Chaos in disordered nonlinear lattices

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Outline

• Dynamical Systems:
  ✓ The quartic Klein-Gordon (KG) disordered lattice
  ✓ The disordered nonlinear Schrödinger equation (DNLS)

• Numerical methods

• Different dynamical behaviors
  ✓ Single site excitations
  ✓ Block excitations

• Summary
Interplay of disorder and nonlinearity

Waves in disordered media – Anderson localization
[Billy et al., Nature (2008)].

Waves in nonlinear disordered media – localization or delocalization?


Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL (2008)].
The Klein – Gordon (KG) model

\[ H_K = \sum_{l=1}^{N} \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} (u_{l+1} - u_l)^2 \]

with fixed boundary conditions \( u_0 = p_0 = u_{N+1} = p_{N+1} = 0 \). Typically \( N = 1000 \).

Parameters: \( W \) and the total energy \( E \). \( \tilde{\varepsilon}_l \) chosen uniformly from \( \left[ \frac{1}{2}, \frac{3}{2} \right] \).

The discrete nonlinear Schrödinger (DNLS) equation

\[ H_D = \sum_{l=1}^{N} \varepsilon_l |\psi_l|^2 + \frac{\beta}{2} |\psi_l|^4 - (\psi_{l+1} \psi_l^* + \psi_{l+1}^* \psi_l)^2, \quad \psi_l = \frac{1}{\sqrt{2}} (q_l + ip_l) \]

where \( \varepsilon_l \) chosen uniformly from \( \left[ -\frac{W}{2}, \frac{W}{2} \right] \) and \( \beta \) is the nonlinear parameter.
Scales

Linear case: \( \omega_v^2 \in \left[ \frac{1}{2}, \frac{3}{2} + \frac{4}{W} \right] \), width of the squared frequency spectrum:

\[
\Delta_K = 1 + \frac{4}{W}
\]

\((\Delta_D = W + 4)\)

Average spacing of squared eigenfrequencies of NMs within the range of a localization volume:

\[
d_K \approx \frac{\Delta_K}{V}
\]

Nonlinearity induced squared frequency shift of a single site oscillator

\[
\delta_l = \frac{3E_l}{2\tilde{\varepsilon}_l} \propto E \quad (\delta_l = \beta |\psi_l|^2)
\]

The relation of the two scales \( d_K \leq \Delta_K \) with the nonlinear frequency shift \( \delta_l \) determines the packet evolution.
Distribution characterization

We consider normalized energy distributions in normal mode (NM) space

\[ z_\nu \equiv \frac{E_\nu}{\sum_m E_m} \quad \text{with} \quad E_\nu = \frac{1}{2} \left( \dot{A}_\nu^2 + \omega_\nu^2 A_\nu^2 \right) , \]

where \( A_\nu \) is the amplitude of the \( \nu \)th NM.

Second moment:

\[ m_2 = \sum_{\nu=1}^{N} (\nu - \bar{\nu})^2 z_\nu \quad \text{with} \quad \bar{\nu} = \sum_{\nu=1}^{N} \nu z_\nu \]

Participation number:

\[ P = \frac{1}{\sum_{\nu=1}^{N} z_\nu^2} \]

measures the number of stronger excited modes in \( z_\nu \). Single mode \( P=1 \), equipartition of energy \( P=N \).
Lyapunov Exponents

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it [see e.g. Ch.S., LNP, (2010)].

Consider an orbit in the 2N-dimensional phase space with initial condition $x(0)$ and an initial deviation vector from it $v(0)$. Then the mean exponential rate of divergence is:

$$\sigma(x(0), v(0)) = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

$\sigma_1 = 0 \rightarrow$ Regular motion

$\sigma_1 \neq 0 \rightarrow$ Chaotic motion
Consider an $N$ degree of freedom autonomous Hamiltonian systems of the form:

$$H(\vec{q}, \vec{p}) = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + V(\vec{q})$$

As an example, we take the Hénon-Heiles system:

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$$

Hamilton equations of motion:

$$\begin{align*}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y
\end{align*}$$

Variational equations:

$$\begin{align*}
\delta x &= \delta p_x \\
\delta y &= \delta p_y \\
\delta p_x &= -(1 + 2y)\delta x - 2x\delta y \\
\delta p_y &= -2x\delta x + (-1 + 2y)\delta y
\end{align*}$$
Symplectic integration schemes

If the Hamiltonian $H$ can be split into two integrable parts as $H=A+B$, a symplectic scheme for integrating the equations of motion from time $t$ to time $t+\tau$ consists of approximating the operator $e^{\tau L_H}$, i.e. the solution of Hamilton equations of motion, by

$$e^{\tau L_H} = e^{\tau(L_A + L_B)} \approx \prod_{i=1}^{j} e^{c_i \tau L_A} e^{d_i \tau L_B}$$

for appropriate values of constants $c_i, d_i$.

So the dynamics over an integration time step $\tau$ is described by a series of successive acts of Hamiltonians $A$ and $B$.


$$SABA_2 = e^{\left[\frac{(3-\sqrt{3})}{6} \tau\right] L_A} e^{\frac{\tau}{2} L_B} e^{\frac{\sqrt{3} \tau}{3} L_A} e^{\frac{\tau}{2} L_B} e^{\left[\frac{(3-\sqrt{3})}{6} \tau\right] L_A}$$
Tangent Map (TM) Method

We use symplectic integration schemes for the integrating the equations of motion AND THE VARIATIONAL EQUATIONS.

The Hénon-Heiles system can be split as:

\[ A = \frac{1}{2}(p_x^2 + p_y^2), \quad B = \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3, \]

The system of the Hamilton equations of motion and the variational equations is split into two integrable systems which correspond to Hamiltonians A and B.
Tangent Map (TM) Method


\[
\begin{align*}
& e^{\tau L_A} : \\
& \begin{cases}
  x' = x + p_x \tau \\
  y' = y + p_y \tau \\
  p_x' = p_x \\
  p_y' = p_y
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
& e^{\tau L_{AV}} : \\
& \begin{cases}
  x' = x + p_x \tau \\
  y' = y + p_y \tau \\
  p_x' = p_x \\
  p_y' = p_y \\
  \delta x' = \delta x + \delta p_x \tau \\
  \delta y' = \delta y + \delta p_y \tau \\
  \delta p_x' = \delta p_x \\
  \delta p_y' = \delta p_y
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
& e^{\tau L_B} : \\
& \begin{cases}
  x' = x \\
  y' = y \\
  p_x' = p_x - x(1 + 2y)\tau \\
  p_y' = p_y + (y^2 - x^2 - y)\tau
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
& e^{\tau L_{BV}} : \\
& \begin{cases}
  x' = x \\
  y' = y \\
  p_x' = p_x - x(1 + 2y)\tau \\
  p_y' = p_y + (y^2 - x^2 - y)\tau \\
  \delta x' = \delta x \\
  \delta y' = \delta y \\
  \delta p_x' = \delta p_x - [(1 + 2y)\delta x + 2x\delta y] \tau \\
  \delta p_y' = \delta p_y + [-2x\delta x + (-1 + 2y)\delta y] \tau
\end{cases} \\
\end{align*}
\]
Symplectic Integrator $SABA_2C$

The integrator

$$SABA_2 = e^{-\frac{(3-\sqrt{3})}{6} \tau L_A} e^{\tau L_B} e^{\frac{\sqrt{3}\tau}{3} L_A} e^{\tau L_B} e^{-\frac{(3-\sqrt{3})}{6} \tau L_A}$$

has only small positive steps and its error is of order $O(\tau^2)$.

In the case where $A$ is quadratic in the momenta and $B$ depends only on the positions the method can be improved by introducing a corrector $C$, having a small negative step:

$$-\tau^3 \frac{c}{2} L\{\{A,B\},B\}$$

with $c = \frac{2 - \sqrt{3}}{24}$.

Thus the full integrator scheme becomes: $SABAC_2 = C (SABA_2) C$ and its error is of order $O(\tau^4)$. 
Symplectic Integrator SABA$_2^C$
for the KG system

We apply the SABA$_2^C$ integrator scheme to the KG Hamiltonian by using the splitting:

\[
A = \sum_{l=1}^{N} \frac{p_l^2}{2}
\]

\[
B = \sum_{l=1}^{N} \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} (u_{l+1} - u_l)^2
\]

with a corrector term which corresponds to the Hamiltonian function:

\[
C = \left\{ \{A, B\}, B \right\} = \sum_{l=1}^{N} \left[ u_l (\tilde{\varepsilon}_l + u_l^2) - \frac{1}{W} (u_{l-1} + u_{l+1} - 2u_l) \right]^2
\]
Different Dynamical Regimes


Weak Chaos Regime: \( \delta < d, \quad m_2 \sim t^{1/3} \)

Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina PRB (1998) – Pikovskiy, Shepelyansky, PRL (2008)].

Intermediate Strong Chaos Regime: \( d < \delta < \Delta, \quad m_2 \sim t^{1/2} \rightarrow m_2 \sim t^{1/3} \)

Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

Selftrapping Regime: \( \delta > \Delta \)

Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].
Single site excitations

DNLS $W=4$, $\beta=0.1, 1, 4.5$    KG $W=4$, $E=0.05, 0.4, 1.5$

No strong chaos regime

In weak chaos regime we averaged the measured exponent $\alpha (m_2 \sim t^\alpha)$ over 20 realizations:

$\alpha=0.33\pm0.05$ (KG)
$\alpha=0.33\pm0.02$ (DLNS)

Single site excitations \((E=0.4)\)

Lyapunov exponents
KG: Weak Chaos (E=0.4)
KG: Selftrapping (E=1.5)

$\tau = 100000000.00$

**Second Moment**

- Slope 1/3

**Participation Number**

**Energy Distribution**

- Slope -1

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Single site excitations: Different spreading regimes

KG $W = 4, E = 0.4, 1.5$

Average over 20 realizations
Crossover from strong to weak chaos

We consider compact initial wave packets of width $L=V$ [Laptyeva, Bodyfelt, Krimer, Ch. S., Flach, EPL (2010) – Bodyfelt, Laptyeva, Ch. S., Krimer, Flach, PRE (2011)]
Crossover from strong to weak chaos

\[ \alpha(\log t) = \frac{d \langle \log m_2 \rangle}{d \log t} \]

DNLS $\beta = 0.04, 0.72, 3.6$  
KG $E/L = 0.01, 0.2, 0.75$

$W=4, L=21$

Average over 1000 realizations!
Block excitations \((E/L=0.2)\)
Lyapunov exponents
KG: Weak Chaos (E/L=0.01)
KG: Strong Chaos (E/L=0.2)

Second Moment

Participation Number

Energy Distribution
KG: Selftrapping \((E/L=0.75)\)
Block excitations:
Different spreading regimes

W = 4, E/L = 0.01, 0.2, 0.75
Average over 20 realizations
Summary

• We predicted theoretically and verified numerically the existence of three different dynamical behaviors:
  ✓ Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$
  ✓ Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2}$ $\rightarrow$ $m_2 \sim t^{1/3}$
  ✓ Selftrapping Regime: $\delta>\Delta$

• Generality of results: a) Two different models: KD and DNLS, b) Predictions made for DNLS are verified for both models.

• Our results suggest that Anderson localization is eventually destroyed by the slightest amount of nonlinearity, since spreading does not show any sign of slowing down.

• Questions under investigation:
  ✓ What is the actual chaotic nature of spreading?
  ✓ What is the final fate of the wave packet?