Eigenvalues of Laplacian Operator on Bounded domain

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Contents

1. Introduction

1.1 Notations and Definitions

1.1.1 Definition. We define the gradient of the scalar function $f(x_1, x_2, \cdots, x_n)$ by the vector field of the partial derivatives of f and denoted by ∇f i.e

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right)$$

1.1.2 Definition. The divergence of the vector field $F = (F_1, F_2, \cdots, F_n)$, in the coordinates $(x_1, x_2, \cdots, x_n)$ is defined by

$$\text{div}F = \nabla \cdot F = \frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2} + \cdots + \frac{\partial F}{\partial x_n}$$

one of the impotent properties of divergence is the linearity

$$\begin{aligned} \text{div}(F + H) &= \text{div}H + \text{div}H, \\ \text{div}(\phi F) &= \phi(\text{div}F) + \langle \nabla\phi, F\rangle. \end{aligned} \tag{1.1.1}$$

1.1.3 Definition. Let $\phi \in C^k(M)$ $, k \geq 2$,where M is bounded domain in $\mathbb{R}^n$ we define the Laplacian operator of ϕ, by

$$\Delta\phi = \text{div}(\nabla\phi).$$

Taking $\phi,\ \psi \in C^k(M)$ $, k \geq 2$, we get

$$\begin{aligned} \Delta(\phi + \psi) &= \Delta\phi + \Delta\psi, \\ \text{div}(\phi(\nabla\psi)) &= \psi(\Delta\phi) + \langle \nabla\phi, \nabla\psi\rangle, \\ \Delta(\phi\psi) &= \psi(\Delta\phi) + 2\langle \nabla\phi, \nabla\phi\rangle + \phi(\Delta\psi). \end{aligned} \tag{1.1.2}$$

The Divergence Theorem . Let F be continuously differentiable, compactly supported vector field on M, then

$$\int_M (\text{div}F)\,\mathrm{d}\,x = \int_{\partial M} \langle \nu, F\rangle dA. \tag{1.1.3}$$

where ν is outward normal vector and dA is an unit surface on the boundary of M proof of this theorem in "I. Chavel, Eigenvalues in riemannian geometry,"

Green formulas . Applying divergence Theorem into equation (1.1.2), such that at least $\psi \in C^1(M)$, $\phi \in C^2(M)$, we get

$$\int_M \text{div}(\phi(\nabla\psi))\,\mathrm{d}\,x = \int_M (\psi(\Delta\phi) + \langle \nabla\phi, \nabla\psi\rangle)\,\mathrm{d}\,x = \int_{\partial M} \langle \nu, \phi(\nabla\psi)\rangle dA.$$

Which implies that the first Green's formula given by

$$\int_M (\psi(\Delta\phi) + \langle \nabla\phi, \nabla\psi\rangle)\,\mathrm{d}\,x = \int_{\partial M} \phi\langle \nu, \nabla\psi\rangle dA. \tag{1.1.4}$$

Taking $\psi \in C^2(M)$ and replacing ϕ by ψ in equation (1.1.4) we get

$$\int\limits_M (\phi(\Delta\psi) + \langle\nabla\psi, \nabla\psi\rangle)\, \mathrm{d}\, x = \int\limits_{\partial M} \psi\langle\nu, \nabla\phi\rangle dA. \tag{1.1.5}$$

Immediately from equations (1.1.4) and (1.1.5), we deduce the second Green's formula

$$\int\limits_M ((\psi(\Delta\phi) - \phi(\Delta\psi))\, \mathrm{d}\, x = \int\limits_{\partial M} (\phi\langle\nu, \nabla\psi\rangle - \psi\langle\nu, \nabla\phi\rangle)dA. \tag{1.1.6}$$

As special case of the divergence theorem, if the vector field $F = 0$ in the boundary, then

$$\int\limits_M (\mathrm{div} F)\, \mathrm{d}\, x = 0 \tag{1.1.7}$$

Therefore the Green's formulas are given as

$$\int\limits_M (\psi(\Delta\phi) + \langle\nabla\phi, \nabla\psi\rangle)\, \mathrm{d}\, x = 0, \tag{1.1.8}$$

And

$$\int\limits_M ((\psi(\Delta\phi) - \phi(\Delta\psi))\, \mathrm{d}\, x = 0. \tag{1.1.9}$$

1.1.4 Definition.

(a) Let $L^2(M)$ be the space of those measurable functions for which the Lebesgue integral of the square of the absolute value of the functions is finite.

$$\int_M |f|^2\, \mathrm{d}\, x < +\infty$$

For $f,\ h \in M$, the inner product on $L^2(M)$, is defined as

$$(f, g) = \int_M f g\, \mathrm{d}\, x,$$

With the associated norm

$$||f||^2 = \int_M |f|^2\, \mathrm{d}\, x,$$

(b) The space of the measurable vector fields is denote by $\mathcal{L}^2(M)$, and we define the inner product of two vector fields $F,\ G$ on M by

$$(F, G) = \int_M \langle F, G\rangle\, \mathrm{d}\, x,$$

with the associated norm

$$||F||^2 = \int_M |F|^2\, \mathrm{d}\, x.$$

Now let the real valued function $\phi \in C^1(M)$ and the vector field $F \in C^1(M)$ be compactly supported. Then applying equation (1.1.1) into (1.1.3) we get

$$\int_M \{\phi(\text{div}F) + \langle \nabla\phi, F\rangle\} \,\mathrm{d}\,x = 0.$$

The definition above yields that

$$(\nabla\phi, F) = -(\phi, \text{div}F). \tag{1.1.10}$$

1.1.5 Definition. We define the weak derivative of a function $f \in L^2(M)$, by the vector field $Y \in \mathcal{L}^2(M)$ such that if

$$(Y, X) = -(f, \text{div}X).$$

for every compactly supported vector field $X \in C^1(M)$. The weak derivative is denoted $Y = \nabla f$

1.1.6 Remark. If the weak derivative exist, then it is unique.

We define the Sobolev space $\mathcal{H}_1(M)$ of order one to be subspace of $L^2(M)$.

$$\mathcal{H}_1(M) = \{f \in L^2(M) : \nabla f \in \mathcal{L}^2(M)\}$$

With the inner product defined as

$$(f, g)_1 = (f, g) + (\nabla f, \nabla h)$$

The associated norm is

$$||f||_1^2 = ||f||^2 + ||\nabla f||^2$$

also we note that

$$f \in C^\infty(M) : ||f||_1^2 \leq \infty$$

As consequence to the Sobolev space $\mathcal{H}_1(M)$ of order one we defined the symmetric bilinear form (energy integral) as

$$D[f, h] = (\nabla f, \nabla h) \tag{1.1.11}$$

for $f, h \in \mathcal{H}_1(M)$.

Laplace operator is self-adjoint operator . For $\phi, \psi \in L^2(M)$, using integration by part twice we get

$$\langle \phi, \triangle\psi\rangle = \int_M \phi \triangle \psi \,\mathrm{d}\,x = \underbrace{\oint_{\partial M} \phi(\triangledown\psi)\cdot v \,\mathrm{d}\,x}_{=0} - \int_M \triangledown\phi \cdot \triangledown\psi \,\mathrm{d}\,x = -\int_M \triangledown\phi \cdot \triangledown\psi \,\mathrm{d}\,x$$

But

$$-\int_M \triangledown\phi \cdot \triangledown\psi \,\mathrm{d}\,x = \underbrace{-\oint_{\partial M} \psi(\triangledown\phi)\cdot v \,\mathrm{d}\,x}_{=0} + \int_M \psi \triangle \phi \,\mathrm{d}\,x = \langle \psi, \triangle\phi\rangle$$

Therefore

$$\langle \phi, \triangle\psi\rangle = \langle \triangle\phi, \psi\rangle$$

Dirichlet boundary condition.

$$-\Delta\phi = f \text{ in } M \tag{1.1.12}$$
$$\phi = 0 \text{ on } \partial M \tag{1.1.13}$$

this mean that ϕ is unique solution of the variation problem

$$\begin{cases} \phi \in \mathcal{H}_1(M), & \forall\psi \in \mathcal{H}_1(M), \\ \int_M(\nabla\phi\cdot\nabla\psi)\,\mathrm{d}\,x = \int_M(f\psi)\,\mathrm{d}\,x \end{cases} \tag{1.1.14}$$

Existence and uniqueness of a solution for (1.1.14) follows from the Lax-Milgram Theorem, and poincaré inequality. We denote by

$$\begin{cases} A^D : L^2(M) \longrightarrow \mathcal{H}_1(M), \\ A^D : f \longmapsto \phi. \quad \text{where } \phi \text{ is solution of (1.1.14)} \end{cases} \tag{1.1.15}$$

Neumann boundary condition. . We consider ϕ as the solution of the Neumann problem and let f function in $L^2(M)$

$$-\Delta\phi = f \quad \text{in} \quad M \tag{1.1.16}$$
$$\nabla\phi\cdot\nu = 0 \quad \text{on} \quad \partial M$$

where ν is toward unit normal vector on M . this mean that ϕ is unique solution of the variation problem

$$\begin{cases} \phi \in \mathcal{H}_1(M), & \forall\psi \in \mathcal{H}_1(M), \\ \int_M(\nabla\phi\cdot\nabla\psi)\,\mathrm{d}\,x = \int_M(f\psi)\,\mathrm{d}\,x \end{cases} \tag{1.1.17}$$

Existence and uniqueness of a solution for (1.1.17) follows from the Lax-Milgram Theorem, and poincaré inequality. We denote by

$$\begin{cases} A^N : L^2(M) \longrightarrow \mathcal{H}_1(M), \\ A^N : f \longmapsto \phi. \quad \text{where } \phi \text{ is solution of (1.1.17)} \end{cases} \tag{1.1.18}$$

1.2 Abstract of the Spectral Theory

Let H be a Hilbert space endowed with inner product$(\bullet,\bullet)$ and let the operator T be linear continuous map from $H \longmapsto H$. We say that

(i) T is positive if $\forall\phi \in H \qquad (T\phi,\phi) \geq 0$

(ii) T is self-adjoint if $\forall\phi,\psi \in H \qquad (T\phi,\psi) = (\psi, T\phi)$

(ii) T is compact if the image of any bounded set is relatively compact (i.e has compact closure in H)

1.2.1 Theorem. *Let $A : H \longrightarrow H$ be compact, self-adjoint and positive operator on Hilbert space H. Then there is finite or infinite sequence $\{\lambda_n\}_{n=0}^N$ or $\{\lambda_n\}_{n=0}^\infty$ of real eigenvalues $\lambda_n \neq 0$ and corresponding orthonormal sequence $\{e_n\}_{n=0}^N$ or $\{e_n\}_{n=0}^\infty$ respectively in H such that:*

(i) $A(e_n) = \lambda_n e_n \qquad \forall n$

(ii) $(\ker(A))^\perp = \mathit{span}\Big(\{e_n\}_{n=0}^N \ \mathit{or}\, \{e_n\}_{n=0}^\infty\Big)$

(iii) if $N = \infty$, then $\lim\limits_{n\longrightarrow\infty} \lambda_n = 0$.

Application to the Laplace operator.

Dirichlet boundary condition . Applying the above theorem to $H = L^2(M)$ with the operator

$$\begin{cases} A^D : L^2(M) \longrightarrow \mathcal{H}_1(M), \\ A^D : f \longmapsto \phi. \end{cases}$$

(i) A^D is positive, let $\phi \in L^2(M)$. and $\phi = A^D(\lambda\phi)$ such that ϕ is the solution of (1.1.14). We get

$$(\lambda\phi, A^D(\lambda\phi)) = \int_M (\lambda\phi, \phi)\,\mathrm{d}\,x = \int_M |\nabla\phi|^2\,\mathrm{d}\,x \geq 0.$$

(ii) A^D is self-adjoint, let $\phi, \psi \in L^2(M)$ and $\phi = A^D(f)$, $\psi = A^D(\lambda\psi)$

$$(\lambda\phi, A^D(\lambda\psi)) = \int_M (\lambda\phi, \psi)\,\mathrm{d}\,x = \int_M \nabla\phi \cdot \nabla\psi\,\mathrm{d}\,x = \int_M (\lambda\psi, \phi)\,\mathrm{d}\,x = (\lambda\psi, A^D(\lambda\phi))$$

(iii) the compactness follows from the following theorem

1.2.2 Theorem. *(I) For any bounded open set M, the embedding $\mathcal{H}_1(M) \longmapsto L^2(M)$ is compact.*

(II) If M is a bounded open set, with the Lipschitz boundary the embedding $\mathcal{H}_1(M) \longmapsto L^2(M)$ is compact.

As consequence of the spectral theorem there exists a Hilbert basis $\{\phi_n\}$ of $L^2(M)$, and a positive sequence $\{\nu_n\}$ converging to zero, such that for every n,

$$A^D\phi_n = \nu_n\phi_n$$

Since ν_n is positive for all n and the operator A^D is positive, then

$$(\phi_n, A^D\phi_n) = \nu_n\|\phi_n\|^2 \geq 0$$

Now from equation(1.1.17) we have

$$\nu_n\left(\int_M (\nabla\phi_n \cdot \nabla\psi)\,\mathrm{d}\,x\right) = \int_M \phi_n\psi\,\mathrm{d}\,x$$

Therefore $-\Delta\phi_n = \dfrac{1}{\nu_n}\phi_n$ which implies that $\lambda_n = \dfrac{1}{\nu_n}$. Which implies that there exist orthonormal basis $\{\phi_n\}$ of $L^2(M)$, such that $-\Delta\phi_n = \lambda_n\phi_n$

1.3 Rayleigh Quotient

For the function $\phi \in C^2(M)$, we its the define Rayleigh quotient as

$$\mathcal{R}(\phi) = \frac{D[\phi, \phi]}{\|\phi\|^2}$$

To connect the Rayleigh quotient with the eigenvalue problem, consider $R(\phi+\epsilon\psi)$ for $\psi \in C^2(M)$ and $\phi \in C^2(M)$. Then find the critical point of

$$f(\epsilon) = \frac{D[\phi+\epsilon\psi, \phi+\epsilon\psi]}{\|\phi+\epsilon\psi\|^2} \tag{1.3.1}$$

at $\epsilon \longrightarrow 0$ where ϵ is any constant. Using equation (1.1.11) into (1.3.1)

$$\begin{aligned} f(\epsilon) &= \frac{(\nabla(\phi+\epsilon\psi), \nabla(\phi+\epsilon\psi))}{\|\phi+\epsilon\psi\|^2} \\ &= \frac{\int_M (\nabla(\phi+\epsilon\psi))^2 \,\mathrm{d}\,x}{\int_M (\phi+\epsilon\psi)^2 \,\mathrm{d}\,x} \end{aligned}$$

Then the derivative with respect to ϵ equal to zero at $\epsilon = 0$

$$0 = f'(0) = \frac{2\left(\int_M (\phi)^2 \,\mathrm{d}\,x\right)\left(\int_M (\nabla(\phi)\cdot\nabla(\psi)) \,\mathrm{d}\,x\right) - 2\left(\int_M (\nabla\phi)^2 \,\mathrm{d}\,x\right)\left(\int_M (\phi\psi) \,\mathrm{d}\,x\right)}{\left(\int_M (\phi)^2 \,\mathrm{d}\,x\right)^2}$$

Arranging the equation above we get

$$\frac{\left(\int_M (\nabla(\phi)\cdot\nabla(\psi)) \,\mathrm{d}\,x\right)}{\left(\int_M (\phi)^2 \,\mathrm{d}\,x\right)} = \frac{\left(\int_M (\nabla\phi)^2 \,\mathrm{d}\,x\right)}{\left(\int_M (\phi)^2 \,\mathrm{d}\,x\right)^2}\left(\int_M (\phi\psi) \,\mathrm{d}\,x\right)$$

Therefore

$$\begin{aligned} \int_M (\nabla(\phi)\cdot\nabla(\psi) \,\mathrm{d}\,x &= \frac{\left(\int_M (\nabla\phi)^2 \,\mathrm{d}\,x\right)}{\left(\int_M (\phi)^2 \,\mathrm{d}\,x\right)}\left(\int_M \phi\psi) \,\mathrm{d}\,x\right) \\ &= \mathcal{R}(\phi)\left(\int_M (\phi\psi) \,\mathrm{d}\,x\right) \end{aligned}$$

Now using equation (1.1.5), we get

$$\mathcal{R}(\phi)\left(\int_M (\phi\psi) \,\mathrm{d}\,x\right) = \int_M (\nabla(\phi)\cdot\nabla(\psi)) \,\mathrm{d}\,x = -\int_M (\Delta(\phi)\psi) \,\mathrm{d}\,x$$

Therefore

$$\int_M (\Delta\phi + \mathcal{R}(\phi)\phi)\psi \,\mathrm{d}\,x = 0. \tag{1.3.2}$$

which gives us $\Delta\phi + \mathcal{R}(\phi)\phi = 0$, because equation (1.3.2) is satisfied $\forall \psi \in C^2(M)$. Thus any critical point ϕ is an eigenfunction of $-\Delta$ with the eigenvalue $\mathcal{R}(\phi)$.

2. The Second Chapter

2.1 Theorems on the eigenvalue problems

Neumann eigenvalue problem. For $\partial M \neq \emptyset$ and $\overline{M}$ compact and connected, find all real numbers λ for which there exists a non-trivial solution $\phi \in C^2(M) \cap C^1(\overline{M})$ to

$$\triangle\phi + \lambda\phi = 0, \tag{2.1.1}$$

which satisfy the boundary condition

$$v\phi = 0, \qquad \text{on } \partial M.$$

Here v is an outward unit normal vector on ∂M

Dirichlet eigenvalue problem . For $\partial M \neq \emptyset$ and $\overline{M}$ compact and connected, find all real numbers λ for which there exists a non-trivial solution $\phi \in C^2(M) \cap C^0(\overline{M})$ to

$$\triangle\phi + \lambda\phi = 0,$$

satisfying the boundary condition

$$\phi = 0, \qquad \text{on } \partial M.$$

Orthogonality of the eigenfunctions. Let ν, μ be two eigenvalues of respective eigenfunctions ϕ, ψ such that $\nu \neq \mu$ Then ϕ and ψ are orthogonal to each other, i.e $\langle\phi, \psi\rangle = 0$. Since Laplace operator is self-adjoint operator, then we have

$$\langle\phi, \triangle\psi\rangle = \langle\psi, \triangle\phi\rangle$$

we deduce that

$$\mu\langle\phi, \psi\rangle = \nu\langle\psi, \phi\rangle$$

Which implies that

$$(\mu - \nu)\langle\phi, \psi\rangle = 0$$

Which implies the claim.

2.1.1 Remark. The first eigenvalue of Neumann eigenvalue problem is $\lambda_1 = 0$ corresponding to to constant eigenfunction.

2.1.2 Theorem. *From equation (2.1.1), the set of the eigenvalues consist of the sequence*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \uparrow +\infty$$

and each associated eigenspace is finite dimensional. $L^2(M)$ is the direct sum of the all eigenspaces. Furthermore, each eigenfunction is C^∞ on $\overline{M}$.

Proof. First, we show that the sequence of the eigenvalues are non-negative, since the eigenfunction ϕ in $C^2(M) \cap C^0(\overline{M})$ (Dirichlet), or ϕ in $C^2(M) \cap C^1(\overline{M})$ (Neumann), using Green formula, i.e

$$-\int_M \phi \bigtriangleup \phi \,\mathrm{d}\,x = \int_M \langle \nabla\phi, \nabla\phi \rangle \,\mathrm{d}\,x$$

This implies that

$$\lambda \int_M \phi.\phi \,\mathrm{d}\,x = \int_M |\nabla\phi|^2 \,\mathrm{d}\,x$$

Therefore

$$\lambda = ||\phi||^{-2} \int_M |\nabla\phi|^2 \,\mathrm{d}\,x \geq 0. \tag{2.1.2}$$

Since the integration (2.1.2) is zero only if ϕ is constant. From the remark above we conclude that $\lambda_1 \geq 0$. Now we can list the set of the eigenvalues, considering the multiplicity of the eigenvalues, in increasing order.

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \uparrow +\infty$$

Second we show that $L^2(M)$ is the direct sum of all eigenspaces. From the spectrum theorem that orthonormal sequence $\{\phi_n\}_{n=0}^{\infty}$ generated the whole space, which means that for all functions $f \in L^2(M)$, Satisfy the following

$$f = \sum_{j=1}^{\infty} (f, \phi_j)\phi_j, \qquad \text{in } L^2(M),$$

With corresponding norm

$$||f||^2 = \sum_{j=1}^{\infty} (f, \phi_j)^2$$

□

2.1.3 Theorem. (Rayleigh's Theorem).*For function $f \in C^\infty(M)\backslash\{0\}$, let $\lambda_1, \lambda_2, \cdots$ be the eigenvalues of Δ, and first eigenvalue λ_1, we have*

$$\lambda_1 = \inf D[f, f]/\|f\|^2$$

if f is eigenfunction of λ_1, we get exactly λ_1. Further more, if we let

$$(f, \phi_1) = \cdots = (f, \phi_{k-1}) = 0$$

for complete orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ of $L^2(M)$, where ϕ_n is an eigenfunction of λ_n we have

$$\lambda_k = \inf D[f, f]/\|f\|^2$$

if f is eigenfunction of λ_k, we get exactly λ_k.

Proof. From equation (**??**) if ϕ is an eigenfunction and $f \in C^{\infty}(M)$, then

$$D[f, \phi] = \lambda(f, \phi)$$

But from the spectral Theorem we have that, any function $f \in C^{\infty}(M)$, can be written as

$$f = \sum_{j=1}^{\infty} \alpha_j \phi_j$$

and where α_j can be represented as $\alpha_j = (f, \phi_j)$. Therefore

$$f = \sum_{j=k}^{\infty} \alpha_j \phi_j$$

with the associated norm

$$\|f\|^2 = \sum_{j=k}^{\infty} \alpha_j^2$$

Since the symmetric bilinear form is defined as inner product of two vector fields, using this linearity we can proceed as follows

$$\begin{aligned}
0 \leq D &\left[f - \sum_{j=k}^{\infty} \alpha_j \phi_j, f = \sum_{j=k}^{\infty} \alpha_j \phi_j \right] \\
&= D[f, f] - 2D \left[f, \sum_{j=k}^{\infty} \alpha_j \phi_j \right] + D \left[\sum_{j=k}^{\infty} \alpha_j \phi_j, \sum_{j=k}^{\infty} \alpha_i \phi_i \right] \\
&= D[f, f] - 2 \sum_{j=k}^{\infty} \alpha_j D\left[f, \phi_j\right] + \sum_{i,j=k}^{\infty} \alpha_j \alpha_i D\left[\phi_j, \phi_i\right] \\
&= D[f, f] + 2 \sum_{j=k}^{\infty} \alpha_j (f, \Delta \phi_j) - \sum_{i,j=k}^{\infty} \alpha_j \alpha_i (\phi_j, \Delta \phi_i) \\
&= D[f, f] - 2 \sum_{j=k}^{\infty} \lambda_j \alpha_j \underbrace{(f, \phi_j)}_{=\alpha_j} + \sum_{i,j=k}^{\infty} \lambda_j \alpha_j \alpha_i \underbrace{(\phi_j, \phi_i)}_{=\delta_{i,j}} \\
&= D[f, f] - 2 \sum_{j=k}^{\infty} \lambda_j \alpha_j^2 + \sum_{i,j=k}^{\infty} \lambda_j \alpha_j^2 \\
&= D[f, f] - \sum_{j=k}^{\infty} \lambda_j \alpha_j^2.
\end{aligned}$$

We deduce that

$$D[f, f] \geq \sum_{j=k}^{\infty} \lambda_j \alpha_j^2$$

Since

$$D[f,f] = (\nabla f, \nabla f) = \int\limits_M |\nabla f|^2 \,\mathrm{d}\,x < \infty.$$

where $\sum\limits_{j=k}^{\infty} \lambda_j \alpha_j^2 < \infty$. Since the eigenvalues are ordered as increasing sequence, λ_k is smallest eigenvalue, hence

$$D[f,f] \geq \sum_{j=k}^{\infty} \lambda_j \alpha_j^2 \geq \lambda_k \sum_{j=k}^{\infty} \alpha_j^2 = \lambda_k \|f\|^2.$$

If f is eigenfunction of λ_k then

$$\|f\|^2 = \sum_{j=k}^{\infty} \alpha_j^2 = \alpha_k = 1.$$

Therefore

$$D[f,f] = (\nabla f, \nabla f) = -(f, \Delta f) = \lambda_k \|f\|^2,$$

which implies the claim. □

2.1.4 Theorem (Max-Min Theorem)**.** . *Suppose that*

$$\mu = \inf D[f,f]/\|f\|^2$$

and define the space $E(M) \subset \mathcal{H}_1$ as the set of functions orthogonal to $\{v_1, v_2, \cdots v_{k-1}\}$ in $\mathcal{H}_1$ Then

$$\mu \leq \lambda_k$$

Proof. Suppose that

$$f = \sum_{j=1}^{k} \alpha_j \phi_j$$

is orthogonal to $\{v_1, v_2, \cdots v_{k-1}\}$ in $\mathcal{H}_1$, and $\{\phi_j\}_{\mathrm{j}=1}^{k}$ is orthonormal sequence, where ϕ_j an eigenfunction of λ_j, this assumption gives us

$$\sum_{j=1}^{k} \alpha_j (\phi_j, v_i) = 0 \quad i = 1, 2, \cdots, k-1. \tag{2.1.3}$$

Thus(2.2.1) is a system of $k-1$ equations in k variables α_j, $\quad j = 1, 2, \cdots, k$, then there exist at least one solution $\{\alpha_j\} \neq 0$ of the system. Therefore

$$\begin{aligned} D[f,f] &= D\left[\sum_{j=1}^{k} \alpha_j \phi_j, \sum_{i=1}^{k} \alpha_i \phi_i\right] = \sum_{i,j=1}^{k} \alpha_i \alpha_j D\left[\phi_j, \phi_i\right] \\ &= -\sum_{i,j=1}^{k} \alpha_i \alpha_j (\phi_j, \Delta \phi_i) = \sum_{j=1}^{k} \lambda_j \alpha_j^2 \leq \lambda_k \sum_{j=1}^{k} \alpha_j^2 = \lambda_k \|f\|^2 \end{aligned}$$

Which implies that

$$\mu = \inf D[f,f]/\|f\|^2 \leq \lambda_k.$$

□

2.2 Domain Monotonicity of the eigenvalues

Dirichlet eigenvalue problem. Suppose that we divide our domain M, into a sequence of sub-domains $\{\Omega_i\}_{i=1}^m$, such that $\Omega_i \cap \Omega_j = \emptyset, i \neq j$ and consider Dirichlet eigenvalue problem on each of this sub-domains. Arrange all the eigenvalues of Ω_i as $\nu_1 \leq \nu_2 \leq, \cdots$, and let the eigenvalues of the whole domain M, under Dirichlet condition are $\lambda_1 \leq \lambda_2 \leq, \cdots$. Then

$$\lambda_k \leq \nu_k \qquad \forall k$$

Proof. Let $\{\psi_j\}_{j=1}^k$ be the orthonormal sequence in $L^2(M)$ of the eigenfunctions of the Dirichlet eigenvalue problem in M and let's take ψ_j to be an eigenfunction of the eigenvalue ν_j in particular sub-domain, in M, and zero outside. Therefore the function

$$f = \sum_{j=1}^{k} \alpha_j \psi_j$$

can be chosen to be orthogonal to $\{\phi_j\}_{j=1}^{k-1}$. If the system

$$\sum_{j=1}^{k} \alpha_j (\phi_j, \psi_i) = 0 \quad i = 1, 2, \cdots, k-1. \tag{2.2.1}$$

has a non-zero solution $\{\alpha_j\} \neq 0$, and hence form Mix-Min Theorem we have

$$\lambda_k \geq D[f, f]/\|f\|^2$$

But

$$D[f,f] = D\left[\sum_{j=1}^{k} \alpha_j \psi_j, \sum_{i=1}^{k} \alpha_i \psi_i\right] = \sum_{i,j=1}^{k} \alpha_i \alpha_j D\left[\psi_j, \psi_i\right]$$
$$= -\sum_{i,j=1}^{k} \alpha_i \alpha_j (\psi_j, \Delta\psi_i) = \sum_{j=1}^{k} \mu_j \alpha_j^2 \leq \nu_k \sum_{j=1}^{k} \alpha_j^2 = \nu_k \|f\|^2$$

Therefore

$$\lambda_k \leq \nu_k \qquad \forall k$$

□

Neumann eigenvalue problem. Suppose we divide our manifold M to complete partitions $\{\Omega_i\}_{i=1}^m$, i.e

$$\overline{M} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \cdots \cup \overline{\Omega}_m.$$

and satisfy the Neumann eigenvalue problem on each of this partitions. Arrange all the eigenvalues of $\{\Omega_i\}_{i=1}^m$ as $\mu_1 \leq \mu_2 \leq, \cdots$. Then we have

$$\mu_k \leq \lambda_k \qquad \forall k$$

Proof. Let $\{\psi_j\}_{j=1}^{k-1}$ be orthonormal sequence of the eigenfunctions in $\mathcal{H}_1(M)$, where that ψ_j is the solution of the Neumann eigenvalue problem, and let's take ψ_j to be an eigenfunction of the eigenvalue μ_j in Ω_i,and zero otherwise. Thus for any function $f \in \mathcal{H}_1(M)$, it is define on all Ω_i. Also by using Max-Min Theorem, we can find function f, s.t

$$f = \sum_{j=1}^{k} \alpha_j \phi_j$$

Orthogonal to $\{\psi_j\}_{j=1}^{k-1}$ (ϕ_j is the eigenfunction for λ_j on M). Therefore

$$\begin{aligned} D[f,f] &= D\left[\sum_{j=1}^{k} \alpha_j \phi_j, \sum_{i=1}^{k} \alpha_i \phi_i\right] = \sum_{i,j=1}^{k} \alpha_i \alpha_j D\left[\phi_j, \phi_i\right] \\ &= -\sum_{i,j=1}^{k} \alpha_i \alpha_j (\phi_j, \Delta\phi_i) = \sum_{j=1}^{k} \lambda_j \alpha_j^2 \leq \lambda_k \sum_{j=1}^{k} \alpha_j^2 = \lambda_k \|f\|^2 \end{aligned}$$

Now if we let f to be orthogonal to $\{\psi_j\}_{j=1}^{k-1}$ in $L^2(M)$, then $f = \sum\limits_{j=1}^{k} \alpha_j \psi_j$ and therefore

$$D[f,f] = \int_M \|\nabla f\|^2 \,\mathrm{d}\,x = \sum_{i=1}^{m} \int_{\Omega_i} \|\nabla f\|^2 \,\mathrm{d}\,x$$

But

$$\int_{\Omega_i} \|\nabla f\|^2 \,\mathrm{d}\,x \geq \mu_k \int_{\Omega_i} |f|^2 \,\mathrm{d}\,x.$$

Thus

$$D[f,f] \geq \mu_k \sum_{i=1}^{m} \int_{\Omega_i} f^2 \,\mathrm{d}\,x = \mu_k \|f\|^2$$

Which implies that

$$\mu_k \leq \lambda_k.$$

□

3. Third Chapter

3.1 Dirichlet eigenvalue problem on a rectangle

Find the all real numbers λ, of non-trivial solution of equation (2.1.1), under Dirichlet boundary conditions, on the rectangle

$$\{0 \leq x \leq a, \;\; 0 \leq y \leq b\}.$$

We can rewrite equation (2.1.1) as follows

$$\begin{aligned} &\phi_{xx} + \phi_{yy} + \lambda\phi = 0 \qquad 0 \leq x \leq a, \;\; 0 \leq y \leq b, \\ &\phi(x,0) = \phi(x,a) = 0 \qquad 0 \leq x \leq a, \\ &\phi(0,y) = \phi(b,y) = 0 \qquad 0 \leq y \leq b, \end{aligned} \tag{3.1.1}$$

By using separation of variables, let

$$\phi(x,y) = X(x) \cdot Y(y)$$

now equation (3.1.1) becomes

$$X_{xx}(x) \cdot Y(y) + X(x) \cdot Y_{yy}(y) + \lambda X(x) \cdot Y(y) = 0$$

we can write it as

$$\frac{X_{xx}(x) + \lambda X(x)}{X(x)} = \frac{-Y_{yy}(y)}{Y(y)}$$

Since the right hand side depends only one y and the left hand side depends only on x. hence both sides are equal to some constant,

$$\frac{X_{xx}(x) + \lambda X(x)}{X(x)} = \frac{-Y_{yy}(y)}{Y(y)} = \mu.$$

We get two ordinary differential equations

$$X_{xx}(x) + (\lambda - \mu)X(x) = 0 \qquad X(0) = X(a) = 0, \tag{3.1.2}$$

$$Y_{yy}(y) + \mu Y(y) = 0 \qquad Y(0) = Y(b) = 0, \tag{3.1.3}$$

Solutions of equation (3.1.3), are given by

$$Y(y) = A\sin\sqrt{\mu}y + B\cos\sqrt{\mu}y$$

Substituting the boundary conditions we get $B = 0$ and

$$\begin{cases} Y(y) = \sin\sqrt{\mu}y \\ \mu = \dfrac{m^2\pi^2}{b^2} \qquad \text{for } \; m = 1, 2, \cdots \end{cases} \tag{3.1.4}$$

By doing the same steps, we find the solution of equation (3.1.2), with $\alpha = \lambda - \mu$, such that

$$\begin{cases} X(x) = \sin\sqrt{\alpha}x \\ \alpha = \dfrac{n^2\pi^2}{n^2} \qquad \text{for } \; m = 1, 2, \cdots \end{cases} \tag{3.1.5}$$

Therefore from equations (3.1.2) and (3.1.3) the solution of equation (3.1.1) obtained by,

$$\begin{cases} \phi_{n,m}(x,y) = X(x)\cdot Y(y) = \sin\frac{n\pi}{a}x\cdot\sin\frac{m\pi}{b}y \\ \lambda_{n,m} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \qquad \text{for } \ n,m = 1,2,\cdots \end{cases} \tag{3.1.6}$$

Since $\sin\frac{n\pi}{a}x\cdot\sin\frac{m\pi}{b}y$ orthonormal basis of $L^2(M)$, then using Fourier series we can write $f \in L^2(M)$, as

$$f(x,y) = \sum_{n,m=1}^{\infty} C_{n,m}\sin\frac{n\pi}{a}x\cdot\sin\frac{m\pi}{b}y$$

Therefore

$$||f||^2 = \sum_{n,m=1}^{\infty} C_{n,m}^2$$

We Generalize the example on rectangle in n-dimension

$$M = \{0 \le x_1 \le \alpha_1,\ \ 0 \le x_2 \le \alpha_2, \cdots, 0 \le x_n \le \alpha_n\}$$

Here equation (2.1.1) can be written as

$$\begin{aligned}
&\phi_{x_1x_1} + \phi_{x_2x_2} + \cdots + \phi_{x_nx_n} + \lambda\phi = 0 \qquad \text{on } M, \\
&\phi(x_1,0,0,\cdots,0) = \phi(x_1,\alpha_2,\cdots,\alpha_n) = 0 \qquad 0 \le x_1 \le \alpha_1, \\
&\phi(0,x_2,0,\cdots,0) = \phi(\alpha_1,x_2,\alpha_3,\cdots,\alpha_n) = 0 \qquad 0 \le x_2 \le \alpha_2, \\
&\vdots \\
&\phi(0,0,\cdots,x_n) = \phi(\alpha_1,\alpha_2,\cdots,x_n) = 0 \qquad 0 \le x_n \le \alpha_n
\end{aligned} \tag{3.1.7}$$

We trying to solve equation (3.1.7), by using septation of variables, then we let

$$\phi(x_1,\cdots,x_2) = X_1(x_1)\cdot\cdots\cdot X_n(x_n) \tag{3.1.8}$$

We obtain equation (3.1.7) as

$$X_{1x_1x_1}(x_1)\cdot X_2(x_2)\cdots X_n(x_n) + \cdots + X_1(x_1)\cdot X_2(x_2)\cdots X_{nx_nx_n}(x_n) + \lambda X_1(x_1)\cdot\cdots\cdot X_n(x_n) = 0$$

Divide the whole terms in above equation by $X_1(x_1)\cdot\cdots\cdot X_n(x_n)$ we get

$$\frac{X_{1x_1x_1}(x_1) + \lambda X_1(x_1)}{X_1(x_1)} + \frac{X_{2x_2x_2}(x_2)}{X_2(x_2)} + \cdots + \frac{X_{nx_nx_n}(x_n)}{X_n(x_n)} = 0 \tag{3.1.9}$$

Since the term i depend only on the variable x_i for $1 \le i \le n$. Let's take a line with constant the variables x_i for $1 \le i \le n-1$, changing only x_n. This will change only the last term, Keeping the others terms constant. Thus the term n is equal to some constant. Namely

$$\frac{X_{nx_nx_n}(x_n)}{X_n(x_n)} = \mu_n$$

Using the boundary conditions for x_n, we get

$$\mu_n = \frac{k_n^2 \pi^2}{\alpha_n^2}.$$

With

$$X_n(x_n) = \sin \frac{k_n \pi x_n}{\alpha_n}.$$

doing the same argument on the term i taking a line with the constant in the all directions except along x_i, using it is boundary conditions, we get

$$\mu_i = \frac{k_i^2 \pi^2}{\alpha_i^2}.$$

With

$$X_i(x_i) = \sin \frac{k_i \pi x_i}{\alpha_i}.$$

Inserting the whole values, $\mu_i = \frac{k_i^2 \pi^2}{\alpha_i^2}, \quad 2 \leq i \leq n$ back to equation(3.1.9), we get

$$\lambda = \frac{k_1^2 \pi^2}{\alpha_1^2} + \frac{k_2^2 \pi^2}{\alpha_2^2} + \cdots + \frac{k_n^2 \pi^2}{\alpha_n^2}.$$

With

$$X_1(x_1) = \sin \frac{k_1 \pi x_1}{\alpha_1}.$$

Hence equation (3.1.8) it can be obtain as

$$\begin{cases} \phi_{k_i}(x_1, \cdots, x_n) = \sin \frac{k_1 \pi x_1}{\alpha_1} \cdot \sin \frac{k_2 \pi x_2}{\alpha_2} \cdots \sin \frac{k_n \pi x_n}{\alpha_n} \\ \lambda_{k_i} = \frac{k_1^2 \pi^2}{\alpha_1^2} + \frac{k_2^2 \pi^2}{\alpha_2^2} + \cdots + \frac{k_n^2 \pi^2}{\alpha_n^2}. \\ 1 \leq i \leq n \end{cases}$$

Since the ϕ_{k_i} for $1 \leq i \leq n$ are orthonormal bases of $L^2(M)$, using Fourier series we obtain

$$f(x_1, \cdots, x_n) = \sum_{k_i=1}^{\infty} C_{k_i} \sin \frac{k_1 \pi_1^x}{\alpha_n} \cdot \sin \frac{k_2 \pi_2^x}{\alpha_2} \cdots \sin \frac{k_n \pi_n^x}{\alpha_n} \quad , 1 \leq i \leq n$$

With associated norm

$$||f||^2 = \sum_{k_i=1}^{\infty} C_{k_i}^2, \qquad 1 \leq i \leq n$$

3.2 Neumann eigenvalue problem on a rectangle

Considering Neumann boundary condition, we have

$$\begin{aligned} &\phi_{xx} + \phi_{yy} + \lambda\phi = 0 \qquad 0 \le x \le a, \;\; 0 \le y \le b, \\ &\phi_y(x,0) = \phi_y(x,a) = 0 \qquad 0 \le x \le a, \\ &\phi_x(0,y) = \phi_x(b,y) = 0 \qquad 0 \le y \le b, \end{aligned} \tag{3.2.1}$$

By doing the same steps in the Dirichlet boundary condition, is easy to see that

$$\begin{cases} \phi_{n,m}(x,y) = X(x)\cdot Y(y) = \cos\dfrac{n\pi}{a}x \cdot \cos\dfrac{m\pi}{b}y \\ \lambda_{n,m} = \dfrac{n^2\pi^2}{a^2} + \dfrac{m^2\pi^2}{b^2} \qquad \text{for } \; n, m = 1, 2, \cdots \end{cases} \tag{3.2.2}$$

Using the same way we can generalise the case of, n-dimension, to get the following solution

$$\begin{cases} \phi_{k_i}(x_1, \cdots, x_n) = \cos\dfrac{k_1\pi x_1}{\alpha_1} \cdot \cos\dfrac{k_2\pi x_2}{\alpha_2} \cdots \cos\dfrac{k_n\pi x_n}{\alpha_n} \\ \lambda_{k_i} = \dfrac{k_1^2\pi^2}{\alpha_1^2} + \dfrac{k_2^2\pi^2}{\alpha_2^2} + \cdots + \dfrac{k_n^2\pi^2}{\alpha_n^2}. \\ 1 \le i \le n \end{cases}$$

3.3 Dirichlet eigenvalue problem on a Disk

Find the all real numbers λ, of non-trivial solution of equation (2.1.1), under Dirichlet boundary conditions, on the rectangle

$$M = \{0 \le r < a, \;\; 0 \le \theta \le 2\pi\}.$$

We can rewrite equation (2.1.1) as follows

$$\begin{aligned} &\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = -\lambda\phi \qquad 0 \le r < a, \;\; 0 \le \theta \le 2\pi, \\ &\phi(a,\theta) = 0 \qquad 0 \le \theta \le 2\pi. \end{aligned} \tag{3.3.1}$$

By using separation of variables, let

$$\phi(r,\theta) = R(r)\cdot\Phi(\theta)$$

now equation (3.3.1) becomes

$$\Phi\left(R_{rr} + \frac{1}{r}R_r\right) + \frac{1}{r^2}R\Phi_{\theta\theta} = -R\lambda\Phi$$

rearranging to separate the variables, we have

$$\frac{r^2\left(R_{rr} + \dfrac{1}{r}R_r + \lambda R\right)}{R} = -\frac{\Phi_{\theta\theta}}{\Phi}$$

Since the right hand side depends only one θ and the left hand side depends only on r. hence both sides are equal to some constant. Therefore we set the following ODEs

$$R_{rr} + \frac{1}{r}R_r + (\lambda - \frac{\mu}{r^2})R = 0 \qquad R(a) = 0,\ \ 0 \le r < a, \tag{3.3.2}$$

$$\Phi_{\theta\theta} + \mu\Phi = 0 \qquad \Phi(0) = \Phi(2\pi),\ \ 0 \le \theta \le 2\pi, \tag{3.3.3}$$

Solutions of equation (3.3.3), are given by

$$\Phi(\theta) = A\sin\sqrt{\mu}\theta + B\cos\sqrt{\mu}\theta$$

The boundary condition gives us $\sqrt{\mu} = n,\ \ n \in \mathbb{N}$. Thus we can write

$$\Phi_n(\theta) = A_n \sin n\theta + B_n \cos n\theta,\ \ n \in \mathbb{N}$$

Now equation (3.3.2) becomes

$$R_{rr} + \frac{1}{r}R_r + (\lambda - \frac{n^2}{r^2})R = 0 \qquad R(a) = 0,\ \ 0 \le r < a \tag{3.3.4}$$

using change of variables $x = \sqrt{\lambda}r$, we get

$$\frac{\mathrm{d}\,R}{\mathrm{dr}} = \frac{\mathrm{d}\,R}{\mathrm{dx}} \cdot \frac{\mathrm{d}\,x}{\mathrm{dr}} = \sqrt{\lambda}\frac{\mathrm{d}\,R}{\mathrm{dx}}, \qquad \frac{\mathrm{d}^2\,R}{\mathrm{dr}^2} = \lambda\frac{\mathrm{d}^2\,R}{\mathrm{dx}^2}$$

Thus equation (3.3.4), becomes

$$x^2R_{xx} + xR_x + (x^2 - n^2)R = 0 \qquad R(\sqrt{\lambda}a) = 0,\ \ 0 \le x < \sqrt{\lambda}a$$

This is Bessel differential equation, more information see[A treatise on the theory of Bessel functions] which has the solution $J_n(x)$, where $J_n(x)$ is the Bessel function of the first kind of order n is defined by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}$$

Then the solution of equation (3.3.2), where $x = \sqrt{\lambda}r$ is given by $R(r) = J_n(\sqrt{\lambda}r)$. Therefore

$$\phi(r,\theta)_n = J_n(\sqrt{\lambda}r) \cdot A_n \sin n\theta + B_n \cos n\theta,\ \ n \in \mathbb{N}$$

Since from Dirichlet boundary condition we have $\phi(a,\theta) = 0$, this implies that $J_n(\sqrt{\lambda}a) = 0\ \forall n \in \mathbb{N}$, which deduce that $\sqrt{\lambda}a$ is solution of the Bessel function. But $J_n(x)$ has infinitely many positive solutions see that in figure 3.1, and then we can order them as follows

$$0 < \alpha_{n,1}, < \alpha_{n,2}, \cdots < \alpha_{n,m} < \alpha_{n,m+1} < \cdots$$

Thus $\sqrt{\lambda_{n,m}}a = \alpha_{n,m}$, and therefore the eigenvalues of the Dirichlet eigenvalue problem is given by

$$\lambda_{n,m} = \left(\frac{\alpha_{n,m}}{a}\right)^2, \quad n, m \in \mathbb{N}$$

With associated eigenfunction.

$$\phi_{n,m}(r,\theta) = J_n\left(\left(\frac{\alpha_{n,m}}{a}\right)r\right) \cdot (A_{n,m}\sin n\theta + B_{n,m}\cos n\theta)\,, \quad n \in \mathbb{N}$$

the orthogonality of the eigenfunctions, follows from the spectral theorem, so that for all function $h(r,\theta) \in L^2(M)$, we can write the Fourier series as

$$h(r,\theta) = \sum_{n,m=1}^{\infty} J_n\left(\left(\frac{\alpha_{n,m}}{a}\right)r\right) \cdot (A_{n,m}\sin n\theta + B_{n,m}\cos n\theta)\,, \quad n, m \in \mathbb{N}$$

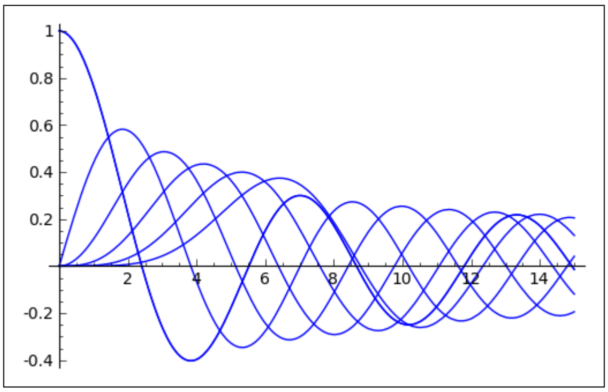


Figure 3.1: Bessel function of the first kind $J_n(x), \quad n = 0, 1, \cdots, 5$

3.4 Neumann eigenvalue problem on the Disk

In the case of the Neumann boundary condition

$$\phi_{n,m}(r,\theta) = J_n\left(\sqrt{\mu}r\right) \cdot \left(A_{n,m}\sin n\theta + B_{n,m}\cos n\theta\right), \quad n \in \mathbb{N}$$
$$\frac{\partial \phi_{n,m}}{\partial r}(r,\theta) = 0 \quad \text{on the boundary i,e} \quad \frac{\partial \phi_{n,m}}{\partial r}(a,\theta) = 0$$

Therefore

$$(\sqrt{\lambda})J_n'\left(\sqrt{\mu}a\right) \cdot \left(A_{n,m}\sin n\theta + B_{n,m}\cos n\theta\right) = 0$$

since the eigenvalue is not zero, then

$$J_n'\left(\sqrt{\mu}a\right) = 0$$

Also the derivative of Bessel function has infinitely many positive solutions, are ordered as

$$0 < \alpha'_{n,1}, < \alpha'_{n,2}, \cdots < \alpha'_{n,m} < \alpha'_{n,m+1} < \cdots \qquad n > 0$$

$$0 = \alpha'_{n,1}, < \alpha'_{n,2}, \cdots < \alpha'_{n,m} < \alpha'_{n,m+1} < \cdots \qquad n = 0$$

Thus the eigenvalues of the Neumann eigenvalue problem is given by

$$\mu_{n,m} = \left(\frac{\alpha'_{n,m}}{a}\right)^2, \quad n,m \in \mathbb{N}$$

With associated eigenfunction.

$$\phi_{n,m}(r,\theta) = J_n\left(\left(\frac{\alpha'_{n,m}}{a}\right)r\right) \cdot \left(A_{n,m}\sin n\theta + B_{n,m}\cos n\theta\right), \quad n \in \mathbb{N}$$

4. The Second Squared Chapter

4.1 Weyl's asymptotic formula

In equation(1), Let $N(\lambda)$ be the number of the eigenvalues $\leq \lambda$, counted with multiplicity . Then

$$N(\lambda) \sim \frac{\omega_n(\text{vol } M)\lambda^{\frac{n}{2}}}{(2\pi)^n} \qquad \text{as } \lambda \to +\infty$$

Where ω_n is the volume of the unit disk in $\mathbb{R}^n$, and vol M is the volume of the manifold. On the other hand

$$\lambda_k^{\frac{n}{2}} \sim \frac{(2\pi)^n k}{(\text{vol } M)\omega_n} \qquad \text{as } k \to +\infty$$

Volume of the Ball in n-dimension. Let's define the area of the sphere (cantered in the origin), in n-dimension with radius r, to be $A(S^{n-1})r^{n-1}$. And the volume of the ball (cantered in the origin), in n-dimension with radius r, to be $V(B^n)r^n$. This implies that

$$\begin{aligned} V(B^n) &= \int_0^R A(S^{n-1})r^{n-1}dr \\ &= A(S^{n-1})\frac{R^n}{n} \end{aligned} \tag{4.1.1}$$

If we have function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is function of $r = \left(\sum_j^n x_j^2\right)^{\frac{1}{2}}$,then.

$$\int_0^\infty f(r)A(S^{n-1})r^n dr = \int_{-\infty}^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(x_1, \cdots, x_n)dx_1 \cdots dx_n$$

Taking $f(r) = e^{-r^2}$, then we have,

$$A(S^{n-1}) \int_0^\infty e^{-r^2} r^n dr = \int_{-\infty}^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(x_1, \cdots, x_n)dx_1 \cdots dx_n$$

In the integral on the left hand side, by substituting $u = r^2$, then $\mathrm{d}u = 2rdr$ we get

$$A(S^{n-1}) \int_0^\infty e^{-r^2} r^n dr = \frac{A(S^{n-1})}{2} \int_0^\infty e^{-u} u^{\frac{n-1}{2}} dr = A(S^{n-1})\frac{\Gamma(\frac{n}{2})}{2} \tag{4.1.2}$$

and the right hand side of the integral is,

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty f(x_1, \cdots, x_n)dx_1 \cdots dx_n = \left(\int_{-\infty}^\infty e^{-x^2} dx\right)^n = \pi^{\frac{n}{2}} \tag{4.1.3}$$

From equations (4.1.2) and (4.1.3) we get

$$A(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \tag{4.1.4}$$

From (4.1.1) and (4.1.4) we get

$$V(B^n) = \frac{\pi^{\frac{n}{2}} R^n}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}} R^n}{(\frac{n}{2})!} \tag{4.1.5}$$

Volume of the Ellipsoid in n-dimension. The equation of the ellipsoid in n-dimension is given as

$$\sum_{i=1}^{n} \frac{x_i^2}{\alpha_i^2} = 1$$

By substituting $\xi_i = \dfrac{x_i}{\alpha_i}$ in equation (4.1.3), we get

$$\begin{aligned}\prod_{i=1}^{n} \alpha_i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\xi_1, \cdots, \xi_n) d\xi_1 \cdots d\xi_n \\ = \prod_{i=1}^{n} \alpha_i \left(\int_{-\infty}^{\infty} e^{-\xi^2} dx \right)^n = \prod_{i=1}^{n} \alpha_i \pi^{\frac{n}{2}}\end{aligned}$$

Hence from equation (4.1.5), with radius $R = 1$ the volume of ellipsoid $V(E^n)$ given by,

$$V(E^n) = \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!} \prod_{i=1}^{n} \alpha_i$$

4.2 proof of Weyl's Asymptotic Formula on rectangle in n-dimension

From the above example, we had our eigenvalue given by the formula

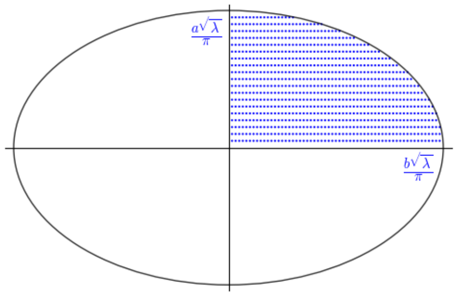
$$\lambda_{n,m} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \qquad \text{for } n, m = 1, 2, \cdots$$

But this in two dimension where $M = \{0 \le x \le a, \quad 0 \le y \le b\}$. We count the total number, of eigenvalues with multiplicity $\lambda_{n,m} \le \lambda$. which mean that

$$\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \le \lambda$$

Rearranging the above equation we get

$$\frac{n^2}{\left(\frac{a\sqrt{\lambda}}{\pi}\right)^2} + \frac{m^2}{\left(\frac{b\sqrt{\lambda}}{\pi}\right)^2} \le 1. \tag{4.2.1}$$



But equation (4.2.1) is an equation of an ellipse. From the diagram of an ellipse and visualise where the eigenvalues are in the diagram. Since the eigenvalues for all $m, n = 1, 2, \cdots$, it implies that all the eigenvalues are located in the first quadrant of the diagram, the total number of the eigenvalues asymptomatically occupies quarter of the area of the ellipse. The total area of our ellipse is

$$\pi\left(\frac{b\sqrt{\lambda}}{\pi}\cdot\frac{a\sqrt{\lambda}}{\pi}\right) = \frac{ab\lambda}{\pi}$$

and so. the number of the eigenvalues is approximately equal to a quarter of this area , which is thus

$$N(\lambda) \sim \frac{ab\lambda}{4\pi}$$

Comparing this result in $\mathbb{R}^2$ with Weyl's Formula, we set for Vol $M = ab$ and $w_2 = \pi$ therefore

$$N(\lambda) \sim \frac{ab\lambda}{4\pi}$$

We Now generalise the proof to $\mathbb{R}^n$. Where our rectangle is

$$M = \{0 \le x_1 \le \alpha_1,\ \ 0 \le x_2 \le \alpha_2, \cdots, 0 \le x_n \le \alpha_n\}$$

and it is volume given as

$$\text{Vol } M = \prod_{i=1}^{n} \alpha_i$$

The eigenvalues are given as

$$\lambda_{k_i} = \frac{k_1^2\pi^2}{\alpha_1^2} + \frac{k_2^2\pi^2}{\alpha_2^2} + \cdots + \frac{k_n^2\pi^2}{\alpha_n^2} \qquad \text{for } k_i = 1, 2, \cdots \quad \text{where } i = 1, 2, \cdots, n$$

Also counting total number, of the eigenvalues with the multiplicity $\lambda_{k_i} \le \lambda$ where $i = 1, 2, \cdots, n$ we have

$$\frac{k_1^2\pi^2}{\alpha_1^2} + \frac{k_2^2\pi^2}{\alpha_2^2} + \cdots + \frac{k_n^2\pi^2}{\alpha_n^2} \le \lambda \qquad \text{for } k_i = 1, 2, \cdots \quad \text{where } i = 1, 2, \cdots, n$$

And rearranging, gives us

$$\frac{k_1^2}{\left(\frac{\alpha_1\sqrt{\lambda}}{\pi}\right)^2} + \frac{k_2^2}{\left(\frac{\alpha_2\sqrt{\lambda}}{\pi}\right)^2} + \cdots + \frac{k_n^2}{\left(\frac{\alpha_n\sqrt{\lambda}}{\pi}\right)^2} \le 1 \qquad \text{for } k_i = 1, 2, \cdots \quad \text{where } i = 1, 2, \cdots, n$$

But this is the equation of the an ellipsoid in n-dimension. Also since $k_i$are positive where $i = 1, 2, \cdots, n$. it implies that the total number of the eigenvalues occupies the volume of $\frac{1}{2^n}$ of the total volume of the ellipsoid. The volume of our ellipsoid is given as

$$V(E^n) = \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!} \prod_{i=1}^{n} \left(\frac{\alpha_i \sqrt{\lambda}}{\pi} \right) \quad \text{where } i = 1, 2, \cdots, n$$

Therefore $\frac{1}{2^n}$ of this volume is

$$N(\lambda) \sim \frac{1}{2^n} \left\{ \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!} \prod_{i=1}^{n} \left(\frac{\alpha_i \sqrt{\lambda}}{\pi} \right) \right\} \quad \text{where } i = 1, 2, \cdots, n$$

rearranging the equation above we get

$$\begin{aligned} N(\lambda) \sim & \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!} \prod_{i=1}^{n} \alpha_i \cdot \frac{\lambda^{\frac{n}{2}}}{(2\pi)^n} \quad \text{where } i = 1, 2, \cdots, n \\ = & \frac{\omega_n (\text{vol } M) \lambda^{\frac{n}{2}}}{(2\pi)^n} \end{aligned}$$

Therefore the proof of Weyl's Formula on a rectangle in $\mathbb{R}^n$ is complete.

4.3 Nodal domain theorem

4.3.1 Definition. Let $\phi_k : M \longrightarrow R$ be an eigenfunction of Laplacian operator, we define the nodal sets (lines) by the set of points in M, such that

$$\mathcal{N} = \{x \in M : \phi_k(x) = 0 \qquad k = 1, 2, \cdots\}.$$

and the nodal domain is define by $M \backslash \mathcal{N}$.

4.3.2 Example. The eigenfunction, of Dirichlet eigenvalue problem, on the interval $(0, a)$, is given by

$$\phi_k(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{k\pi}{a} x\right), \quad k = 1, 2, \cdots$$

and therefore the nodal set given by

$$\{x = \frac{na}{k} : \quad n = 1, 2, \cdots k - 1\}$$

From the Dirichlet boundary condition, note that $x = 0$ and $x = a$ are not in the nodal set.

4.3.3 Example. Consider the example on rectangle with $a = b = \pi$ then we get

$$\begin{cases} \phi_{m,n}(x, y) = \sin nx \sin my \\ \lambda_{m,n} = n^2 + m^2, \quad m, n = 1, 2, \cdots \end{cases}$$

Now we can order the eigenvalues as increasing sequence with multiplicity. Note that, there are eigenvalues with multiplicity more than one; for example $\lambda_{1,2} = \lambda_{2,1} = 5$ and $\lambda_{1,3} = \lambda_{3,1} = 10$, so the common eigenfunctions of these kind of eigenvalues we can be written as a linear combination. For example, the eigenfunction of the eigenvalue $\lambda = 5$ is $A \sin x \sin 2y + B \sin 2x \sin y$ and the eigenfunction of the eigenvalue $\lambda = 10$ is $A \sin x \sin 3y + B \sin 3x \sin y$. Therefore, we can order the eigenvalues with the corresponding eigenfunctions as follows:

λ_n	ϕ_n
2	$A\sin x \sin y$
5	$A\sin x \sin 2y + B\sin 2x \sin y$
8	$A\sin 2x \sin 2y$
10	$A\sin x \sin 3y + B\sin 3x \sin y$
13	$A\sin 2x \sin 3y + B\sin 3x \sin 2y$
$\vdots$	$\vdots$

The nodal lines of the eigenfunction with the eigenvalues of the multiplicity one are given by:

$$\mathcal{N} = \left\{(x,y) : x = \frac{\pi}{n'}, y = \frac{\pi}{m'}, \ \ (1 \leq n' < n, \ \ n = kn'), \ \ (1 \leq m' < m, \ \ m = lm') \ \ k,l \in \mathbb{N}\right\}.$$

but the nodal lines of the eigenfunction with the eigenvalues of the multiplicity two, are coming from the zeros of the eigenfunctions which is linear combination of two functions. Now if one of the two coefficients is zero, we obtain the case of one multiplicity, but if we have a case other than zero it becomes too complicated. As examples in Figure 2 and Figure 3, we show the nodal lines and nodal domains of the eigenvalues $\lambda = 10$ and $\lambda = 13$ respectively for different values of the coefficients.

Figure 4.1: The nodal lines of the eigenvalues $\lambda = 10$

(a) $\sin 3x \sin y$ (b) $\sin x \sin 3y - \sin 3x \sin y$ (c) $\sin x \sin 3y - \frac{1}{3}\sin 3x \sin y$

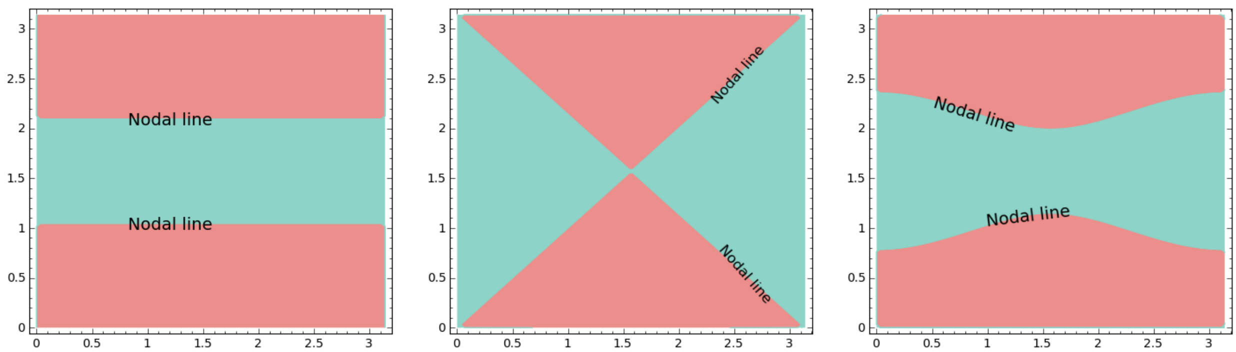
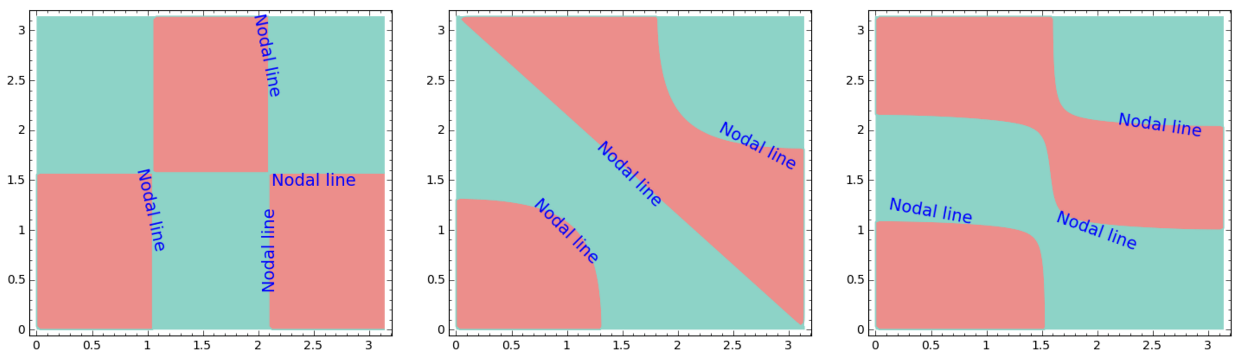


Figure 4.2: The nodal lines of the eigenvalues $\lambda = 13$

(a) $\sin 3x \sin 2y$ (b) $\sin 3x \sin 2y + \sin 2x \sin 3y$ (c) $\frac{1}{3}\sin 3x \sin 2y + 3\sin 2x \sin 3y$



4.4 Courant Nodal domain Theorem

4.4.1 Theorem. *Let ϕ_k, $k \geq 2$ be the k-th eigenfunction of the Laplacian operator,corresponding to the k-th eigenvalue with the multiplicity, then ϕ_k has at least two(2) and at most k nodal domains.*

4.4.2 Corollary. The first eigenfunction ϕ_1 has constant sign and the corresponding eigenvalue has multiplicity 1, and the second eigenfunction ϕ_2 divides the domain M precisely into 2 pieces.

Proof. From the Max-Min theorem we have that

$$\lambda_1 = \frac{\int_M \|\nabla\phi_1\|^2}{\|\phi_1\|^2}$$

Now let's take $|\phi_1|$ in the state of ϕ_1, we get the same value for λ_1 and then $\phi_1 = |\phi_1|$, this implies the claim that the first eigenfunction has constant sign and its corresponding eigenvalue has multiplicity 1. From the orthogonality of ϕ_2 to ϕ_1 we have that

$$\int_M \phi_1\phi_2 dx = 0,$$

Since we have shown that ϕ_1 has constant sign, ϕ_2 has to change its sign in M and since ϕ_2 is a continuous function then it has zero some where in the domain M and it divides the domain M precisely into two pieces. ☐

4.4.3 Corollary. Let $\Omega_k \subset M$ be nodal domain of an eigenfunction ϕ_k corresponding to the eigenvalue λ_k, under Dirichlet boundary condition. Then

$$\lambda_k(M) = \lambda_1(\Omega)$$

Proof. Since Ω_k is the nodal domain of the eigenfunction ϕ_k, then

$$\begin{aligned} -\Delta\phi_k =& \lambda_k\phi_k \quad \text{in} \quad \Omega_k \\ \phi_k =& 0 \quad \text{on} \quad \partial\Omega_k \end{aligned}$$

and since ϕ_k has constant sign in Ω_k, this means that ϕ_k is the first eigenfunction in Ω_k,so that $\lambda_k(M)$ is first eigenvalue in Ω_k ☐

References

[AST92] Alan Adolphson, Steven Sperber, and Marvin Tretkoff (eds.), *p-adic methods in number theory and algebraic geometry*, Contemporary Mathematics, no. 133, American Mathematical Society, Providence, RI, 1992.

[Bea06] Alan F. Beardon, *From problem solving to research*, 2006, Unpublished manuscript.

[Dav99] M. C. Davey, *Error-correction using low-density parity-check codes*, Phd, University of Cambridge, 1999.

[Lam86] Leslie Lamport, *LaTeX: A document preparation system*, Addison-Wesley, 1986.

[Mac86] D. J. C. MacKay, *Statistical testing of high precision digitisers*, Tech. Report 3971, Royal Signals and Radar Establishment, Malvern, Worcester. WR14 3PS, 1986.

[Mac95] D. J. C. MacKay, *A free energy minimization framework for inference problems in modulo 2 arithmetic*, Fast Software Encryption (Proceedings of 1994 K.U. Leuven Workshop on Cryptographic Algorithms) (B. Preneel, ed.), Lecture Notes in Computer Science Series, no. 1008, Springer, 1995, pp. 179–195.

[MN95] D. J. C. MacKay and R. M. Neal, *Good codes based on very sparse matrices*, Available from `http://www.inference.phy.cam.ac.uk/`, 1995.

[Sha48] C. E. Shannon, *A mathematical theory of communication*, Bell Sys. Tech. J. **27** (1948), 379–423, 623–656.

[Sha93] ______, *The best detection of pulses*, Collected Papers of Claude Shannon (N. J. A. Sloane and A. D. Wyner, eds.), IEEE Press, New York, 1993, pp. 148–150.

[Web] Webots, *http://www.cyberbotics.com*, Commercial Mobile Robot Simulation Software.