Abstract
We study the evolution of dynamics in the Fermi-Pasta-Ulam-α model in order to classify and characterize the behavior of the system. This classification is divided into three main time intervals, called stages, in which there is a qualitative change in the dynamics of the system. We compare every stage in the FPU dynamics with those of Toda’s system. Thus we can use Toda as a tool to distinguish between the chaotic and integrable behavior in the FPU-α system.

Introduction
The one dimensional FPU-α lattice with fixed boundary conditions is described by the Hamiltonian
\[ H_{FPU}^{\alpha} = \frac{1}{2} \sum_{i=1}^{N-1} \left( p_i^2 + \sum_{k=1}^{\alpha} \left[ \frac{a}{\omega_k} (q_i - q_{i+k}) \right]^2 \right) \]
with \( x(0) = x(N) = 0 \).

The Toda lattice, described by the Hamiltonian function
\[ H_T = \frac{1}{2} \sum_{i=1}^{N-1} \omega_i |p_i|^2 - \frac{N}{4} \omega_1 \]
can be regarded as an approximation of the FPU-α Hamiltonian (1) of order \( \alpha^3 \), since
\[ H_T \approx H_{FPU}^{\alpha} - \frac{1}{2} \sum_{i=1}^{N-1} \sum_{k=1}^{\alpha} \frac{\omega_k}{\omega_1} (q_i - q_{i+k})^2. \]

At the harmonic limit \( \alpha = 0 \) is
\[ H_T = H_{FPU}^{\alpha} = H_1 = \sum_{i=1}^{N-1} E_i \]
where \( E_i \) are the harmonic energies and \( \omega_k = 2 \sin \frac{\alpha \pi}{N} \) the harmonic frequencies for both systems.

In the present work, we excite the first normal mode with \( k = 1 \) of the FPU-α and Toda systems, with initial conditions \( q_i = A \sin \frac{\alpha \pi i}{N}, q_i = 0 \), for \( i = 1, 2, \ldots, N - 1 \).

STAGE I
The harmonic energies of both systems are characterized by a sharp power law growth in time, due to acoustic resonances.

STAGE II
The harmonic energy spectrum saturates around a constant profile. FPU-α and Toda systems exhibit in Fourier space the natural packet formation, in which the total energy is exchanged among few low-frequency modes [1], and the exponential energy localization of the tail modes [2], i.e. higher-frequency modes. This stage holds for FPU-α up to the time that the energy spectra of tail modes slowly grow and the system tends to reach equipartition.

The exponential localization, of the averaged Toda energy spectra of the tail modes, are given by
\[ \log \frac{\sigma}{\| \cdot \|} = -\alpha + \theta \]
where
\[ \sigma = \frac{C_1}{\sqrt{N\alpha^{\alpha^2}}} \]
and
\[ \theta = \log \frac{C_2}{N^{\alpha^{\alpha^2}}} \]
with \( C_1 \approx 3^{\frac{1}{12}} \) and \( C_2 \approx 6 \) (See Fig. 4).

By \( \sigma \) and \( \theta \) we denote the moments and the tail energy, defined for the averaged harmonic energies \( E_i \).

The exponents of these powers laws are numerically evaluated and appear in Figs.6 and 7(a). Both of them fluctuate very strongly indicating a dense region of invariant objects. As \( E \) increases, these objects are destroyed and the system tends to equipartition linearly in time (\( \sigma \propto t \)).

Figure 6: FPU-α model with \( N = 32, \alpha = 0.33, \) Evolution of the least squares linear fit, of moments \( m_j = 1, \ldots, 16 \) versus the total energy of the system \( E \). Same line colours with Fig.5 are used. i) The same for the slope \( \gamma_j \) of moments \( m_j = 1, \ldots, 16, \)

Conclusions
The energy transfer in the FPU-α model from the lower frequency modes to the tail modes is initially very sharp, after that stops for a certain time window [3] and then starts again with a linear in time process, that leads the system to equipartition. Comparison with the Toda model shows that only the last part is due to non-integrability of FPU-α.

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References