Chaotic behavior of disordered nonlinear lattices

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Outline

• **Disordered lattices:**
  - The quartic Klein-Gordon (KG) model
  - The disordered nonlinear Schrödinger equation (DNLS)
  - Different dynamical behaviors

• **Chaotic behavior of the KG model**
  - Lyapunov exponents
  - Deviation Vector Distributions

• **Numerical methods**
  - Symplectic Integrators
  - Tangent Map method
  - Different integration schemes for DNLS

• **Summary**
Interplay of disorder and nonlinearity


Waves in nonlinear disordered media – localization or delocalization?


Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL (2008)]
The Klein – Gordon (KG) model

\[ H_K = \sum_{l=1}^{N} \frac{p_{l}^2}{2} + \frac{\varepsilon_l}{2} u_{l}^2 + \frac{1}{4} u_{l}^4 + \frac{1}{2W} (u_{l+1} - u_{l})^2 \]

with fixed boundary conditions \( u_0 = p_0 = u_{N+1} = p_{N+1} = 0 \). Typically \( N = 1000 \).

Parameters: \( W \) and the total energy \( E \). \( \varepsilon_l \) chosen uniformly from \( \left[ \frac{1}{2}, \frac{3}{2} \right] \).

**Linear case** (neglecting the term \( u_l^4/4 \))

Ansatz: \( u_l = A_l \exp(i \omega t) \). Normal modes (NMs) \( A_{\nu, l} \) - Eigenvalue problem:

\[ \lambda A_l = \varepsilon_l A_l - (A_{l+1} + A_{l-1}) \] with \( \lambda = W \omega^2 - W - 2 \), \( \varepsilon_l = W(\varepsilon_l - 1) \)

The discrete nonlinear Schrödinger (DNLS) equation

We also consider the system:

\[ H_D = \sum_{l=1}^{N} \varepsilon_l \left| \psi_l \right|^2 + \frac{\beta}{2} \left| \psi_l \right|^4 - \left( \psi_{l+1} \psi_l^* + \psi_{l+1}^* \psi_l \right) \]

where \( \varepsilon_l \) chosen uniformly from \( \left[ -\frac{W}{2}, \frac{W}{2} \right] \) and \( \beta \) is the nonlinear parameter.

Conserved quantities: The energy and the norm \( S = \sum_l \left| \psi_l \right|^2 \) of the wave packet.
Distribution characterization

We consider normalized energy distributions in normal mode (NM) space

\[ z_v \equiv \frac{E_v}{\sum_m E_m} \text{ with } E_v = \frac{1}{2} \left( \dot{A}_v^2 + \omega_v^2 A_v^2 \right) \], where \( A_v \) is the amplitude of the \( v \)-th NM (KG) or norm distributions (DNLS).

Second moment:

\[ m_2 = \sum_{v=1}^{N} \left( v - \bar{v} \right)^2 z_v \text{ with } \bar{v} = \sum_{v=1}^{N} v z_v \]

Participation number:

\[ P = \frac{1}{\sum_{v=1}^{N} z_v^2} \]

measures the number of stronger excited modes in \( z_v \).

Single mode \( P=1 \). Equipartition of energy \( P=N \).
Scales

Linear case: $\omega^2_v \in \left[ \frac{1}{2}, \frac{3}{2} + \frac{4}{W} \right]$, width of the squared frequency spectrum:

$$\Delta_K = 1 + \frac{4}{W} \quad (\Delta_D = W + 4)$$

Localization volume of an eigenstate:

$$V \sim \frac{1}{\sum_{l=1}^{N} A_{v,l}^4}$$

Average spacing of squared eigenfrequencies of NM within the range of a localization volume:

$$d_K \approx \frac{\Delta_K}{V}$$

Nonlinearity induced squared frequency shift of a single site oscillator

$$\delta_l = \frac{3E_l}{2\tilde{c}_l} \propto E \quad (\delta_l = \beta |\psi_l|^2)$$

The relation of the two scales $d_K \leq \Delta_K$ with the nonlinear frequency shift $\delta_l$ determines the packet evolution.
Different Dynamical Regimes


$\Delta$: width of the frequency spectrum, $d$: average spacing of interacting modes, $\delta$: nonlinear frequency shift.

Weak Chaos Regime: $\delta<d$, $m^2 \sim t^{1/3}$
Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina, PRB (1998) – Pikovsky, & Shepelyansky, PRL (2008)].

Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m^2 \sim t^{1/2} \rightarrow m^2 \sim t^{1/3}$
Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

Selftrapping Regime: $\delta>\Delta$
Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].
Single site excitations

DNLS $W=4, \beta=0.1, 1, 4.5$  
KG $W = 4, E = 0.05, 0.4, 1.5$

No strong chaos regime

In weak chaos regime we averaged the measured exponent $\alpha (m_2 \sim t^\alpha)$ over 20 realizations:

$\alpha=0.33\pm0.05$ (KG)  
$\alpha=0.33\pm0.02$ (DLNS)

Flach et al., PRL (2009)  
S. et al., PRE (2009)
KG: Different spreading regimes
Crossover from strong to weak chaos (block excitations)

Average over 1000 realizations!

\[ \alpha(\log t) = \frac{d \langle \log m_2 \rangle}{d \log t} \]

DNLS $\beta = 0.04, 0.72, 3.6$  
KG $E = 0.01, 0.2, 0.75$

$W=4$

$\alpha = 1/2$

$\alpha = 1/3$

Laptyeva et al., EPL (2010)  
Bodyfelt et al., PRE (2011)
Lyapunov Exponents (LEs)

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

Consider an orbit in the 2N-dimensional phase space with initial condition \( x(0) \) and an initial deviation vector from it \( v(0) \). Then the mean exponential rate of divergence is:

\[
\text{mLCE} = \lambda_1 = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|\mathbf{v}(t)\|}{\|\mathbf{v}(0)\|}
\]

\( \lambda_1 = 0 \rightarrow \text{Regular motion } \propto (t^{-1}) \)

\( \lambda_1 \neq 0 \rightarrow \text{Chaotic motion} \)
KG: LEs for single site excitations (E=0.4)
KG: Weak Chaos (E=0.4)
KG: Weak Chaos

Individual runs
Linear case
E=0.4, W=4

Average over 50 realizations

Single site excitation E=0.4, W=4
Block excitation (21 sites)  
E=0.21, W=4
Block excitation (37 sites)  
E=0.37, W=3

\[ \alpha = \frac{d \log \langle \Lambda \rangle}{d \log t} \]

S. et al. PRL (2013)
Deviation Vector Distributions (DVDs)

Deviation vector:

\[ v(t) = (\delta u_1(t), \delta u_2(t), \ldots, \delta u_N(t), \delta p_1(t), \delta p_2(t), \ldots, \delta p_N(t)) \]

DVD: 

\[ w_l = \frac{\delta u_l^2 + \delta p_l^2}{\sum_l (\delta u_l^2 + \delta p_l^2)} \]
Deviation Vector Distributions (DVDs)

Individual run
E=0.4, W=4

Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.
Consider an \( N \) degree of freedom autonomous Hamiltonian system having a Hamiltonian function of the form:

\[
H(q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N)
\]

The time evolution of an orbit (trajectory) with initial condition

\[
P(0) = (q_1(0), q_2(0), \ldots, q_N(0), p_1(0), p_2(0), \ldots, p_N(0))
\]

is governed by the Hamilton’s equations of motion

\[
\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}
\]
Autonomous Hamiltonian systems

Let us consider an \( N \) degree of freedom autonomous Hamiltonian systems of the form:

\[
H(q, p) = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + V(q)
\]

As an example, we consider the Hénon-Heiles system:

\[
H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3
\]

Hamilton equations of motion:

\[
\begin{align*}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y
\end{align*}
\]

Variational equations:

\[
\begin{align*}
\delta \dot{x} &= \delta p_x \\
\delta \dot{y} &= \delta p_y \\
\delta \dot{p}_x &= -(1 + 2y)\delta x - 2x\delta y \\
\delta \dot{p}_y &= -2x\delta x + (-1 + 2y)\delta y
\end{align*}
\]
Symplectic Integrators (SIs)

Formally the solution of the Hamilton equations of motion can be written as:
\[ \frac{d\tilde{X}}{dt} = \{H, \tilde{X}\} = L_H \tilde{X} \Rightarrow \tilde{X}(t) = \sum_{n \geq 0} \frac{t^n}{n!} L_H^n \tilde{X} = e^{tL_H} \tilde{X} \]

where \( \tilde{X} \) is the full coordinate vector and \( L_H \) the Poisson operator:
\[
L_H f = \sum_{j=1}^{N} \left\{ \frac{\partial H}{\partial p_j} \frac{\partial f}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial f}{\partial p_j} \right\}
\]

If the Hamiltonian H can be split into two integrable parts as \( H = A + B \), a symplectic scheme for integrating the equations of motion from time \( t \) to time \( t + \tau \) consists of approximating the operator \( e^{\tau L_H} \) by
\[
e^{\tau L_H} = e^{\tau(L_A + L_B)} = \prod_{i=1}^{j} e^{c_i \tau L_A} e^{d_i \tau L_B} + O(\tau^{n+1})
\]

for appropriate values of constants \( c_i, d_i \). This is an integrator of order \( n \). So the dynamics over an integration time step \( \tau \) is described by a series of successive acts of Hamiltonians A and B.
Symplectic Integrator SABA$_2$C

The operator $e^{\tau L_H}$ can be approximated by the symplectic integrator [Laskar & Robutel, Cel. Mech. Dyn. Astr. (2001)]:

$$SABA_2 = e^{c_1 \tau L_A} \ e^{d_1 \tau L_B} \ e^{c_2 \tau L_A} \ e^{d_1 \tau L_B} \ e^{c_1 \tau L_A}$$

with $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$, $c_2 = \frac{\sqrt{3}}{3}$, $d_1 = \frac{1}{2}$.

The integrator has only small positive steps and its error is of order 2.

In the case where $A$ is quadratic in the momenta and $B$ depends only on the positions the method can be improved by introducing a corrector $C$, having a small negative step:

$$C = e^{-\tau^3 \frac{c}{2} L_{\{A,B\},B}}$$

with $c = \frac{2 - \sqrt{3}}{24}$.

Thus the full integrator scheme becomes: $SABAC_2 = C (SABA_2) C$ and its error is of order 4.
Tangent Map (TM) Method


The Hénon-Heiles system can be split as: \[ A = \frac{1}{2} (p_x^2 + p_y^2) \quad B = \frac{1}{2} (x^2 + y^2) + x^2 y - \frac{1}{3} y^3 \]

\[
\begin{align*}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y
\end{align*}
\]

\[
\begin{align*}
\delta x &= \delta p_x \\
\delta y &= \delta p_y \\
\delta p_x &= -(1 + 2y) \delta x - 2x \delta y \\
\delta p_y &= -2x \delta x + (-1 + 2y) \delta y
\end{align*}
\]

\[
\begin{align*}
\dot{\vec{u}} &= \mathbf{A}(\vec{p}) \Rightarrow \frac{d\vec{u}}{dt} = L_{AV} \vec{u} \Rightarrow e^{\tau L_{AV}} \quad \begin{cases} 
x' &= x + p_x \tau \\
y' &= y + p_y \tau \\
p_x' &= p_x \\
p_y' &= p_y \\
\delta x' &= \delta x + \delta p_x \tau \\
\delta y' &= \delta y + \delta p_y \tau \\
\delta p_x' &= \delta p_x \\
\delta p_y' &= \delta p_y
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= 0 \\
\dot{y} &= 0 \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y \\
\delta x &= 0 \\
\delta y &= 0 \\
\delta p_x &= -(1 + 2y) \delta x - 2x \delta y \\
\delta p_y &= -2x \delta x + (-1 + 2y) \delta y
\end{align*}
\]

\[
\begin{align*}
\frac{d\vec{u}}{dt} &= \mathbf{B}(\vec{q}) \Rightarrow e^{\tau L_{BV}} \quad \begin{cases} 
x' &= x \\
y' &= y \\
p_x' &= p_x - x(1 + 2y) \tau \\
p_y' &= p_y + (y^2 - x^2 - y) \tau \\
\delta x' &= \delta x \\
\delta y' &= \delta y \\
\delta p_x' &= \delta p_x - [(1 + 2y) \delta x + 2x \delta y] \tau \\
\delta p_y' &= \delta p_y + [-2x \delta x + (-1 + 2y) \delta y] \tau
\end{cases}
\end{align*}
\]
The KG model

We apply the SABAC$_2$ integrator scheme to the KG Hamiltonian by using the splitting:

$$H_K = \sum_{l=1}^{N} \left( \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} (u_{l+1} - u_l)^2 \right)$$

with a corrector term which corresponds to the Hamiltonian function:

$$C = \left\{ \{A, B\}, B \right\} = \sum_{l=1}^{N} \left[ u_l (\tilde{\varepsilon}_l + u_l^2) - \frac{1}{W} (u_{l-1} + u_{l+1} - 2u_l) \right]^2.$$
The DNLS model

A 2\textsuperscript{nd} order SABA Symplectic Integrator with 5 steps, combined with approximate solution for the \textit{B} part (Fourier Transform): \textit{SIFT}\textsuperscript{2}

\[ H_D = \sum_l \varepsilon_l |\psi_l|^2 + \frac{\beta}{2} |\psi_l|^4 - \left( \psi_{l+1}\psi^*_l + \psi^*_{l+1}\psi_l \right), \quad \psi_l = \frac{1}{\sqrt{2}} (q_l + ip_l) \]

\[ H_D = \sum_l \left( \frac{\varepsilon_l}{2} \left( q_l^2 + p_l^2 \right) + \frac{\beta}{8} \left( q_l^2 + p_l^2 \right)^2 \right) - q_nq_{n+1} - p_np_{n+1} \]

\[ e^{\tau L_A}: \begin{cases} 
q'_l = q_l \cos(\alpha_l \tau) + p_l \sin(\alpha_l \tau), \\
p'_l = p_l \cos(\alpha_l \tau) - q_l \sin(\alpha_l \tau), \\
\alpha_l = \varepsilon_l + \beta(q_l^2 + p_l^2)/2 
\end{cases} \]

\[ e^{\tau L_B}: \begin{cases} 
\varphi_q = \sum_{m=1}^{N} \psi_m e^{2\pi i q(m-1)/N} \\
\varphi'_q = \varphi_q e^{2i \cos(2\pi(q-1)/N)\tau} \\
\psi'_l = \frac{1}{N} \sum_{q=1}^{N} \varphi'_q e^{-2\pi i l(q-1)/N} 
\end{cases} \]
The DNLS model

Symplectic Integrators produced by Successive Splits (SS)

\[ H_D = \sum_l \left( \frac{\varepsilon_l}{2} \left( q_{l}^2 + p_{l}^2 \right) + \frac{\beta}{8} \left( q_{l}^2 + p_{l}^2 \right)^2 \right) - q_n q_{n+1} - p_n p_{n+1} \]

\[ \begin{align*}
q'_{l} &= q_l \cos(\alpha_l \tau) + p_l \sin(\alpha_l \tau), \\
p'_{l} &= p_l \cos(\alpha_l \tau) - q_l \sin(\alpha_l \tau),
\end{align*} \]

Using the SABA_2 integrator we get a 2^{nd} order integrator with 13 steps, SS^2:

\[ SS^2 = e^{\left[ \frac{(3-\sqrt{3})}{6} \tau \right]} L_A e^{\frac{\tau}{2} L_B} e^{\frac{\tau}{3} L_A} e^{\left[ \frac{(3-\sqrt{3})}{6} \tau \right]} L_A \]

\[ \tau' = \tau / 2 \]
Non-symplectic methods for the DNLS model

In our study we also use the DOP853 integrator which is an explicit non-symplectic Runge-Kutta integration scheme of order 8.

Three part split symplectic integrators for the DNLS model

Three part split symplectic integrator of order 2, with 5 steps: $ABC^2$

$$H_D = \sum_l \left( \frac{\varepsilon_l}{2} \left( q_i^2 + p_i^2 \right) + \frac{\beta}{8} \left( q_i^2 + p_i^2 \right)^2 \right) - q_n q_{n+1} - p_n p_{n+1}$$

$$ABC^2 = e^{2L_A} e^{2L_B} e^{\tau L_C} e^{2L_B} e^{2L_A}$$

This low order integrator has already been used by e.g. Chambers, MNRAS (1999) – Goździewski et al., MNRAS (2008).
2nd order integrators: Numerical results

ABC\(^2\) \(\tau=0.005\)
SS\(^2\) \(\tau=0.02\)
DOP853 \(\delta=10^{-16}\)
SIFT\(^2\) \(\tau=0.05\)

\(E_r\): relative energy error
\(S_r\): relative norm error
\(T_c\): CPU time (sec)

Composition Methods: 4\textsuperscript{th} order SIs

Starting from any 2\textsuperscript{nd} order symplectic integrator $S^\text{2nd}$, we can construct a 4\textsuperscript{th} order integrator $S^\text{4th}$ using the composition method proposed by Yoshida [Phys. Lett. A (1990)]:

$S^\text{4th}(\tau) = S^\text{2nd}(x_1 \tau) \times S^\text{2nd}(x_0 \tau) \times S^\text{2nd}(x_1 \tau), \quad x_0 = -\frac{2^{1/3}}{2 - 2^{1/3}}, \quad x_1 = \frac{1}{2 - 2^{1/3}}$

In this way, starting with the 2\textsuperscript{nd} order integrators $SS^2$, $SIFT^2$ and $ABC^2$ we construct the 4\textsuperscript{th} order integrators:

$SS^4$ with 37 steps \quad $SIFT^4$ with 13 steps \quad $ABC^4_{[Y]}$ with 13 steps


$S^\text{4th}(\tau) = S^\text{2nd}(p_2 \tau) \times S^\text{2nd}(p_2 \tau) \times S^\text{2nd}((1 - 4p_2) \tau) \times S^\text{2nd}(p_2 \tau) \times S^\text{2nd}(p_2 \tau)$

$p_2 = \frac{1}{4 - 4^{1/3}}, \quad 1 - 4p_2 = -\frac{4^{1/3}}{4 - 4^{1/3}}$

Starting with the 2\textsuperscript{nd} order integrators $ABC^2$ we construct the 4\textsuperscript{th} order integrator: $ABC^4_{[S]}$ with 21 steps.
More $4^{\text{th}}$ order SIs


Approximating the solution of the $B$ part by a Fourier Transform we construct the $4^{\text{th}}$ order integrators:

- $\text{SIFT}^4_{864}$ with 43 steps
- $\text{SIFT}^4_{1064}$ with 49 steps

Using successive splits for the $B$ part and implementing the SABA$_2$ integrator for its integration, we construct the $4^{\text{th}}$ order integrators:

- $\text{SS}^4_{864}$ with 49 steps
- $\text{SS}^4_{1064}$ with 55 steps
4th order integrators: Numerical results (I)

SIFT^4 \tau=0.125
SIFT^2 \tau=0.05
ABC^4_{[S]} \tau=0.1
SS^4 \tau=0.1
ABC^4_{[Y]} \tau=0.05

E_r: relative energy error
S_r: relative norm error
T_c: CPU time (sec)

4th order integrators: Numerical results (II)

SIFT$^4_{1064}$ $\tau=0.25$

ABC$^4_{[Y]}$ $\tau=0.05$

SIFT$^4_{864}$ $\tau=0.25$

SS$^4_{1064}$ $\tau=0.25$

SS$^4_{864}$ $\tau=0.25$

$E_r$: relative energy error

$S_r$: relative norm error

$T_c$: CPU time (sec)

High order composition methods (I)

Using a composition technique introduced by Yoshida [Phys. Let. A (1990)] we construct the 6th order symplectic integrator $ABC^6_Y$ having 29 steps:

$$ABC^6(\tau) = ABC^2(w_3\tau) \times ABC^2(w_2\tau) \times ABC^2(w_1\tau) \times ABC^2(w_0\tau) \times ABC^2(w_1\tau) \times ABC^2(w_2\tau) \times ABC^2(w_3\tau)$$

whose coefficients

$$w_1 = -1.17767998417887$$
$$w_2 = 0.235573213359357$$
$$w_3 = 0.784513610477560$$
$$w_0 = 1 - 2(w_1 + w_2 + w_3)$$

cannot be given in analytic form.
High order composition methods (II)

In addition, following the works of Kahan & Li, Math Comput. (1997), and Sofroniou & Spaletta, Optim. Methods Softw. (2005) we implement some efficient high order composition methods, considering as the basic block the 2\textsuperscript{nd} order ABC\textsuperscript{2} integrator.

- \(\text{ABC}^{6}_{[KL]}\) with 37 steps
- \(\text{ABC}^{6}_{[SS]}\) with 45 steps
- \(\text{ABC}^{8}_{[Y]}\) with 61 steps
- \(\text{ABC}^{8}_{[KL]}\) with 69 steps
- \(\text{ABC}^{8}_{[SS]}\) with 77 steps
- \(\text{ABC}^{10}_{[SS]}\) with 125 steps
High order integrators: Numerical results (I)

SS$^4_{864}$ $\tau=0.015625$

ABC$^6_{[Y]}$ $\tau=0.03$

ABC$^6_{[KL]}$ $\tau=0.04$

ABC$^6_{[SS]}$ $\tau=0.125$

$E_r$: relative energy error

$S_r$: relative norm error

$T_c$: CPU time (sec)

High order integrators: Numerical results (II)

![Graphs showing numerical results](image)

**ABC**

$\tau = 0.0625$

$\tau = 0.125$

$\tau = 0.2$

$E_r$: relative energy error

$S_r$: relative norm error

$T_c$: CPU time (sec)

Summary (I)

• We presented three different dynamical behaviors for wave packet spreading in 1d nonlinear disordered lattices:
  ✓ Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$
  ✓ Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2}$ → $m_2 \sim t^{1/3}$
  ✓ Selftrapping Regime: $\delta>\Delta$

• Generality of results:
  ✓ Two different models: KD and DNLS,
  ✓ Predictions made for DNLS are verified for both models.

• Lyapunov exponent computations show that:
  ✓ Chaos not only exists, but also persists.
  ✓ Slowing down of chaos does not cross over to regular dynamics.
  ✓ Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.

• Our results suggest that Anderson localization is eventually destroyed by nonlinearity, since spreading does not show any sign of slowing down.
Summary (II)

• We presented several efficient integration methods suitable for the integration of the DNLS model, which are based on symplectic integration techniques.

• The construction of symplectic schemes based on 3 part split of the Hamiltonian was emphasized (ABC methods).

• Algorithms based on the integration of the B part of Hamiltonian via Fourier transforms, i.e. methods SIFT\(^2\), SIFT\(^4\), SIFT\(^4_{864}\) and SIFT\(^4_{1064}\) succeeded in keeping the relative norm error \(S_r\) very low. **Drawback:** they require the number of lattice sites to be \(2^k, k \in \mathbb{N}^*\).

• We hope that our results will initiate future research both for the theoretical development of new, improved 3 part split integrators, as well as for their applications to different dynamical systems.
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Thank you for your attention