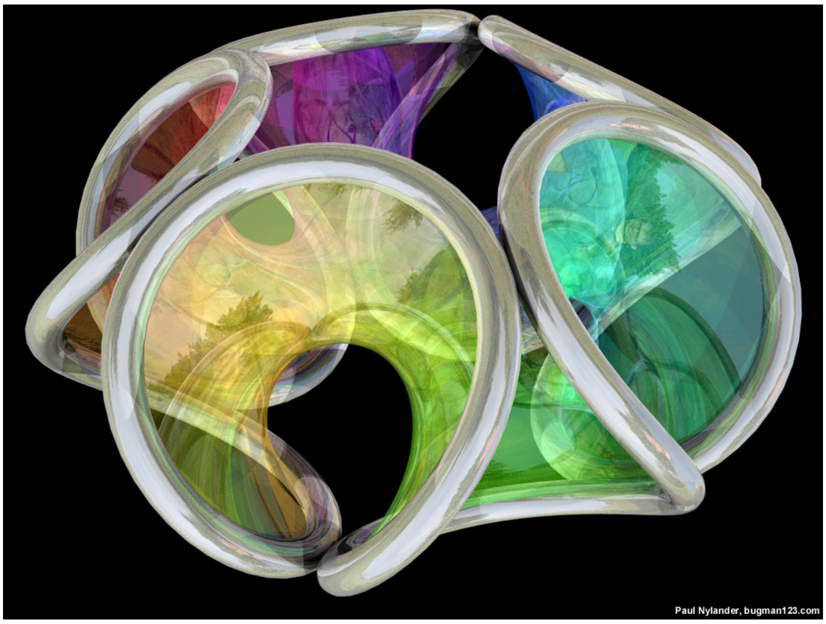

Figure 1: This picture shows *Ennepper's surface of the fourth kind* as a soap film. This surface is a well-known example of a minimal surface. This picture was sourced from [Nyl12].

The *Lorentz-Minkowski space* (or simply *Minkowski space*) in three dimensions, $\mathbf{L}^3$, is a space with a flat metric defined on it.[2] It differs from regular Euclidean 3-space as its metric is no longer positive definite. It is thus called a *Pseudo-Riemannien space*. It's four-dimensional counterpart describes flat space-time and is thus very useful in physics.[3]

[1] See [Opr04].

[2] This simply means that $\mathbf{L}^3$, like $\mathbf{R}^3$ is a flat space, not a curved space.

[3] For more details on flat and curved spaces, the difference between *Riemannien* and *Pseudo-Riemannien spaces* as well as the use of four-dimensional Minkowski space $\mathbf{L}^4$ in physics see [Car04].

Besides the relevance in physics, maximal surfaces in $\mathbf{L}^3$ have many mathematically interesting properties. These properties are what this project will be focusing on. Before they can be explored in greater detail, some background information is required. The relevant background material is discussed within the following chapter.

2 Background Information

2.1 The Minkowski Metric

Let us consider a 3-dimensional Euclidean coordinate system, $\mathbf{R}^3$, with axes x , y and t. In it, a *vector* can be given as follows:

$$\vec{v} = (v_x, v_y, v_t)$$

It is a well known fact, that the *inner product* of two vectors, say $\vec{v} = (v_x, v_y, v_t)$ and $\vec{w} = (w_x, w_y, w_t)$, is defined as follows:

$$\vec{v} \cdot \vec{w} \equiv v_x w_x + v_y w_y + v_t w_t \tag{1}$$

Using matrix multiplication, equation (1) can equivalently be given as:

$$\begin{aligned} \vec{v} \cdot \vec{w} &= \vec{v} I_3 \vec{w}^T \\ \Leftrightarrow & \begin{bmatrix} v_x & v_y & v_t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_t \end{bmatrix} \end{aligned}$$

where I_3 denotes the *identity matrix* in 3D and $\vec{w}^T$ is the transpose of $\vec{w}$.

By this reasoning, one can formulate the following definition:

Definition 2.1.1. *Let $g_E : \mathbf{R}^3 \times \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be a function, such that:*

$$g_E(\vec{v}, \vec{w}) = \vec{v} I_3 \vec{w}$$

*(in other words, g_E sends two vectors to their inner product, $(\vec{v}, \vec{w}) \mapsto \vec{v} \cdot \vec{w}$). This function g_E defines the **Euclidean metric**[4] in $\mathbf{R}^3$.*

Therefore, the metric space $(\mathbf{R}^3, g_E)$ with g_E as defined above, is just the 3-dimensional Euclidean space with the usual inner product defined on it.

In *Minkowski space*, the *metric* g is defined in a similar way:

[4] It is easy to prove that g_E is in fact a metric. This proof is omitted here, but readers are welcome to convince themselves of this fact.

Definition 2.1.2. *Let $g : \mathbf{R}^3 \times \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be a function, such that:*

$$\begin{aligned} g(\vec{v}, \vec{w}) &= \begin{bmatrix} v_x & v_y & v_t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} w_x \\ w_y \\ w_t \end{bmatrix} \\ &= v_x w_x + v_y w_y - v_t w_t \end{aligned}$$

The above is more commonly written as:

$$g(\vec{v}, \vec{w}) = \vec{v} g \vec{w}$$

where:

$$g \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

*g is known as the **Minkowski metric**.*

The metric space $(\mathbf{R}^3, g)$, which will be denoted as $\mathbf{L}^3$, is called the *Minkowski space*, where the metric g as defined in **Definition 2.1.2** is understood.

Thus, losely speaking, the t-direction yields a “negative contribution”.

In $\mathbf{L}^3$, one defines a *lightcone*[5]. This lightcone is a cone formed by the rotation of the line $l(s)$ defined by:

$$l(s) = (0, 0, 0) + s(0, 1, 1)$$

about the t-axis, where s is a parameter.

The following terminology is used:

- a vector v, which has the property that $g(v, v) < 0$, is said to be *timelike*
- a vector v, which has the property that $g(v, v) > 0$, is said to be *spacelike*
- a vector v, which has the property that $g(v, v) = 0$, is said to be *lightlike*

[5]It is called the *lightcone* due to physical reasons which will not be explained here but can be found in [Car04].

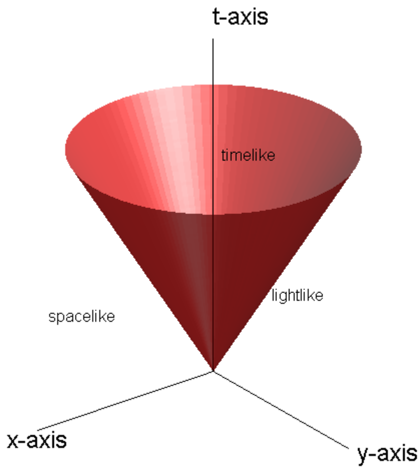

Figure 2: This diagram depicts the ***lightcone*** and shows the various regions in $\mathbf{L}^3$ relative to the lightcone. (Graphic generated in MATLAB).

Similarly, one speaks of a *timelike* (resp. *spacelike*) surface, if its tangent plane is a *timelike* (resp. *spacelike*) subspace of Minkowski space at every point. This terminology is visualized in Figure 2.[6]

In this project, primarily *spacelike* surfaces are considered.

2.2 Notation

In this project, unless stated otherwise, a subscript notation for derivatives will be used, in other words, f_x will denote the partial derivative of a function f with respect to x:

$$f_x \equiv \frac{\partial f}{\partial x}$$

A similar notation will be used for higher order derivatives:

$$f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y} \equiv f_{yx} \qquad f_{yy} \equiv \frac{\partial^2 f}{\partial y^2}$$

[6]This information has been sourced from [Car04].

and so forth.

2.3 Important Theorems

In this section, the *Inverse Function Theorem* will be stated without a proof.[7]

Theorem 2.3.1. *Let E be an open set with $E \subset \mathbf{R}^n$ and let $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be a continuously differentiable map. Furthermore, let the Jacobian of f at $\mathbf{a}$ be nonzero in the neighbourhodd of a point $\mathbf{a} \in E$. Then:*

1. *there exist two open sets U and V with $U, V \subset \mathbf{R}^n$ such that $\mathbf{a} \in U$, $f(\mathbf{a}) \in V$, $f : U \longrightarrow V$ is injective, and $f(U) = V$*
2. *the inverse of f, namely $g : V \longrightarrow U$, which exists due to* 1., *defined by:*
$$g(f(\mathbf{x})) = \mathbf{x} \qquad \text{for every } \mathbf{x} \in U$$
is continuously differentiable on V.

The next important Theorem that will be used later on is *Green's Theorem*[8], which reads:

Theorem 2.3.2. *Consider a plane. Let $\mathcal{C}$ be a region within this plane and let $\partial\mathcal{C}$ be its closed contour.*
Let $P(u, v)$ and $Q(u, v)$ be smooth functions on $\mathcal{C}$. Then, the following relation holds:

$$\int\int \left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} \right) \mathrm{d}u \, \mathrm{d}v = \int_C P \, \mathrm{d}v - \int_C Q \, \mathrm{d}u$$

The definitions in the following, (sections 2.4 – 2.9), are adapted from [Opr04], unless stated otherwise.

[7] A proof can be found in [Rud76].
[8] A proof of Green's Theorem can be found in [Rud76].

2.4 Curves

It is assumed that the reader is familiar with the concept of a curve and thus said concept will not be explained here. In this section, some results that will prove useful in later sections, will be stated.

Definition 2.4.1. *A curve $\gamma(u)$ is said to have* ***unit speed*** *if its derivative is of unit length for every value of u, in other words*

$$|\dot{\gamma}(u)| = 1 \qquad \text{for all } u \text{ in the domain}$$

where $\dot{\gamma}(u) \equiv \frac{\mathrm{d}\gamma(u)}{\mathrm{d}u}$. If the above holds, the curve $\gamma(u)$ is said to have a ***unit speed parameterization****.*

The following result provides information about the existence of a unit speed parameterization for a given curve:

Result 2.4.2. *Every regular curve has a unit speed parameterization.*

Proof. [9] Let $\gamma(u)$ be a regular curve. Then its length-function l is:

$$l(u) = \int_0^u |\dot{\gamma}(t)| \, \mathrm{d}t$$

By the *Fundamental Theorem Of Calculus*, it follows that:

$$\frac{\mathrm{d}l(u)}{\mathrm{d}u} = |\dot{\gamma}(u)| > 0 \tag{2}$$

Since $\frac{\mathrm{d}l(u)}{\mathrm{d}u}$ is strictly greater than zero, by the *Inverse Function Theorem*, its inverse function $u(l)$ exists and is differentiable. Its derivative is related to $\frac{\mathrm{d}l(u)}{\mathrm{d}u}$ as follows:

$$\frac{\mathrm{d}u(l)}{\mathrm{d}l} = \frac{1}{\frac{\mathrm{d}l(u)}{\mathrm{d}u}} > 0 \tag{3}$$

[9] This proof is based on a proof found in [Opr04].

Let $\lambda(l) \equiv \gamma(u(l))$. Thus, $\lambda(l)$ is clearly a reparameterization of $\gamma(u)$ and hence $\lambda(l)$ and $\gamma(u)$ describe the same curve.

By the chain rule, the following can be noted:

$$\begin{aligned}
\frac{\mathrm{d}\lambda}{\mathrm{d}l} &= \frac{\mathrm{d}\gamma}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}l} \\
\left|\frac{\mathrm{d}\lambda}{\mathrm{d}l}\right| &= \left|\frac{\mathrm{d}\gamma}{\mathrm{d}u}\right|\left|\frac{\mathrm{d}u}{\mathrm{d}l}\right| \\
&= \frac{\mathrm{d}l}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}l} && \text{(by equation (2))} \\
&= \frac{\mathrm{d}l}{\mathrm{d}u}\frac{1}{\frac{\mathrm{d}l}{\mathrm{d}u}} && \text{(by equation (3))} \\
&= 1
\end{aligned}$$

Hence, $\lambda(l)$ has unit speed and is thus a unit speed parameterization of $\gamma(u)$.

□

Let us define the following quantities:

Definition 2.4.3. *Let $\gamma(u)$ be a regular curve. Then, one defines:*

1. *The* ***unit tangent vector*** *$\vec{T}(u)$ to $\gamma(u)$ at a particular u is defined as follows:*

$$\vec{T}(u) \equiv \frac{\dot{\gamma}(u)}{|\dot{\gamma}(u)|}$$

2. *The* ***unit normal vector*** *$\vec{N}(u)$ to $\gamma(u)$ at a particular u is defined as follows:*

$$\vec{N}(u) \equiv \frac{\vec{T}'(u)}{\kappa(u)}$$

where $\vec{T}'(u) \equiv \frac{\mathrm{d}\vec{T}}{\mathrm{d}u}$ and

$$\kappa(u) \equiv |\vec{T}'(u)|$$

$\kappa(u)$ is said to be the ***curvature*** *of γ at u.*

3. *The* ***unit binormal vector*** *$\vec{B}(u)$ to $\gamma(u)$ at a particular u is defined as follows:*

$$\vec{B}(u) \equiv \vec{T}(u) \times \vec{N}(u)$$

Therefore, the vectors $\vec{T}$, $\vec{N}$ and $\vec{B}$ are three ***mutually orthogonal*** *unit vectors. Thus, they form a coordinate system that "moves along" with the curve γ. This coordinate system is known as the* ***Frenet Frame****.*

Note that the *curvature* κ gives a measure of how much the curve γ diverges from a straight line at u.

The following can be verified by direct calculation:[10]

Result 2.4.4. *The following set of equations is known as the* ***Frenet Formulae****:*

$$\begin{aligned} \vec{T}'(u) &= & \kappa\vec{N}(u) & \\ \vec{N}'(u) &= -\kappa\vec{T}(u) & & +\tau\vec{B}(u) \\ \vec{B}'(u) &= & -\tau\vec{N}(u) & \end{aligned}$$

where the primes denote derivatives with respect to u, and

$$\tau(u) \equiv -\vec{N}(u) \cdot \vec{B}'(u)$$

is called the ***torsion*** *of γ at u.*

The *torsion* τ gives a measure of how much the curve γ bends out of the plane spanned by $\vec{T}$ and $\vec{N}$.

2.5 Characteristics of Surfaces

Consider a surface M that can (locally) be perameterized by a smooth, differentiable coordinate patch[11] $\mathbf{x}(u, v)$. On it, we define the following quantities:

[10] These calculations can be found in [Opr04].

[11] It is assumed that the reader understands the notion of a coordinate patch. If however, this is not the case, a definition of this concept can be found in [Car76].

Definition 2.5.1. *The **unit normal vector** of M is defined to be:*

$$\vec{U} \equiv \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$$

Definition 2.5.2. *It will be convenient to define the following short-hand notation:*

$$\begin{aligned} l &= \mathbf{x}_{uu} \cdot \vec{U} \qquad & E &= \mathbf{x}_u \cdot \mathbf{x}_u \\ m &= \mathbf{x}_{uv} \cdot \vec{U} \qquad & F &= \mathbf{x}_u \cdot \mathbf{x}_v \\ n &= \mathbf{x}_{vv} \cdot \vec{U} \qquad & G &= \mathbf{x}_v \cdot \mathbf{x}_v \end{aligned}$$

*The **metric** is defined to be*

$$\tilde{g} \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

(Note that the metric $\tilde{g}$ is not to be confused with the Minkowski metric g. The main difference between the two is that whilst g is constant, $\tilde{g}$ is dependent on the surface and changes with the surface accordingly).

Definition 2.5.3. *The **Second Fundamental Form** is defined to be:*

$$\mathbf{II} \equiv \begin{bmatrix} \mathbf{x}_{uu} \cdot \vec{U} & \mathbf{x}_{uv} \cdot \vec{U} \\ \mathbf{x}_{uv} \cdot \vec{U} & \mathbf{x}_{vv} \cdot \vec{U} \end{bmatrix}$$

The Second Fundamental form gives a measure of how the normal vector of M, namely $\vec{U}$, changes as one goes along M in a particular direction. Therefore, **II** tells us about the curvature of M in a particular direction.[12] This curvature is called the *normal curvature* $k(\vec{u})$ in a particular direction $\vec{u}$ on M.[13]At a particular point $P \in M$, there exist directions $\vec{u}_1$ and $\vec{u}_2$ such that the normal curvature attains its maximum and minimum when considering the directions $\vec{u}_1$ and $\vec{u}_2$ respectively.[14]

One defines the following quantities:

[12]A more detailed discussion on **II** can be found in [Car76].

[13]A formal defintion of *normal curvature* can be found in [Car76].

[14]This is true as the normal curvature $k(\vec{u})$ is a continuous function and thus attains its maximum and minimum on a closed interval. This is made clear in [Opr04].

Definition 2.5.4. *At a particular point $P \in M$, the **principal curvatures** k_1 and k_2 of M at P are defined to be:*

$$k_1 = k(\vec{u}_1) = \min_{\vec{u}} k(\vec{u})$$
$$k_2 = k(\vec{u}_2) = \max_{\vec{u}} k(\vec{u})$$

It turns out that k_1 and k_2 are the eigenvalues of **II**.[15]

We define another kind of curvature:

Definition 2.5.5. *The **mean curvature** H of M at a point $P \in M$ is defined to be:*

$$H = \frac{k_1 + k_2}{2}$$

Using the notation introduced in this section, it can be shown that the following formula for H holds:[16]

$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)} \tag{4}$$

2.6 Maximal Surfaces

At this point, one can define a *maximal surface*:

Definition 2.6.1. *A surface M in $\mathbf{L}^3$ is said to be **maximal**, if it has zero mean curvature, i.e if:*

$$H = 0$$

everywhere on M.

The following result proviedes a characteristic trait of maximal surfaces:

[15] This fact can be seen in [Opr04].
[16] The explicit calculations can be found in [Opr04].

Result 2.6.2. *A graphical spacelike surface M represented by a single smooth coordinate patch $\mathbf{x}(u,v) = (u,v,f(u,v))$ is maximal, if and only if the following equation is satisfied:*

$$f_{uu}(1-f_v^2) + 2f_u f_v f_{uv} + f_{vv}(1-f_u^2) = 0 \tag{5}$$

Equation (5) is referred to as the ***Maximal Surface Equation***.[17]

2.7 Isothermal Coordinates

One can obviously see that the features of a surface, such as its curvature, do not depend on the parameterization of the surface. Thus, one usually chooses to work with a parameterization, that is most suitable for the desired calculation. *Isothermal coordinates* turn out to be a very convenient parameterization to work with:

Definition 2.7.1. *Let M be a surface parameterized by a coordinate patch $\mathbf{x}(u,v)$. Then the parameterization is called* ***isothermal*** *(with* ***isothermal parameters*** *u and v), if the following two properties are satisfied:*

$$\begin{aligned} &1.) \quad \mathbf{x}_u \cdot \mathbf{x}_u = \mathbf{x}_v \cdot \mathbf{x}_v \iff E = G \\ &2.) \quad \mathbf{x}_u \cdot \mathbf{x}_v = 0 \qquad\quad\ \iff F = 0 \end{aligned}$$

Next, a result regarding the existence of isothermal coordinates on a maximal surface will be stated without proof:[18]

Result 2.7.2. *Let M be a maximal smooth surface in $\mathbf{L}^3$. Then M can be locally parameterized by isothermal coordinates.*

[17] The more general *Maximal Surface Equation* for smooth spacelike surfaces in $\mathbf{L}^n$ can be found in [Che76].

[18] A proof of the more general result, namely that isothermal coordinates exist locally on every smooth Lorentzian surface is given in [Lar96].

2.8 Complex Variables and Liouville's Theorem

The proof of the *Calabi-Bernstein Theorem* (**Theorem 6.1**) at the end of this project relies on *Liouville's Theorem*, which reads as follows:

Theorem 2.8.1. *Every entire, bounded, complex valued function is a constant function.*

Proof. [19] Let $f(z) : \mathbf{C} \longrightarrow \mathbf{C}$ be an entire and bounded complex-valued function. Since f is bounded, there exists an $N \in \mathbf{R}^+$ such that:

$$|f(z)| \leq N \qquad \text{for all } z \in \mathbf{C}$$

As f is an entire function it is analytic everywhere. In particular, f is analytic in a region $\mathcal{C}$ and along its boundary $\partial\mathcal{C}$, where $\mathcal{C}$ is defined to be a circle of redius R centered at some arbitrary fixed $z_0 \in \mathbf{C}$. Thus, the first derivative of f at z_0 can be expressed by means of the *Cauchy Integral Formula for Derivatives*[20] as follows:

$$f'(z_0) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{(z - z_0)^2}\, \mathrm{d}z$$

where $f'(z_0) \equiv \frac{\mathrm{d}f}{\mathrm{d}z}\big|_{z=z_0}$.

Then certainly, the following holds:

$$\begin{aligned} 0 \leq |f'(z_0)| &= \left| \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{(z - z_0)^2}\, \mathrm{d}z \right| \\ &\leq \frac{1}{2\pi} \int_{\mathcal{C}} \frac{|f(z)|}{|z - z_0|^2}\, \mathrm{d}z \\ &\leq \frac{1}{2\pi} \int_{\mathcal{C}} \frac{N}{R^2}\, \mathrm{d}z \\ &= \frac{1}{2\pi} \frac{N}{R^2} \int_{\mathcal{C}} \mathrm{d}z \\ &= \frac{1}{2\pi} \frac{N}{R^2} 2\pi R \\ &= \frac{N}{R} \end{aligned}$$

[19] This proof is based on the proof of Liouville's Theorem found in [Bro04].

[20] An informal proof of this formula can be found in [Bro04].

As f is an entire function, the above argument holds not only on and inside the region $\mathcal{C}$, but on the entire complex plane, in other words, on and inside $\mathcal{C}$ as $R \longrightarrow \infty$. Therefore:

$$\begin{aligned} 0 \leq \quad |f'(z_0)| \quad &\leq \frac{N}{R} \\ \lim_{R\to\infty} 0 \leq \lim_{R\to\infty} |f'(z_0)| &\leq \lim_{R\to\infty} \frac{N}{R} \\ 0 \leq \quad |f'(z_0)| \quad &\leq 0 \end{aligned}$$

Hence:

$$\begin{aligned} |f'(z_0)| &= 0 \\ \Longrightarrow \quad f'(z_0) &= 0 \end{aligned}$$

Since z_0 was chosed arbitrarily, $f'(z) = 0$ *for all* $z \in \mathbf{C}$. Therefore, $f(z)$ is a constant function.

□

2.9 Gauss Map

The *Gauss map* is named after the German mathematician Carl Friedrich Gauß. The Gauss map, usually denoted by G, is a mapping from a surface M to the unit sphere S_2, which is defined as follows:

$$G : M \longrightarrow S_2 \qquad \text{with} \qquad G(P) = \vec{U}(P)$$

where P is a point on M and $\vec{U}(P)$ is the unit normal vector of M at P. So, the Gauss map maps a point $P \in M$ to its unit normal vector $\vec{U}(P)$. Since a vector is independent of its position, it might as well be moved to the origin of the coordinate system. Since $\vec{U}(P)$ has unit length for all $P \in M$, it thus points to a point on the unit sphere S_2. The Gauss map will prove useful in the proof of the *Calabi-Bernstein Theorem* (**Theorem 6.1**).

3 Surfaces with Maximum Area

In $\mathbf{R}^3$, consider a contour $\partial\mathcal{C}$. Clearly, there are infinitely many surfaces M that have $\partial\mathcal{C}$ as their boundary. But it is physically reasonable to say that there exists one surface, say M^*, such that M^* has the least area of all the surfaces M with boundary $\partial\mathcal{C}$.[21]
Let us define an area functional as follows:

Definition 3.1. *Let $\partial\mathcal{C}$ be a contour in a 3-dimensional space. Denote the set of all surfaces M bounded by $\partial\mathcal{C}$ by $\mathbf{M}(\partial\mathcal{C})$. Then, one can define a functional:*

$$A : \mathbf{M}(\partial\mathcal{C}) \longrightarrow \mathbf{R}$$

such that $A(M)$ = area of M. A is called the ***area functional****.*

Clearly, since M^* has the least area of all $M \in \mathbf{M}(\partial\mathcal{C})$, it is thus a critical point of A. Therefore, $A'(M) = 0$.[22]

In Minkowski space $\mathbf{L}^3$, we have already seen that there is a "negative contribution" from the t-axis. Therefore, a surface area in $\mathbf{L}^3$ can always be decreased by extending the surface into the t-direction. This means that the concept of *minimum area* is not well defined in $\mathbf{L}^3$. There is however a possibility of maximizing the area of a surface M bounded by a contour $\partial\mathcal{C}$. Thus in $\mathbf{L}^3$ a surface with maximum area will be a critical point of the area functional A.

In this section, a result regarding the correspondence of *surfaces with maximum area* and *maximal surfaces* will be illustrated.

Theorem 3.2. *Every graphical surface in $\mathbf{L}^3$ that has maximum area is a maximal surface.*

[21] This is strictly speaking not true. In fact, there exist contours that bound several least area surfaces. It turns out that only if the total curvature of the contour is less than 4π, the least-area surface that it bounds is unique. This Theorem has been proven by Nitsche (see [Nit73]). For the purpose of this project, it is assumed that $\partial\mathcal{C}$ has total curvature less than 4π and hence M^* is unique.

[22] The derivative notation is sloppy as it does not express what the derivative is taken with respect to. This however will become clear in the proof of **Theorem 3.2**.

Proof. This proof makes use of the *calculus of variations.*[23] The general strategy of a proof using the calculus of variations follows:

- We consider a certain quantity, say Q, subject to a set of boundary conditions that extremize a certain functional, say ϕ.
- Next, one considers a variation of Q, namely $Q_\varepsilon \equiv Q + \varepsilon P$ where P is an auxiliary function and $\varepsilon \ll 1$, such that Q_ε satisfies the same boundary conditions as Q.
- Since Q is assumed to extremize f, it is certainly true that:

$$\left.\frac{\mathrm{d}\phi(Q_\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \frac{\mathrm{d}\phi(Q)}{\mathrm{d}\varepsilon} = 0 \tag{6}$$

- Now, equation (6) will be reformulated using algebraic manipulations in order to infer information about the function Q.

Consider a graphical surface M with boundary $\partial\mathcal{C}$ that can be described by a single patch of the form:

$$\mathbf{x}(u,v) = (u, v, f(u,v)) \tag{7}$$

where $f(u,v)$ is a function of u and v.

The partial derivatives of $\mathbf{x}$ are:

$$\begin{aligned} \mathbf{x}_u &= (1, 0, f_u) \\ \mathbf{x}_v &= (0, 1, f_v) \\ \mathbf{x}_u \times \mathbf{x}_v &= (-f_u, -f_v, 1) \\ |\mathbf{x}_u \times \mathbf{x}_v| &= \sqrt{f_u^2 + f_v^2 - 1} \end{aligned}$$

Let the patch $\mathbf{x}$ as described above maximize the area $\mathcal{A}$ of M. $\mathcal{A}$ can be described by the following surface integral:

$$A(M) \equiv \mathcal{A} = \iint |\mathbf{x}_u \times \mathbf{x}_v| \,\mathrm{d}u\,\mathrm{d}v = \iint \sqrt{f_u^2 + f_v^2 - 1}\,\mathrm{d}u\,\mathrm{d}v$$

[23] A detailed chapter on the *calculus of variations* (or *variational calculus*) can be found in [Arf05].

where $A(M)$ is the area functional as described in **Definition 3.1**.

Consider a variation of the patch $\mathbf{x}$, namely $\mathbf{x}_\varepsilon \equiv (u, v, f(u,v) + \varepsilon h(u,v))$. Here, $h(u,v)$ is an auxiliary function and $\varepsilon \ll 1$. For $\mathbf{x}_\varepsilon$ to describe a surface M_ε which lies in the domain of the area functional A, this surface needs to be bounded by $\partial\mathbf{C}$. To ensure this condition, we set $h(x,y) = 0$ for all $x, y \in \partial\mathcal{C}$, as then $\mathbf{x}_\varepsilon$ reduces to $\mathbf{x}$ at $\partial\mathcal{C}$ which, by definition, describes a surface bounded by $\partial\mathcal{C}$.

The following calculations can be obtained:

$$\begin{aligned}
(\mathbf{x}_\varepsilon)_u &= (1, 0, f_u + \varepsilon h_u) \\
(\mathbf{x}_\varepsilon)_v &= (0, 1, f_v + \varepsilon h_v) \\
(\mathbf{x}_\varepsilon)_u \times (\mathbf{x}_\varepsilon)_v &= (-(f_u + \varepsilon h_u), -(f_v + \varepsilon h_v), 1) \\
|(\mathbf{x}_\varepsilon)_u \times (\mathbf{x}_\varepsilon)_v| &= \sqrt{(f_u + \varepsilon h_u)^2 + (f_v + \varepsilon h_v)^2 - 1} = \\
&= \sqrt{f_u^2 + f_v^2 - 1 + 2\varepsilon (f_u h_u + f_v h_v) + \varepsilon^2 (h_u^2 + h_v^2)}
\end{aligned}$$

The following short hand notation will be used:

$$A(M_\varepsilon) \equiv A(\varepsilon) \tag{8}$$

Then, the area functional of a surface M_ε described by $\mathbf{x}_\varepsilon$ is:

$$\begin{aligned}
A(\varepsilon) &= \iint |(\mathbf{x}_\varepsilon)_u \times (\mathbf{x}_\varepsilon)_v| \, \mathrm{d}u \, \mathrm{d}v \\
&= \iint \sqrt{f_u^2 + f_v^2 - 1 + 2\varepsilon (f_u h_u + f_v h_v) + \varepsilon^2 (h_u^2 + h_v^2)} \, \mathrm{d}u \, \mathrm{d}v
\end{aligned}$$

Clearly, at $\varepsilon = 0$, $M_\varepsilon = M$ and hence:

$$\left. \frac{\mathrm{d}A(\varepsilon)}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} = 0$$

Therefore, we consider the following:

$$\begin{aligned}
\frac{\mathrm{d}A(\varepsilon)}{\mathrm{d}\varepsilon} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left\{ \iint \sqrt{f_u^2 + f_v^2 - 1 + 2\varepsilon (f_u h_u + f_v h_v) + \varepsilon^2 (h_u^2 + h_v^2)} \, \mathrm{d}u \, \mathrm{d}v \right\} \\
&= \iint \frac{\partial}{\partial \varepsilon} \left\{ \sqrt{f_u^2 + f_v^2 - 1 + 2\varepsilon (f_u h_u + f_v h_v) + \varepsilon^2 (h_u^2 + h_v^2)} \right\} \mathrm{d}u \, \mathrm{d}v \\
&= 2 \iint \frac{f_u h_u + f_v h_v + \varepsilon (h_u^2 + h_v^2)}{\sqrt{f_u^2 + f_v^2 - 1 + 2\varepsilon (f_u h_u + f_v h_v) + \varepsilon^2 (h_u^2 + h_v^2)}} \, \mathrm{d}u \, \mathrm{d}v
\end{aligned} \tag{9}$$

where the derivative could be brought into the integral by *Leibniz' Rule of Differentiation under the Integral* (see [Fla73]).
Equation (9) implies:

$$\left.\frac{\mathrm{d}A(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 2\iint \frac{f_u h_u + f_v h_v}{\sqrt{(f_u)^2 + (f_v)^2 - 1}}\,\mathrm{d}u\,\mathrm{d}v = 0$$

Hence:

$$\iint \frac{f_u h_u + f_v h_v}{\sqrt{(f_u)^2 + (f_v)^2 - 1}}\,\mathrm{d}u\,\mathrm{d}v = 0 \tag{10}$$

The next objective is to use Green's Theorem (**Theorem 2.3.2**).This can be accomplished as follows: Let $k = \sqrt{(f_u)^2 + (f_v)^2 - 1}$ and define P and Q as follows:

$$P \equiv \frac{1}{k} f_u h \qquad\qquad Q \equiv \frac{1}{k} f_v h$$

Differentiating P and Q with respect tu u and v respectively, we obtain:

$$\begin{aligned}\frac{\partial P}{\partial u} &= \frac{\partial}{\partial u}\left(\frac{f_u h}{\sqrt{(f_u)^2 + (f_v)^2 - 1}}\right) = \\ &= \frac{f_{uu} h + f_u h_u}{\sqrt{f_u^2 + f_v^2 - 1}} - \frac{f_u h (f_u f_{uu} + f_v f_{uv})}{\sqrt[3]{f_u^2 + f_v^2 - 1}} = \\ &= \frac{f_u h_u}{k} + \frac{h(k^2 f_{uu} - f_u^2 f_{uu} - f_u f_v f_{uv})}{k^3}\end{aligned}$$

Similarly, for $\frac{\partial Q}{\partial v}$, we obtain:

$$\frac{\partial Q}{\partial v} = \frac{f_v h_v}{k} + \frac{h(k^2 f_{vv} - f_v^2 f_{vv} - f_u f_v f_{uv})}{k^3}$$

Therefore, we obtain that:

$$\begin{aligned}&\iint \left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v = \\ &\quad = \iint \frac{f_u h_u + f_v h_v}{k}\,\mathrm{d}u\,\mathrm{d}v + \\ &\qquad + \iint \frac{h}{k^3}\left(k^2 f_{uu} - f_u^2 f_{uu} - f_u f_v f_{uv} + k^2 f_{vv} - f_v^2 f_{vv} - f_u f_v f_{uv}\right)\mathrm{d}u\,\mathrm{d}v = \\ &\quad = \iint \frac{f_u h_u + f_v h_v}{k}\,\mathrm{d}u\,\mathrm{d}v + \\ &\qquad + \iint \frac{h}{k^3}\left(f_{uu}(k^2 - f_u^2) - 2 f_u f_v f_{uv} + f_{vv}(k^2 - f_v^2)\right)\mathrm{d}u\,\mathrm{d}v\end{aligned} \tag{11}$$

The first integral of equation (11) is zero, by equation (10). By Green's Theorem, the following can be obtained:

$$\iint \frac{f_u h_u + f_v h_v}{k}\,\mathrm{d}u\,\mathrm{d}v + \iint \frac{h}{k^3}\left(f_{uu}(k^2 - f_u^2) - 2f_u f_v f_{uv} + f_{vv}(k^2 - f_v^2)\right)\,\mathrm{d}u\,\mathrm{d}v$$
$$= \int_{\partial\mathcal{C}} \frac{f_u h}{\sqrt{f_u^2 + f_v^2 - 1}}\,\mathrm{d}v - \int_{\partial\mathcal{C}} \frac{f_v h}{\sqrt{f_u^2 + f_v^2 - 1}}\,\mathrm{d}u = 0 \tag{12}$$

The above integral, equation (12), has to be equal to zero, as we are integrating along the $\partial\mathcal{C}$ and the auxiliary function h was defined to be zero at the boundary. Combining this result with equation (10) yield the following:

$$\iint \frac{h}{k^3}\left(f_{uu}(k^2 - f_u^2) - 2f_u f_v f_{uv} + f_{vv}(k^2 - f_v^2)\right)\,\mathrm{d}u\,\mathrm{d}v = 0 \tag{13}$$

Equation (13) must hold *for all* functions $h(u, v)$. Therefore, the integrand of euqation (13) must be identically equal to zero. Thus, the following equations can be obtained:

$$\begin{aligned}
0 &= \frac{h}{k^3}\left(f_{uu}(k^2 - f_u^2) - 2f_u f_v f_{uv} + f_{vv}(k^2 - f_v^2)\right)\\
&= f_{uu}(k^2 - f_u^2) - 2f_u f_v f_{uv} + f_{vv}(k^2 - f_v^2)\\
&= f_{uu}(f_u^2 + f_v^2 - 1 - f_u^2) - 2f_u f_v f_{uv} + f_{vv}(f_u^2 + f_v^2 - 1 - f_v^2)\\
&= f_{uu}(f_v^2 - 1) - 2f_u f_v f_{uv} + f_{vv}(f_u^2 - 1)\\
&= f_{uu}(1 - f_v^2) + 2f_u f_v f_{uv} + f_{vv}(1 - f_u^2) \qquad (14)
\end{aligned}$$

But (14) is just the *Maximal Surface Equation* (see **Result 2.6.2**, equation (5)). Therefore, a graphical surface of maximum area satisfies the *Maximal Surface Equation* and is thus a *maximal surface*.

□

4 Surfaces of Rotation

It is a well-known fact that a surface of rotation is generated by rotating a curve about an axis of rotation. In this section, a characteristic result about maximal spacelike surfaces of rotation about the t-axis will be proven:

Theorem 4.1. *Every maximal spacelike surface of rotation (about the t-axis) M in L^3 is congruent to (a part of) the following:*

1. *(x,y) - plane*
2. *catenoid of the first kind*

Proof. [24]

1. Obviously, the *(x,y)-plane* is a surface of rotation, as it easily can be obtained by for example rotating the line $l(s)$, where s is a parameter, defined by:
$$l(s) = (0,0,0) + s(1,0,0)$$
about the t-axis.

   A plane has zero normal curvature everywhere as the unit normal vector $\vec{U}$ clearly does not change when traversing along a particular direction, *regardless of the direction.* For the principal curvatures, this means that $k_1 = 0 = k_2$. Therefore, the mean curvature of the *(x,y)-plane* is:
$$H = \frac{k_1 + k_2}{2} = \frac{0+0}{2} = 0$$

   Therefore, by definition, the *(x,y)-plane* is a *maximal surface of rotation.*

   In the following, it is assumed that M is *not* a plane.

[24] This proof is based on the proof of the corresponding Theorem for general maximal surfaces of rotation in $\mathbf{L}^3$ in [Kob83].

2. Consider a rotation of a curve $\rho(t)$ about the t-axis. Then M can be represented by a single smooth coordinate patch $\mathbf{x}(t, \theta)$ as follows:

$$M = \mathbf{x}(t, \theta) = (\rho(t)\cos\theta, \rho(t)\sin\theta, t) \tag{15}$$

(Clearly, surfaces of rotation preserve cylindrical symmetry, which is why cylindrical coordinates were chosen in the above). Note that $\frac{\mathrm{d}\rho}{\mathrm{d}t} > 1$ is required for M to be spacelike.[25]

The partial derivatives of $\mathbf{x}$ with respect to θ and t become:

$$\begin{aligned} \mathbf{x}_t &= (\dot\rho\cos\theta, \dot\rho\sin\theta, 1) \\ \mathbf{x}_\theta &= (-\rho\sin\theta, \rho\cos\theta, 0) \end{aligned}$$

$$\begin{aligned} \mathbf{x}_{tt} &= (\ddot\rho\cos\theta, \ddot\rho\sin\theta, 0) \\ \mathbf{x}_{t\theta} &= (-\dot\rho\sin\theta, \dot\rho\cos\theta, 0) = \mathbf{x}_{\theta t} \\ \mathbf{x}_{\theta\theta} &= (-\rho\cos\theta, -\rho\sin\theta, 0) \end{aligned}$$

where $\dot\rho \equiv \frac{\mathrm{d}\rho}{\mathrm{d}t}$

Now, the following quantities can be calculated:

$$\begin{aligned} E = \mathbf{x}_t \cdot \mathbf{x}_t &= \quad \dot\rho^2\cos^2\theta + \dot\rho^2\sin^2\theta - 1 &&= \dot\rho^2 - 1 \\ F = \mathbf{x}_t \cdot \mathbf{x}_\theta &= -\rho\dot\rho\cos\theta\sin\theta + \rho\dot\rho\cos\theta\sin\theta &&= 0 \\ G = \mathbf{x}_\theta \cdot \mathbf{x}_\theta &= \quad \rho^2\sin^2\theta + \rho^2\cos^2\theta &&= \rho^2 \end{aligned}$$

By definition, the *metric* becomes:

$$\tilde g = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \dot\rho^2 - 1 & 0 \\ 0 & \rho^2 \end{bmatrix}$$

Next, the normal vector $\vec N$ and the unit normal vector $\vec U$ will be calculated:

$$\begin{aligned} \vec N &= \mathbf{x}_t \times \mathbf{x}_\theta = (-\rho\cos\theta, -\rho\sin\theta, \dot\rho\rho) \\ |\vec N| &= \sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta - \rho^2\dot\rho^2} = \rho\sqrt{1-\dot\rho^2} \end{aligned}$$

$$\vec U = \frac{\vec N}{|\vec N|} = \frac{(-\cos\theta, -\sin\theta, \dot\rho)}{\sqrt{1-\dot\rho^2}} = \alpha(-\cos\theta, -\sin\theta, \dot\rho)$$

[25]Reader are asked to convince themselves of this fact as it will not be proven here.

where $\alpha \equiv \frac{1}{\sqrt{1-\dot{\rho}^2}}$.

Now, we are in a position to calculate the *Second Fundamental Form* **II**:

$$\mathbf{II} = \begin{bmatrix} \mathbf{x}_{tt} \cdot \vec{U} & \mathbf{x}_{t\theta} \cdot \vec{U} \\ \mathbf{x}_{t\theta} \cdot \vec{U} & \mathbf{x}_{\theta\theta} \cdot \vec{U} \end{bmatrix} = \alpha \begin{bmatrix} -\ddot{\rho} & 0 \\ 0 & \rho \end{bmatrix}$$

In the above, it is important to note that **II** is with respect to the metric $\tilde{g}$. Therefore, one needs to trace **II** with respect to $\tilde{g}$ rather than with respect to g:

$$\tilde{g}^{-1}\mathbf{II} = \alpha \begin{bmatrix} \frac{1}{\dot{\rho}^2-1} & 0 \\ 0 & \frac{1}{\rho^2} \end{bmatrix} \begin{bmatrix} -\ddot{\rho} & 0 \\ 0 & \rho \end{bmatrix} = \alpha \begin{bmatrix} \frac{-\ddot{\rho}}{\dot{\rho}^2-1} & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} = \alpha \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \equiv \mathbf{II}^*$$

Since we desire M to be maximal, the mean curvature H has to be equal to zero, by the definition of maximal surfaces. This leads to the following differential equation:

$$\begin{aligned} & H = \frac{1}{2} tr(\mathbf{II}^*) = \frac{\alpha}{2}\left(\frac{-\ddot{\rho}}{\dot{\rho}^2-1} + \frac{1}{\rho}\right) = 0 \\ \Longrightarrow \quad & -\ddot{\rho}\rho - 1 + \dot{\rho}^2 = 0 \\ \Longrightarrow \quad & \ddot{\rho}\rho - \dot{\rho}^2 = -1 \qquad (16) \end{aligned}$$

where we keep in mind that $\dot{\rho} > 1$. Hence, also $\dot{\rho}^2 > 1$. It follows that $\dot{\rho}^2 - 1 > 0$. This is why α could be cancelled out in the above.

Equation (16) is a second order, non-homogeneous differential equation. Its general solution ρ_G is of the following form:[26]

$$\rho_G = \rho_H + \rho_P$$

where ρ_H denotes the general solution of the homogeneous version of equation (16) and ρ_P represents a particular solution to the nonhomogeneous equation (16)

First, lets find ρ_H. The homogeneous version of equation (16) reads:

$$\ddot{\rho}\rho - \dot{\rho}^2 = 0 \qquad (17)$$

[26] This can be seen in more detail in [Pol06].

Note that $\rho = 0$ for all t in the domain is a solution to the above equation (the surface corresponding to this solution is in fact the (x, y)-plane). In the following, it will be assumed that there exists a t in the domain for which $\rho \neq 0$ in order to find non-trivial solutions to equation (17).

The following change of variables is introduced: Let $z(\rho) = \frac{\dot{\rho}}{\rho}$. Then, we obtain:

$$\begin{aligned} \dot{\rho} &= z\rho \\ \ddot{\rho} &= \frac{\mathrm{d}z}{\mathrm{d}\rho}\dot{\rho}\rho + z\dot{\rho} \\ &= \dot{\rho}\left[z'\rho + z\right] \end{aligned}$$

where $z' \equiv \frac{\mathrm{d}z}{\mathrm{d}\rho}$

Substituting the above into equation (17), the following can be obtained:

$$\begin{aligned} \ddot{\rho}\rho - \dot{\rho}^2 = 0 \Longrightarrow \quad & \ddot{\rho} - \frac{\dot{\rho}}{\rho}\dot{\rho} &= 0 \\ \Longleftrightarrow \quad & \dot{\rho}\left[z'\rho + z\right] - z\dot{\rho} &= 0 \\ \Longrightarrow \quad & z'\dot{\rho}\rho &= 0 \end{aligned}$$

Since $\rho \neq 0$ by assumption, it follows that:

$$\begin{aligned} & z'\dot{\rho} = 0 \\ \Longleftrightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}t}\left(z(\rho(t))\right) = 0 \\ \Longleftrightarrow \quad & z = C_1 \qquad \text{(where } C_1 \text{ is a constant in } \mathbf{R}\text{)} \\ \Longleftrightarrow \quad & \frac{\dot{\rho}}{\rho} = C_1 \\ \Longleftrightarrow \quad & \dot{\rho} = C_1\rho \end{aligned} \tag{18}$$

Equation (18) is a seperable ordinary differential equation and can be solved as follows:

$$\begin{aligned} \dot{\rho} &= C_1\rho \\ \frac{\mathrm{d}\rho}{\mathrm{d}t} &= C_1\rho \\ \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= C_1 \\ \int \frac{1}{\rho}\,\mathrm{d}\rho &= \int_0^t C_1\,\mathrm{d}s && \text{(by the Chain Rule)} \\ \ln\rho &= C_1 t + C_2 && \text{(where } C_2 \text{ is a constant in } \mathbf{R}\text{)} \\ \Longrightarrow \quad \rho(t) &= C_3 e^{C_1 t} && \text{(where } C_3 \equiv e^{C_2} \text{ is a constant in } \mathbf{R}\text{)} \end{aligned}$$

Therefore,

$$\rho_H(t) = C_3 e^{C_1 t} \tag{19}$$

where C_1 and C_3 are arbitrary constants in $\mathbf{R}$.[27]

Next, we focus on finding ρ_P. This requires us to find a particular solution to equation (16).

Using the *Method of Undetermined Coefficients*[28], a natural *Ansatz* would be a trigonometric function. Recalling that $\dot{\rho}^2 - 1 > 0$ is required, we consider the following *Ansatz*: Let $\rho(t) = \sinh(t)$. We obtain:

$$\begin{aligned}
\rho &= \sinh(t) \\
\dot{\rho} &= \cosh(t) \\
\dot{\rho}^2 &= \cosh^2(t) \\
\ddot{\rho} &= -\sinh(t)
\end{aligned}$$

Substituting the above calculations into equation (16) then gives:

$$\begin{aligned}
-1 &= \ddot{\rho}\rho - \dot{\rho}^2 \\
&= -\sinh(t) \cdot \sinh(t) - \cosh^2(t) \\
&= -\left(\sinh^2(t) + \cosh^2(t)\right) \\
&= -(1) \qquad \checkmark
\end{aligned}$$

Therefore, $\rho_P = \sinh(t)$ is a particular solution of equation (16).
But now, one has to note that:

$$\sinh(t) = \frac{e^t - e^{-t}}{2}$$

Therefore, ρ_P is not linearly independent of ρ_H, which was determined by equation (19). But since equation (16) is a second order ordinary differential equation, its solutions solemnly needs to be a linear combination of *two* linearly independent solutions. One can easily show that e^t and e^{-t} are two linearly independent functions of t[29] and as we have

[27] Note that additional information about the problem is required in order to obtain values for C_1 and C_3.

[28] This method is explained in detail in [Pol06].

[29] This will not be proven in this project, but readers are asked to convince themselves of this fact.

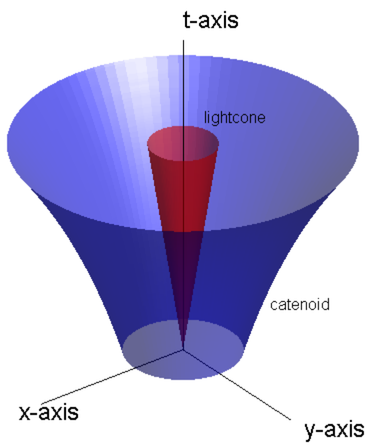

Figure 3: In the above one can see a *catenoid of the first kind*, which is described by equation (20). It is clear that it is a spacelike surface, as it is positioned outside the lightcone. For this particular catenoied, the parameter values $A = 1$ and $B = 1.3$ were used. (Graphic generated in MATLAB).

seen above, their linear combination satisfies equation (16).
In order to keep things general, one still has to include two constants of integration which can be incorporated into ρ_P as follows: Let:

$$\rho_P = \frac{1}{A}\sinh(At + B)$$

where A and B are arbitrary constants in $\mathbf{R}$.[30]

Therefore, ρ_P is a linear combination of two linearly independent functions with two integration constants. This means that $\rho_P = \rho_G$. Thus, the general solution of equation (16) is:

$$\rho_H = \frac{1}{A}\sinh(At + B)$$

Substituting this solution back into the patch $\mathbf{x}(t, \theta)$ (equation (15)), one is left with the following surface:

$$M = \mathbf{x}(t, \theta) = (\frac{1}{A}\sinh(At + B)\cos\theta, \frac{1}{A}\sinh(At + B)\sin\theta, t) \quad (20)$$

[30]The form of these constants was chosen following the example of [Kob83].

Equation (20) describes a *catenoid of the first kind* (which can be seen in Figure 3), as required.

□

5 Ruled Surfaces

Definition 5.1. *A ruled surface*[31] *M is a surface that can be represented by a coordinate patch of the following form:*

$$\mathbf{x}(u,v) = \gamma(u) + v\nu(u) \tag{21}$$

where $\gamma(u)$ is a curve, $\nu(u)$ is a vector and u and v are parameters contained in the intervals I_u and I_v respectively.

A ruled surface can be imagined as follows: One considers a curve $\gamma(u)$, which is also called the *directrix*, and a vector $\nu(u)$. At each point on the curve $\gamma(u)$, i.e. for each $u \in I_u$, let $L(u,v)$ be the line passing through $\gamma(u)$ with direction $\nu(u)$ and length $v|\nu(u)|$. These lines are also referred to as *rulings*. Then the suface M is the union of all the lines $L(u,v)$:

$$M = \bigcup_{u \in I_u} L(u,v) \tag{22}$$

From now on, the "dot-notation" for derivatives with respect to u will be used throughout this section, for example, let $\dot{\gamma}(u) \equiv \frac{\mathrm{d}\gamma(u)}{\mathrm{d}u}$.
The next result shows useful ways of parameterizing ruled surfaces:

Result 5.2. *Every ruled surface M can be parameterized as follows:*

$$\mathbf{x}(u,v) = \gamma(u) + v\nu(u)$$

such that:

1. $\nu(u) \cdot \nu(u) = 1$
2. $\dot{\gamma}(u) \cdot \dot{\gamma}(u) = 1$
3. $\dot{\gamma}(u) \cdot \nu(u) = 0$

Proof. [32]

[31] A detailed discussion on ruled surfaces can be found in [Car76], which is where the following information has been sourced from.

[32] This proof is based on [Car76].

1. This fact should be clear, so only a motivation (not a proof) will be provided here: from equation (22), the ruled surface is a union of lines $L(u,v)$ as described above. These lines have length $v|\nu(u)|$. If one now parameterizes the surface by means of a vector $\tilde{\nu}(u)$ instead of $\nu(u)$, where $\tilde{\nu}(u)\cdot\tilde{\nu}(u) = |\tilde{\nu}(u)| = 1$, then one can define another parameter, say $\tilde{v}$ such that $\tilde{v} = v|\nu(u)|$. Then, the ruled surface can be equally well described by the following two patches:

$$M = \mathbf{x}(u,v) = \gamma(u) + v\nu(u) = \mathbf{y}(u,v) \equiv \gamma(u) + \tilde{v}\tilde{\nu}(u)$$

where $|\tilde{\nu}(u)| = 1$, as required.

2. This fact follows from **Result 2.4.2**.

3. Let M be a ruled surface described by a coordinate patch as in equation (21). Let $\zeta(u)$ be a curve defined as follows:

$$\zeta(u) = \gamma(u) + r(u)\nu(u) \tag{23}$$

where $r(u)$ is a real-valued function and $\gamma(u)$ and $\nu(u)$ are parameterized such that $\nu(u)\cdot\nu(u) = 1$ and $\dot{\gamma}(u)\cdot\dot{\gamma}(u) = 1$ (which is possible due to *1.* and *2.*). Let $\zeta(u)$ be such that $\dot{\zeta}(u)\cdot\nu(u) = 0$.
The unitlength of $\nu(u)$ implies the following:

Proposition 5.3. *Let $\nu(u)$ be a curve for a parameter $u \in I_u$ with $\nu(u)\cdot\nu(u) = 1$. Then $\nu(u)\cdot\dot{\nu}(u) = 0$*

Proof. [33] This follows from the product rule:
Since $\nu(u)\cdot\nu(u) = 1$, we have that:

$$\begin{aligned} 1 &= \nu(u)\cdot\nu(u) \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}u}\left[\nu(u)\cdot\nu(u)\right] \\ &= \nu(u)\cdot\dot{\nu}(u) + \dot{\nu}(u)\cdot\nu(u) \\ &= 2\nu(u)\cdot\dot{\nu}(u) \end{aligned}$$

Therefore, $\nu(u)\cdot\dot{\nu}(u) = 0$

□

[33] The proof of this Proposition is based on [Opr04].

By differentiating equation (23), the following is obtained:

$$\dot{\zeta}(u) = \dot{\gamma}(u) + \dot{r}(u)\nu(u) + r(u)\dot{\nu}(u)$$

Forming the inner product of $\dot{\zeta}(u)$ with $\nu(u)$ yields:

$$\begin{aligned}
0 &= \dot{\zeta}(u) \cdot \nu(u) && \text{(by definition of } \zeta(u)\text{)}\\
&= \dot{\gamma}(u) \cdot \nu(u) + \dot{r}(u)\,[\nu(u) \cdot \nu(u)] + r(u)\,[\dot{\nu}(u) \cdot \nu(u)]\\
&= \dot{\gamma}(u) \cdot \nu(u) + \dot{r}(u)\,[\nu(u) \cdot \nu(u)] && \text{(by } \textbf{Proposition 5.3}\text{)}
\end{aligned}$$

The above equation can be solved for $r(u)$ to obtain the following result:

$$r(u) = -\int \frac{\dot{\gamma}(u) \cdot \nu(u)}{\nu(u) \cdot \nu(u)}\,\mathrm{d}u \tag{24}$$

Next, define a parameter w as follows: Let $w = v - r(u)$, where $r(u)$ is defined as in equation (24). Furthermore, define a coordinate patch $\mathbf{y}(u, w)$ by:

$$\mathbf{y}(u, w) \equiv \zeta(u) + w\nu(u)$$

Substituting in for $\zeta(u)$ and w yields:

$$\begin{aligned}
\mathbf{y}(u, w) &= \zeta(u) + w\nu(u)\\
&= [\gamma(u) + r(u)\nu(u)] + [v - r(u)]\,\nu(u)\\
&= \gamma(u) + v\nu(u) + [r(u) - r(u)]\,\nu(u)\\
&= \gamma(u) + v\nu(u)\\
&= \mathbf{x}(u, v)
\end{aligned}$$

Thus, the surface M can also be represented by a coordinate patch $\mathbf{y}(u, w) \equiv \zeta(u) + w\nu(u)$ with $\dot{\zeta}(u) \cdot \nu(u) = 0$, as required.

□

Theorem 5.4. *Every maximal ruled surface in* $\mathbf{L}^3$ *is congruent to (a part of) one of the following*

1. *(x,y)-plane*
2. *heliocoid of the second kind*
3. *conjugate of Enneper's surface of the second kind*

Proof. [34] By definition, a spacelike ruled surface M can be represented by the following coordinate patch:

$$\mathbf{x}(u,v) = \gamma(u) + v\nu(u)$$

where $\gamma(u)$ is the *directrix* and the $\nu(u)$ are the *rulings*.[35]

From **Result 5.2**, a ruled surface can always be parameterized such that the following is true:

$$\nu(u)\cdot\nu(u) = 1 = \dot{\gamma}(u)\cdot\dot{\gamma}(u) \tag{25}$$

and

$$\dot{\gamma}(u)\cdot\nu(u) = 0 \tag{26}$$

From now on, the explicit dependence on u will be surpressed and γ shall denote $\gamma(u)$ (equivalently for $\dot{\gamma}$ and ν).

It follows from equation (26) that ν is the normal vector field to γ. Thus, for a particular u, ν **is the unit normal vector to the curve** γ at that u. This means that γ and ν are mutually orthogonal to each other. So, only γ needs to be found to determine M (as ν will automatically be established due to orthogonality to γ).

Next, let us calculate l, m, n and E, F, G:

$$\begin{aligned}\mathbf{x}_u &= \dot{\gamma} + v\dot{\nu}\\ \mathbf{x}_v &= \nu\end{aligned}$$

$$\begin{aligned}\mathbf{x}_{uu} &= \ddot{\gamma} + v\ddot{\nu}\\ \mathbf{x}_{uv} &= \dot{\nu}\\ \mathbf{x}_{vv} &= 0\end{aligned}$$

Thus, we have that:

$$\vec{U} = \frac{\dot{\gamma}\times\nu + v\dot{\nu}\times\nu}{|\dot{\gamma}\times\nu + v\dot{\nu}\times\nu|}$$

[34] This proof is based, in part, on the proof in [Kob83] and on the proof of the corresponding result in $\mathbf{R}^3$ (*Catalan's Theorem*) as found in [Opr04].

[35] For this terminology, refer to [Opr04].

And:

$$E = 1 + 2v\dot{\gamma}\cdot\dot{\nu} + v^2\dot{\nu}^2$$
$$F = 0$$
$$G = 1$$

$$l = \frac{v^2\{\ddot{\nu}\cdot(\dot{\nu}\times\nu)\} + v\{\ddot{\nu}\cdot(\dot{\gamma}\times\nu) + \ddot{\gamma}\cdot(\dot{\nu}\times\nu)\} + \ddot{\gamma}\cdot(\dot{\gamma}\times\nu)}{|\dot{\gamma}\times\nu + v\dot{\nu}\times\nu|}$$
$$m = \text{undetermined here}$$
$$n = 0$$

Using the above calculations, the formula for the mean curvature H, equation (4), reduces to the following:

$$H = \frac{l}{2E}$$

As M is taken to be a maximal surface, it follows that:

$$H = \frac{l}{2E} = 0 \qquad \Longrightarrow \qquad l = 0$$

Therefore, we have that:

$$v^2\{\ddot{\nu}\cdot(\dot{\nu}\times\nu)\} + v\{\ddot{\nu}\cdot(\dot{\gamma}\times\nu) + \ddot{\gamma}\cdot(\dot{\nu}\times\nu)\} + \ddot{\gamma}\cdot(\dot{\gamma}\times\nu) = 0 \tag{27}$$

Equation (27) is a polynomial of degree 2 in v. As a consequence of the *Fundamental Theorem of Algebra*[36], it follows that equation (27) equals zero if and only if each of the coefficients equals zero. Therefore, one is left with the following set ot equations:

$$\ddot{\gamma}\cdot(\dot{\gamma}\times\nu) = 0 \tag{28a}$$
$$\ddot{\nu}\cdot(\dot{\gamma}\times\nu) + \ddot{\gamma}\cdot(\dot{\nu}\times\nu) = 0 \tag{28b}$$
$$\ddot{\nu}\cdot(\dot{\nu}\times\nu) = 0 \tag{28c}$$

Let's recall the *Frenet Formulae* (**Result 2.4.4**):

$$\ddot{\gamma} = \kappa\nu$$
$$\dot{\nu} = -\kappa\dot{\gamma} + \tau\beta$$
$$\dot{\beta} = -\tau\nu$$

where β is the binormal vector of γ for a partucular u. Again the explicit dependence of β, the curvature κ and the torsion τ on u is understood.

Now, One can arrive to the following conclutions:

[36] The *Fundamental Theorem of Algebra* and an outline of its Proof are given in [Art11].

- Equation (28a) implies that $\ddot{\gamma}$ lies in the plane spanned by $\dot{\gamma}$ and ν (this plane will be denoted by $\text{Span}\{\dot{\gamma}, \nu\}$). From the Frenet Formulae, it follows that $\ddot{\gamma}$ is parallel to ν and by assumption, ν is perpendicular to $\dot{\gamma}$, therefore, $\ddot{\gamma}$ is perpendicular to $\dot{\gamma}$.

- Now, consider equation (28b). It is clear that $\ddot{\gamma}$ is parallel to ν whilst clearly $\nu \times \dot{\nu}$ is perpendicular to ν Therefore, $\ddot{\gamma} \cdot (\nu \times \dot{\nu}) = 0$. Hence equation (28b) reduces to:

$$\ddot{\nu} \cdot (\dot{\gamma} \times \nu) = 0$$

  This implies that $\ddot{\nu} \in \text{Span}\{\dot{\gamma}, \nu\}$.

- From equation (28c), it is clear that $\ddot{\nu} \in \langle \dot{\nu}, \nu \rangle$

As $\ddot{\nu} \in \text{Span}\{\dot{\gamma}, \nu\}$ and $\ddot{\nu} \in \text{Span}\{\dot{\nu}, \nu\}$, it follows that $\ddot{\nu} \in \text{Span}\{\dot{\gamma}, \nu\} \cap \text{Span}\{\dot{\nu}, \nu\}$. From this, one has to consider the following two cases:

Case 1: *$\ddot{\nu}$ is not parallel to ν for all u in the domain*
Since $\ddot{\nu} \in \text{Span}\{\dot{\gamma}, \nu\} \cap \text{Span}\{\dot{\nu}, \nu\}$ but $\ddot{\nu}$ is not parallel to ν for all u in the domain, it follows that $\dot{\gamma} = \frac{1}{a}\dot{\nu}$ for some $a \in \mathbf{R}$. Hence, $\ddot{\nu} \in \text{Span}\{\dot{\gamma} + b\dot{\nu}, \nu\}$ for some $b \in \mathbf{R}$. And for $\vec{U}$, one obtains:

$$\vec{U} = \frac{(\dot{\gamma} \times \nu)(1 + va)}{|\dot{\gamma} \times \nu||1 + va|} = \pm\dot{\gamma} \times \nu$$

where $|\dot{\gamma} \times \nu| = 1$ as $|\nu| = 1$ and $|\dot{\gamma}| = 1$ and $\frac{1+va}{|1+va|} = \pm 1$.
It follows that $\vec{U}$ is a unit normal vector to the plane $\text{Span}\{\dot{\gamma} + b\dot{\nu}, \nu\}$. Differentiating the above yields:

$$\begin{aligned}
\frac{\mathrm{d}\vec{U}}{\mathrm{d}u} &= \pm\left[\ddot{\gamma} \times \nu + \dot{\gamma} \times \dot{\nu}\right] \\
&= \pm\left[\kappa(\nu \times \nu) + \frac{1}{a}(\dot{\nu} \times \dot{\nu})\right] \\
&= \pm\left[\kappa(\vec{0}) + \frac{1}{a}(\vec{0})\right] \\
&= \vec{0}
\end{aligned}$$

Since $\frac{\mathrm{d}\vec{U}}{\mathrm{d}u} = \vec{0}$, it follows that $\vec{U}$ is a constant vector over the entire surface. Therefore, M is congruent to (a part of) a plane.

Case 2: *$\ddot{\nu}$ is parallel to ν for all u in the domain*
In this case, $\ddot{\nu}$ and ν are linearly dependent and one can write $\ddot{\nu} = c\nu$ for some constant c. This and the fact that, by assumption, $\dot{\gamma} \cdot \nu = 0$, implies that:

$$\dot{\gamma} \cdot \ddot{\nu} = 0 \tag{29}$$

Next, the curvature of the curve γ, κ_γ, and its torsion, τ_γ, are considered:

- For the curvature κ_γ, the following can be noted:

$$\begin{aligned} \kappa_\gamma &= \kappa_\gamma \nu \cdot \nu && \text{(as } \nu \cdot \nu = 1 \text{ by assumption)} \\ &= \ddot{\gamma} \cdot \nu && \text{(by the \textit{Frenet Formulae})} \\ &= -\dot{\gamma} \cdot \dot{\nu} \end{aligned}$$

  where the last equality holds due to the following:

$$\begin{aligned} & 0 &&= \dot{\gamma} \cdot \nu \\ \implies \quad & 0 &&= \frac{\mathrm{d}}{\mathrm{d}u} (\dot{\gamma} \cdot \nu) \\ & &&= \ddot{\gamma} \cdot \nu + \dot{\gamma} \cdot \dot{\nu} \\ \implies \quad & \ddot{\gamma} \cdot \nu &&= -\dot{\gamma} \cdot \dot{\nu} \end{aligned}$$

  Differentiating κ_γ with respect to u yields the following:

$$\begin{aligned} \frac{\mathrm{d}\kappa_\gamma}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}u} (-\dot{\gamma} \cdot \dot{\nu}) \\ &= -\ddot{\gamma} \cdot \dot{\nu} - \dot{\gamma} \cdot \ddot{\nu} \\ &= -\kappa_\gamma (\nu \cdot \dot{\nu}) - 0 \\ &= 0 \end{aligned}$$

  where $\dot{\gamma} \cdot \ddot{\nu} = 0$ by equation (29) and $\nu \cdot \dot{\nu} = 0$ by **Proposition 5.3**.

  The above thus implies that κ_γ is a constant.

- Using the *Frenet Formulae*, the following expression for the torsion τ_γ

is obtained:

$$\tau_\gamma = -\nu \cdot \dot{\beta}$$

Also:

$$\begin{aligned} \beta &= \dot{\gamma} \times \nu \\ \Longrightarrow \dot{\beta} &= \ddot{\gamma} \times \nu + \dot{\gamma} \times \dot{\nu} \end{aligned}$$

Therefore:

$$\begin{aligned} \tau_\gamma &= -(\ddot{\gamma} \times \nu) \cdot \nu + (\dot{\gamma} \times \dot{\nu}) \cdot \nu \\ &= -0 - (\dot{\gamma} \times \dot{\nu}) \cdot \nu \\ &= -(\dot{\gamma} \times \nu) \cdot \dot{\nu} \end{aligned} \tag{30}$$

where the last equality holds due to the general rule that, for every vectors a, b and $c \in \mathbf{L}^3$, we have that:

$$(a \times b) \cdot c = (a \times c) \cdot b$$

which can be verified by direct calculation.

Differentiating equation (30) yields the following:

$$\frac{\mathrm{d}\tau_\gamma}{\mathrm{d}u} = -\underbrace{(\ddot{\gamma} \times \nu) \cdot \dot{\nu}}_{\mathcal{E}1} - \underbrace{(\dot{\gamma} \times \dot{\nu}) \cdot \dot{\nu}}_{\mathcal{E}2} - \underbrace{(\dot{\gamma} \times \nu) \cdot \ddot{\nu}}_{\mathcal{E}3} \tag{31}$$

Now, each component of equation (31) is considered seperately:

$\mathcal{E}1$: $\quad (\ddot{\gamma} \times \nu) \cdot \dot{\nu} = (\kappa_\gamma \nu \times \nu) \cdot \dot{\nu}$
$= (\kappa_\gamma \vec{0}) \cdot \dot{\nu}$
$= \vec{0}$

$\mathcal{E}2$: $\quad (\dot{\gamma} \times \dot{\nu}) \cdot \dot{\nu} = \vec{0}$ (as clearly $(\dot{\gamma} \times \dot{\nu}) \perp \dot{\nu}$)

$\mathcal{E}3$: $\quad (\dot{\gamma} \times \nu) \cdot \ddot{\nu} = \vec{0}$ (as $\ddot{\nu} \in \mathrm{Span}\{\dot{\gamma}, \nu\}$ (see page 34), hence $(\dot{\gamma} \times \nu) \perp \ddot{\nu}$.)

This means that $\frac{\mathrm{d}\tau_\gamma}{\mathrm{d}u} = 0$, therefore τ_γ is constant.

Now, the following cases have to be considered:[37]

[37] As can be seen in [Kob83].

Case 1: $|\tau_\gamma| = |\kappa_\gamma| = 0$
Clearly, a ruled surface whose directrix fulfills this criteria is a plane (which is an option that has been covered earlier).

Case 2: $|\tau_\gamma| = |\kappa_\gamma| \neq 0$
This case yields a special kind of heliocoid, namely the conjugate of Ennepers surface of the second kind.

Case 3: $|\tau_\gamma| > |\kappa_\gamma| > 0$
Here, M is a heliocoid of the second kind.

Case 4: $0 < |\tau_\gamma| < |\kappa_\gamma|$
This surface is not spacelike (in fact it is light-like) and thus will not be considered here.

□

6 Calabi-Bernstein Theorem in $\mathbf{L}^3$

In 1914, Bernstein discovered that the only entire solution to the minimal surface equation in $\mathbf{R}^n$ for $n = 3$ is a plane. Later on, it was shown that this result holds for every $n \leqq 7$.
In 1968, Calabi found that the maximal surface equation has a *Bernstein-type property* for $\mathbf{L}^n$ for $n \leqq 4$ (see [Cal70]). Therefore, the *Bernstein-Theorem* in Minkowski space is now commonly refered to as the *Calabi-Bernstein Theorem*. In 1976, it was shown by Cheng and Yau that the Bernstein property holds in $\mathbf{L}^n$ *for all* n (see [Che76]), which is quite surprising as there are non-trivial solutions to the corresponding problem in $\mathbf{R}^n$ for $n > 7$.[38]

In this project, a prove of the *Calabi Bernsetein Theorem* in $\mathbf{L}^3$ for graphical maximal surfaces will be proved.

Theorem 6.1. *Let M be an entire graphical spacelike surface in $\mathbf{L}^3$ such that M is maximal. Then M is a plane.*

Proof. [39] Let M be a graphical surface. Then, by definition, M can be represented by one coordinate patch $\mathbf{x}$ as follows:

$$M = \mathbf{x}(x, y) = (x, y, f(x, y))$$

for some function$f$ whose graph is M.

From **Result 2.7.1** we know that every maximal surface M can be locally represented by isothermal coordinates. Since we are considering a graphical surface, the isothermal coordinates from **Result 2.7.1** can be extended globally over the entire surface such that M is defined by a single isothermal patch[40]$\mathbf{x}(u, v)$, where u and v are the isothermal parameters. Then the map T defined by:

$$T(x, y) \equiv (u, v)$$

[38] Cf. [Che76].
[39] This Proof is based on an outline of the Proof of the corresponding result for minimal surfaces in $\mathbf{R}^3$ in [Opr04].
[40] This is true as the proof in [Lar96] relies on the fact that a smooth (Lorentzian) surface can locally approximated by a graph. Since M is a graphical surface, this fact holds globally for M.

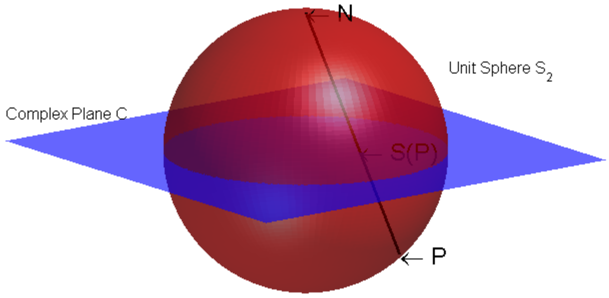

Figure 4: This diagram depicts how a point P on the unit sphere S_2 gets mapped to a poin $\mathcal{S}(P)$ on the complec plane $\mathbf{C}$ by the **Stereographic Projection map** $\mathcal{S} : S_2 \ \{N\} \longrightarrow \mathbf{C}$. (Graphic generated in MATLAB).

which exists by **Result 2.7.1**, has a smooth inverse T^{-1} : $(u, v) - domain \longrightarrow (x, y)-plane$, with $(u, v) \longmapsto (x, y)$, which is defined everywhere (by the *Inverse Function Theorem*, **Theorem 2.3.1**). Hence, T is a *diffeomorphism*[41] between the $(u, v) - domain$ and the $(x, y) - plane$. Thus, one can think of the $(u, v)-domain$ as a plane. Let us identify the $(u, v)-domain$ with the complex plane, i.e. let $(u, v) - domain = \mathbf{C}$. Therefore, one can consider a map $\mathcal{F} : \mathbf{C} \longrightarrow M$, $(u, v) \mapsto (u, v, f(u, v))$.

We have that M is a graphical surface and, by definition of a graph, f is well-defined. Due to that, and since M is spacelike, all the normal vectors $\vec{U}$ are mapped to the upper hemisphere of the sphere S_2 by the Gauss map G (see section 2.9). Equivalently, G maps all $-\vec{U}$ into the lower hemisphere of S_2.

Now, consider the stereographic projection $\mathcal{S} : S_2 \ \{N\} \longrightarrow \mathbf{C}$, where N is the *northpole* of S_2. Since, as was stated above, $G(-\vec{U})$ lies in the lower hemisphere of S_2, we have that $G(-\vec{U}) \neq N$ for all $-\vec{U}$ of M. Therefore, the composite $\mathcal{S} \circ G$ is defined for its entire domain $D \equiv \{-\vec{U} \text{ of } M\}$. Also note that $\mathcal{S} \circ G$ is bounded on its domain D, as $N \not\in D$. In fact, the range of $\mathcal{S} \circ G$ is the unit disc $S_1 \equiv \{z \in \mathbf{C} : |z| \leqq 1\}$, as can be seen Figure 4.

Consider the composite function $\mathcal{H} \equiv \mathcal{S} \circ G \circ \mathcal{F}$. Clearly, $\mathcal{H} : \mathbf{C} \longrightarrow \mathbf{C}$. Also, since $\mathcal{S}$ is bounded, so is $\mathcal{H}$. Furthermore, since each of the functions $\mathcal{F}$, G and $\mathcal{S}$ are entire, so is their composite $\mathcal{H}$. Therefore, $\mathcal{H}$ is a bounded and

[41] The definition of a diffeomorphism can be found in [Opr04].

entire complex function. It follows by Liouville's Theorem (**Theorem 2.8.1**) that $\mathcal{H}$ is a constant function. Clearly, $\mathcal{F}$ and $\mathcal{S}$ are not in general constant functions. So, G has to be constant. Then, the following equivalence chain holds:

$$\begin{aligned} G \text{ is constant} &\iff \vec{U} = c \text{ , for some real constant } c \\ &\iff \nabla f \text{ is a constant vector} \\ &\iff f \text{ is linear} \end{aligned}$$

Hence, since f is linear, M is a plane, as required.

$\square$

7 Conclusion

In this project, general concepts of differential geometry leading up to the definition of a maximal surface in Minkowski 3-space $\mathbf{L}^3$ were introduced. Then, the *Maximal Surface Equation* (equation (5)) was provided.

After this, the following interesting results about maximal surfaces were proven:

- Firslty, it was shown that a surface in $\mathbf{L}^3$ bounded by a closed contour and maximizing the corresponding area functional is necessarily always maximal.

- It was proven that a maximal spacelike surface of rotation is always congruent to either the (x, y)-plane or a catenoid.

- Furthermore, it was shown that a maximal spacelike ruled surface is congruent to either the (x, y)-plane, a heliocoid of the second kind or the conjugate of Ennepper's surface of the second kind.

- Lastly, a Bernstein-type result, namely the *Calabi-Bernstein Theorem* in $\mathbf{L}^3$ was proven. It says that the only entire graphical spacelike surface in $\mathbf{L}^3$ that satisfies the *Maximal Surface Equation* is a plane.

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