Stability of the Trojan asteroids

Ch. Skokos
Research Center for Astronomy, Academy of Athens, 14 Anagnostopoulou str.,
GR-10673, Athens, Greece

Abstract
The problem of stability of the Trojan asteroids is investigated in the light of the
Nekhoroshev theory of stability over large time intervals. We consider the two-
dimensional (2D) planar, and the three-dimensional (3D) spatial restricted three
body problem (Sun–Jupiter–asteroid) as simple models for describing the motion
of an asteroid. Using these models we find regions of effective stability around the
Lagrangian point $L_1$ such that if the initial conditions of an orbit are inside these
regions the orbit is confined in a slightly larger neighborhood of the equilibrium for
a very long time. We prove that stability over the age of the universe is guaranteed
in a realistic region, big enough to include some real asteroids. This significantly
improves previous works on the subject.

1 Introduction
Almost one century after the discovery of the first Trojan asteroid (named 588
Achilles) by Max Wolf in 1906, the problem of the stability of these asteroids
remains open. In recent years this problem has been investigated by a number of
researchers, both numerically and analytically. The numerical investigations deal
mainly, with the evolution in time of a sample of orbits, in sophisticated realistic
models of the solar system, and the statistical study of these orbits (Milani 1993,

The usual approach in analytical studies of the stability of the Trojan asteroids is
to consider simple models for the system such as the two dimensional (2D) planar,
and the three dimensional (3D) spatial restricted three body problem (RTBP)
1996). As an example of a more complicated model for the problem we refer to
the model studied by Gubern & Jorba (2001) where the effect of Saturn on the
motion of the asteroid has been taken into account. The techniques used in these
papers are based in normal forms or first integrals calculations. Roughly speaking
one shows that the system admits a number of approximate integrals, whose time
variation can be controlled to be small for an extremely long time. In this case we
have effective stability, i.e. even when an orbit is not stable, the time needed for it
to leave the neighborhood of the equilibrium is larger than the expected lifetime
of the physical system studied. This is the basis to derive the classical Nekhoroshev’s
estimates (Nekhoroshev 1977). The size of the regions of effective stability found in
the above mentioned papers was not negligible but no real asteroids were actually
found to be inside these regions.

In the present communication we pay attention to some somehow recent results
obtained by Giorgilli & Skokos (1997–hereafter paper I) and Skokos & Dok-
oumetzidis (2001–hereafter paper II), where some real asteroids were found to be
effectively stable.

2 Realistic estimations of the effective stability region
The first result that guaranties the effective stability of real asteroids was provided
in paper I for the 2D RTBP. In the RTBP the Trojan asteroids are located in
Figure 1: Curves of zero velocity for the RTBP on the plane of motion of Jupiter (J) in (a) heliocentric coordinates $Q_1$, $Q_2$ in a uniformly rotating frame with its origin on the Sun (S) and (b) polar coordinates $\rho$, $\theta$, with $Q_1 = \rho \cos \theta$, $Q_2 = \rho \sin \theta$. In both frames the position of the elliptic Lagrangian point $L_4$ is also marked.

the neighborhood of the elliptic equilibrium Lagrangian points $L_4$ and $L_5$. In 2D systems the KAM theory guarantees (under generic conditions of non-resonance and non-degeneracy) the existence of 2D tori that act as confiners for the motion, since, they separate the 3D energy surface. So trajectories initially located near
the elliptic equilibrium point and inside a 2D torus will always remain in the
neighborhood of the equilibrium point, which ensures true stability. So, the result
obtained in paper I underline the fact that Nekhoroshev theory can give meaningful
estimates, but does not take into account the possibility of the so-called Arnold
diffusion which appears only in the 3D spatial problem. In the spatial problem
the tori around the elliptic equilibrium points $L_4$ and $L_5$ are 3D while the energy
manifold is 5D. Thus the tori cannot act as barriers for the motion. So, some orbits
with initial conditions near $L_4$ (or $L_5$) can be driven to regions of the phase space
far away from it. The effective stability around $L_4$ for the 3D RTBP was studied
in paper II and one real asteroid (out of the 98 tested) was found to be inside the
estimated stability region.

Both papers were based on the construction of a normal form for the system in
appropriate coordinates and in estimations of its time variation. The not negligible
improvement of the size of the stability region achieved in these papers, compared
to older attempts, is mainly due to the use of better coordinates. The curves of
zero velocity on the plane of Jupiter’s orbit form the banana-shaped region around $L_4$ shown in figure 1(a). All previous works were based on expansions in cartesian
coordinates around $L_4$. It is evident that cartesian coordinates are not suitable to
describe regions with circular shape. On the other hand the use of polar (for the
2D problem) or cylindrical coordinates (for the 3D problem) are better candidates
as can be seen from figure 1(b).

Also the fact that the normal form was computed to higher orders than in previous
studies, both in the 3D and the 2D case, helped in improving the estimations
of the stability region’s size. In particular in the 3D case the normal form was
computed up to order 29, while in the 2D case up to order 49.

3 Summary

The basic results of papers I and II can be summarized as follows:

- For the first time the estimated size of the effective stability region is big
enough to include real asteroids: 1 asteroid in the spatial case and 4 in the planar case. In the 3D case the region where the most remote asteroid is located (out of the 98 real asteroids checked), is larger by a factor 34 compared to the estimated stability region. This result improves significantly older estimates (Giorgilli et al. 1989, Celletti & Giorgilli 1991) where no real asteroid was inside the stability region and a factor 3,000 was needed for the most remote asteroid to be inside the stability region.

- The radii of the effective stability region in the spatial and planar cases are close to each other for the same order of expansion of the normal form, with the radius computed for the spatial case being always slightly smaller. Thus, Arnold diffusion does not affect the size of the effective stability region significantly.

- The theoretical framework used in papers I and II reached the limits of its effectiveness by providing the best possible results in the 2D case, and comparable results in the 3D case. In particular the optimal order of the expansion of the normal form was found to be equal to 38 in the 2D case, although the expansion of the normal form was performed up to order 49. So the computation of the normal form to orders higher than 38 does not improve the estimations in the 2D case. In order to have a non negligible improvement of the estimated size of the effective stability region, one has to choose better coordinates than the cylindrical ones, in the sense that these coordinates should be more adapted to describe the banana-shaped region of the actual stability region around $L_4$.

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