Constrained portfolio selection with Markov and non-Markov processes and Insiders

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Abstract

Constrained portfolio selection problems in continuous-time are solved via the methods of *continuous-time stochastic dynamic programming* and *the calculus of variations*.

First a portfolio selection problem including *inequality constraints* (in the absence of transaction costs) is considered. The expected utility of terminal wealth over a finite time horizon is maximised for portfolios comprised of a money market security and diffusions. (Merton [101] solved a similar problem but implicitly considered only the equality constraint that the portfolio security weights sum to unity, for the rest of this thesis referred to as the *unity weight constraint*, and maximised the expected utility of *consumption* over a finite time horizon.) A value functional is defined and proofs of its properties are provided. Using a theorem of stochastic dynamic programming Hamilton-Jacobi-Bellman (HJB) equations are derived. Optimal portfolios are given in feedback form in terms of the solutions of the HJB equations and their partial derivatives. An analysis of the *no-constraining (NC) region* of a portfolio is conducted and an example is provided.

Second a financial market comprised of non-Markov securities driven by Lévy processes is considered. A constrained portfolio selection problem for an *insider* with a strictly increasing, concave, at least once-differentiable utility function is solved by maximising the expected utility of terminal wealth over a finite time horizon. An insider is an investor who has more information available about the disturbances in a financial market than an *honest investor*. We generalise to a multidimensional setting the models of ([18], [42], [62]) and this immediately introduces (amongst other things) an explicit unity weight constraint on the portfolio security weights which is not present in these papers. Inequality constraints are also imposed on the insider's portfolio security weights and the resulting constrained portfolio selection problem is solved via the method of calculus of variations. Some analytical solutions are derived and some problems are solved numerically.

Unless otherwise stated all results in this thesis are original. A summary of the results in Sections 2.3-2.6 (of this thesis) was published in [48].

Keywords: control theory, utility maximisation, stochastic dynamic programming, Lévy process, insider, enlargement of filtration.

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Contents

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Chapter 1

Introduction

The most important aspect of Markowitz [96] *and* Roy's [118] work was to show that it is not a security's own risk, perhaps as measured by variance, that is important to an investor, but rather the contribution the security makes to the risk of his entire portfolio. The general problem of choosing the best allocation of assets in a portfolio is called *portfolio selection theory.* [100] There are two main methodologies for the study of portfolio selection problems: one that relies heavily on the theory of *partial differential equations* (the control theoretic approach) and the other on *martingale theory.* [151] (See ([140], pp403) for a comparison of the two approaches.) The main aim of this thesis is to incorporate portfolio security weight constraints into both approaches.

In Chapter 2 (stochastic) dynamic programming is employed to solve a constrained portfolio selection problem where the risky securities are diffusions. Since dynamic programming is employed, the state variables (which in this setting are the risky securities) must be Markov, which is why these are modelled as diffusions. Not a great deal of attention has been given to continuous-time portfolio selection models with weight constraints solved by the method of dynamic programming. One of the early important contributions is a consumption-investment portfolio selection model for diffusions by Robert Merton [101] where the only constraint is a unity weight constraint. In practice portfolio managers are required to impose inequality constraints on the total or relative amount of investment in particular securities. In the presence of constraints other than the unity weight constraint, the solution obtained in [101] cannot be applied and a constrained portfolio selection model must be derived from scratch. In this chapter optimal portfolios are derived where the portfolio weights are constrained between realistic bounds.

Consumption-investment portfolio selection models with constraints have been derived in [150] where a financial market with only two securities (a log-normal risky security and a riskless bond) is considered. The control variables are the portfolio security holdings and the expectation of total utility of consumption over an infinite horizon is maximised. As explained in [150], since the security drifts are constant, short-selling constraints need not be considered.

The model in [140] is the same as that in [150] except that a nonnegativity of wealth constraint is also included. In [140] and [150] no short-selling constraints are considered but in our model we find that short-selling constraints are important in keeping the portfolio weights within realistic bounds. Control problems with constraints have also been considered in ([134], [135]) but these are deterministic control problems whereas our problem is stochastic. The model considered in Chapter 2 differs from those mentioned above in the following ways:

(i) We allow an arbitrary, finite number of securities in an investor's portfolio.

(ii) A riskless (money market) security may or may not be available for investment by the insider.

(iii) Drifts and volatilities of the portfolio securities can be stochastic processes but the portfolio securities must be Markov processes.

(iv) The *weights* of the portfolio securities are the control variables (and not the security holdings).

(v) The expected utility of terminal wealth over a finite time horizon is maximised (rather than the expected utility of consumption over an infinite time horizon).

(vi) Both buying and short-selling constraints are included in the portfolio selection model.

(vii) A positivity of wealth constraint is imposed.

From (v) above, constrained optimal portfolios are derived in Chapter 2 by defining a value functional and using a theorem from stochastic dynamic programming to derive a Hamilton-Jacobi-Bellman (HJB) equation which the value functional should satisfy. Using the *Karush-Kuhn-Tucker* conditions [149] we show that constrained optimal portfolio weights are given in feedback form in terms of the solution of the HJB equations and its partial derivatives. In an example a constrained portfolio selection problem is solved. An analysis of the no-constraining (NC) region of a portfolio is conducted and an example is provided.

In Chapter 3 a financial market comprised of non-Markov securities driven by Lévy processes is considered. The securities are non-Markov since the expected returns, volatilities and jump coefficients of the securities are path dependent. The portfolio selection model in this chapter is a generalisation of that in Chapter 2 in at least four ways viz:

(i) The risky securities are modelled as being non-Markov processes.

(ii) The logarithmic returns of the risky securities are allowed to exhibit jumps.

(iii) The hypothetical investor is modelled as having more information available to him other than that generated by the financial market - the investor is assumed to be an insider.

(iv) Particular types of investment strategy of the insider are penalised. This is accomplished by including *penalty functions* in the objective functionals of the insider portfolio selection problems solved.

A constrained portfolio selection problem for an insider with a strictly increasing, concave, at least once-differentiable utility function is solved. An insider is an investor who has more information available about the disturbances in a financial market than an honest investor. The fundamental difficulty associated with solving a portfolio selection problem for an insider in continuous time in the presence of non-deterministic disturbances, is how to interpret the resulting integrals, which in general are no longer stochastic integrals. We follow closely the models of ([18], [42], [62]) and generalise these to a multidimensional setting and this immediately introduces (amongst other things) an explicit unity weight constraint on the portfolio security weights which is not present in these papers. Inequality constraints on the insider's portfolio security weights are also imposed and the resulting constrained portfolio selection problems are solved via the method of the calculus of variations.

In ([62], [116]) optimal insider portfolios comprised of a money market security and a diffusion are derived by maximising the expected logarithmic utility of terminal wealth of the insider. In [62] penalty functions are included in the objective functional so that optimal insider portfolios are not conspicuous (relative to an optimal honest investor portfolio) and so that the objective functional maximised in [62] is finite. In [116] the authors assume a particular form for the insider's filtration and moreover assume that the Brownian motion disturbance (employed in [116]) is in fact a semimartingale with respect to the insider's filtration. In [31] optimal insider portfolios comprised of a money market security and a diffusion are derived by maximising the expected difference between the logarithmic utility of the terminal wealth of the insider and the logarithmic utility of the terminal wealth of an honest investor. In ([18], [79]) optimal insider portfolios comprised of a money market security and a diffusion are derived by maximising the expected utility of the insider's terminal wealth, where the utility function need only be concave and at least once differentiable. In [79] however the coefficients in the stochastic differential equation of the diffusion are modelled as being anticipative. In ([51], [57]) optimal portfolios comprised of a money market security and risky securities driven by independent Brownian motions and (compound) Poisson processes are derived by maximising the sum of the expected utility of intertemporal consumption and the expected utility of terminal wealth of the insider. In ([42], [78], [110]) optimal insider portfolios comprised of a money market security and a risky security driven by a Lévy process are derived by maximising the expected logarithmic utility of terminal wealth of the insider. In [110] however the coefficients in the stochastic differential equation of the risky security are assumed to be anticipative. The portfolio selection model considered in Chapter 3 differs from those mentioned above in the following ways:

(i) We allow an arbitrary, finite number of securities in an insider's portfolio.

(ii) A riskless (money market) security may or may not be available for investment by the insider.

(iii) The risky securities are driven by Lévy processes and can be non-Markov.

(iv) Drifts, volatilities and jump coefficients of the risky securities can be non-Markov processes but these must not be anticipative.

(v) The expected utility of terminal wealth over a finite time horizon is maximised (rather than the expected utility of consumption over an infinite time horizon).

(vi) Penalty functions are introduced into the objective functionals so that optimal insider portfolios are not conspicuous (relative to optimal honest investor portfolios) and so that the objective functionals maximised are finite.

(vii) In particular penalty functions are included in the case where the insider has logarithmic utility and the securities are driven by Lévy processes (with jumps).

(viii) The insider portfolio selection problem is solved for general utility where the securities are driven by Lévy processes (with jumps).

(ix) Both buying and short-selling constraints are included in the portfolio selection models.

(x) A positivity of wealth constraint is imposed.

Chapter 3 contains only theoretical results. In Chapter 4 some analytical solutions are derived and some problems are solved numerically.

1.1 The organisation of this thesis

In Chapter 2 a portfolio selection problem including inequality constraints is solved. Constrained optimal portfolios comprised of a money market security and diffusions are derived by maximising the expected utility of terminal wealth over a finite time horizon. A value functional is defined for the constrained portfolio selection problem and proofs of its properties are provided. Using a theorem of stochastic dynamic programming HJB (Hamilton-Jacobi-Bellman) equations are derived. Constrained optimal portfolios are given in feedback form in terms of the solutions of the HJB equations and their partial derivatives. An analysis of the no-constraining (NC) region of a portfolio is conducted and an example is provided.

In Chapter 3 a non-Markov financial market driven by Lévy processes is considered. A constrained portfolio selection problem for an insider with a strictly increasing, concave, at least once-differentiable utility function is solved by maximising the expected utility of terminal wealth over a finite time horizon.

We generalise to a multidimensional setting the models of ([18], [42], [62]) and this immediately introduces (amongst other things) an explicit unity weight constraint on the portfolio security weights which is not present in these papers. Inequality constraints are also imposed on the insider's portfolio security weights and the resulting constrained portfolio selection problem is solved via the method of calculus of variations. In Chapter 4 some analytical solutions are derived and some problems are solved numerically.

Chapter 5 is the Conclusion. Derivations of some equations are listed as appendices. For the rest of this thesis the following are required:

- The variable $T \in \mathbb{R}^+$ will be fixed and will denote the time horizon of the portfolio selection problems.
- All matrices will be in bold and all vectors will be columns. $\mathbf{A}^T$ will denote the transpose of the matrix $\mathbf{A}$.
- Let $\mathbf{v} := (v_1, \ldots, v_d) \in \mathbb{R}^d, d \in \mathbb{N}$. Then the matrix $diag(\mathbf{v})$ will denote a diagonal matrix with its diagonal elements equal to that of $\mathbf{v}$, in other words with $\mathbf{V} := diag(\mathbf{v})$ we have that

$$V_{ij} := \begin{cases} v_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

- Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where $\mathbb{F} := \{\mathcal{F}_t\}_{0 \le t \le T}$ is a filtration.
- The variable ω will denote a sample point of Ω.
- The symbol $:=$ will denote that the expression on the left (of this symbol) is defined as the expression on the right (of this symbol).
- The variables $N, N_S, N_B, N_q \in \mathbb{N}$ and the sets $\mathcal{N} := \{1, \ldots, N\}, \mathcal{N}_S := \{1, \ldots, N_S\}, \mathcal{N}_B := \{1, \ldots, N_B\}, \mathcal{N}_q := \{1, \ldots, N_q\}$.
- The vector $\mathbf{S} := (S_1, \ldots, S_{N_S})$ will denote N_S risky securities and where required S_0 will denote a money market security or another risky security.
- For each $i \in \mathcal{N}_S$ the variable $\pi_i = \pi_i(t, \omega), 0 \le t \le T, \omega \in \Omega$ will denote the portfolio security weight of S_i and $\boldsymbol{\pi} := (\pi_1, \ldots, \pi_{N_S})$ will denote a portfolio.
- $\mathcal{B}(\mathbb{R}^{N_S})$ are all Borel sets in $\mathbb{R}^{N_S}$.
- The symbol $\varnothing$ will denote the empty set.
- The symbol $\blacksquare$ will denote the end of a proof, theorem, proposition, lemma or corollary.
- The symbol $\blacklozenge$ will denote the end of a definition, assumption or remark.

We now progress to the main part of this thesis where constrained optimal portfolios comprised of Markov processes (in particular diffusions) are derived.

Chapter 2

Constrained portfolio selection with Markov processes

A constrained portfolio selection model is derived for diffusions where inequality constraints are imposed on the portfolio security weights. A value functional is defined for the problem of maximising the expected utility of terminal wealth over a finite time horizon and properties of the value functional are proved. Using the method of stochastic dynamic programming HJB (Hamilton-Jacobi-Bellman) equations are derived. Optimal portfolio weights are given in feedback form in terms of the solution of the HJB equations and its partial derivatives. In an example a constrained portfolio selection problem is solved. An analysis of the no-constraining (NC) region of a portfolio is conducted and an example is provided. A summary of the results in Sections 2.3-2.6 was published in [48].

2.1 Introduction

Optimal portfolios in the presence of short-selling constraints have been derived in [71] and [72]. The fundamental difference between these two papers and our model is that there the authors employ the *martingale theoretic approach* to solving constrained portfolio selection problems. The martingale theoretic approach to portfolio optimisation allows the state variables to be non-Markov processes whereas the dynamic programming approach applied in this chapter requires all state variables to be Markov. (In Chapter 3 a constrained portfolio selection model including non-Markov processes is derived.) Also in [71] the sum of the expected utility of terminal wealth and expected utility of intertemporal consumption is maximised. In [72] the same model as in [71] is used except that more general constraints are dealt with other than those relating only to short-selling. In our model we do not maximise the expected utility of intertemporal consumption and we maximise only the expected utility of terminal wealth. We do this since the future liabilities (or consumption) of an institutional portfolio, for example a portfolio managed by an asset manager, are known *a priori*, hence an optimum consumption rule need not be determined. (For example the fee agreement between the asset manager and client are defined upfront.) The inclusion of intertemporal consumption better describes the investment behaviour of an individual - see [22] and [105].

In [150] optimal portfolios with only short-selling constraints are derived by maximising the expected total utility of consumption over an infinite horizon. The investor's opportunity set if comprised of a lognormal risky security and a riskless bond. The model in [140] is the same as that in [150] except that a nonnegativity of wealth constraint is also included. In [140] and [150] no short-selling constraints are considered but in our model we find that short-selling constraints are important in keeping the portfolio weights within realistic bounds. Control problems with constraints have also been considered in ([134], [135]) but these are deterministic control problems whereas our problem is stochastic. The model considered in this chapter differs from those mentioned above in the following ways:

(i) We allow an arbitrary, finite number of securities in an investor's portfolio.

(ii) A riskless (money market) security may or may not be available for investment by the insider.

(iii) Drifts and volatilities of the portfolio securities can be stochastic processes but the portfolio securities must be Markov processes.

(iv) The *weights* of the portfolio securities are the control variables (and not the security holdings).

(v) The expected utility of terminal wealth over a finite time horizon is maximised (rather than the expected utility of consumption over an infinite time horizon).

(vi) Both buying and short-selling constraints are included in the portfolio selection model.

(vii) A positivity of wealth constraint is imposed.

The rest of this chapter is organised as follows: in Section 2.2 the financial market model is defined, in Section 2.3 the portfolio selection problem with constraints is solved, in Section 2.4 a procedure for calculating constrained optimal portfolios is provided, in Section 2.5 an example is solved and in Section 2.6 an analysis of the no-constraining (NC) region of a portfolio is conducted.

2.2 The Financial Market Model

Suppose we have a financial market comprised of risky securities $\mathbf{S} := (S_1, \ldots, S_{N_S})$ satisfying for all $0 \leq t \leq T$

$$\frac{dS_i(t)}{S_i(t)} = \xi_i(t, \mathbf{S})dt + \sigma_i(t, \mathbf{S})dB_i(t), \quad i = 1, \ldots, N_S. \tag{2.1}$$

The requirements of this financial market model are:

(i) The expected security returns $\boldsymbol{\xi} := (\xi_1, \ldots, \xi_{N_S})$ and volatilities $\boldsymbol{\sigma} := (\sigma_1, \ldots, \sigma_{N_S})$ must satisfy certain regularity conditions [117] which ensure the existence of solution of (2.1).

(ii) The N_S Brownian motions $\mathbf{B} := (B_1, \ldots, B_{N_S})$ are correlated with

$$\mathbb{E}[dB_i(t)dB_j(t)] = \rho_{ij}(t)dt, \quad i, j = 1, \ldots, N_S$$

and $\rho_{ij}(t) \in (-1, 1)$ almost surely for all $0 \leq t \leq T$ is the correlation between B_i and B_j at time t.

(iii) For all $0 \leq t \leq T$ the *covariance matrix* of the returns of the securities $\mathbf{S}$

$$\bar{\boldsymbol{\sigma}}(t) \equiv [\sigma_{ij}(t)] := [\rho_{ij}(t)\sigma_i(t)\sigma_j(t)] \tag{2.2}$$

must be positive definite. (This is actually only stated for completeness since covariance matrices are by definition positive definite.)

(iv) For all $0 \leq t \leq T$ let

$$W(t) := \sum_{i=1}^{N_S} N_i(t)S_i(t) \tag{2.3}$$

be the investor's portfolio wealth value at time t, where $N_i(t)$ is the number of units of S_i held in the portfolio at time t. For all $0 \leq t \leq T, i \in \mathcal{N}_S$ let $\pi_i(t) := \frac{N_i(t)S_i(t)}{W(t)}$ be the time-t portfolio weight of S_i. Then assuming the investor's portfolio is *self-financing* [101] the evolution of the wealth process W is described by the stochastic differential equation

$$dW = \sum_{i=1}^{N_S} \pi_i W \xi_i dt + \sum_{i=1}^{N_S} \pi_i W \sigma_i dB_i, \tag{2.4}$$

where we have suppressed the notational dependence of the variables in (2.4) for simplicity.

(v) The hypothetical investor takes prices as given, the shares of the securities are infinitely divisible, short sales are permitted with full use of the proceeds, taxes on capital gains are zero and transaction costs are zero.

For the portfolio selection problem we consider, derivative securities may also be included in the investor's opportunity set $\{S_1, \ldots, S_{N_S}\}$ where the only requirement is that the price processes of the derivatives are Markov. For example using the Greeks [63] one can show that European vanilla call options priced according to the Black-Scholes model have price processes that are Markov. In (2.4) we require that for all $0 \leq t \leq T$

$$\begin{aligned} \sum_{i=1}^{N_S} \pi_i(t) &= \Upsilon(t) \quad \text{and} & (2.5) \\ \mathbf{a}(t, W, \mathbf{S}) \leq \boldsymbol{\pi}(t) &\leq \mathbf{b}(t, W, \mathbf{S}), & (2.6) \end{aligned}$$

almost surely where $\Upsilon(t) \in \mathbb{R}$ and $\boldsymbol{\pi}(t) := (\pi_1(t), \ldots, \pi_{N_S}(t))$. In (2.6) we have that the variables

$$\begin{aligned} \mathbf{a}(t) &= \mathbf{a}(t, W, \mathbf{S}) := (a_1(t, W, \mathbf{S}), \ldots, a_{N_S}(t, W, \mathbf{S})) \quad \text{and} \\ \mathbf{b}(t) &= \mathbf{b}(t, W, \mathbf{S}) := (b_1(t, W, \mathbf{S}), \ldots, b_{N_S}(t, W, \mathbf{S})) \end{aligned}$$

are exogenously given bounds for the portfolio weights $\boldsymbol{\pi}(t)$. We also require that $a_i(t) < b_i(t)$ for all $0 \leq t \leq T, i \in \mathcal{N}_S$. Also the bounds $\mathbf{a}$ and $\mathbf{b}$ must be such that if any set of constraints (2.6) is active for $\boldsymbol{\pi}(t)$, then we must have that $\sum_{i=1}^{N_S} \pi_i(t) = \Upsilon(t)$ almost surely. In (2.5) the function Υ is almost always identically one requiring the portfolio security weights $\boldsymbol{\pi}(t)$ to sum to unity at each time t. The function Υ need not always be identically one. The reason for this is that a money market security has zero volatility and this will result in the covariance matrices $\bar{\boldsymbol{\sigma}}(t), 0 \leq t \leq T$ of the security returns being singular and the analysis in Section 2.3.3 below (where no security is assumed to be riskless) then cannot be applied. Even if we try (to incorporate riskless securities in (the portfolio selection model in Section 2.3.3 by) taking the limit as the volatilities tend to zero, we shall find that the solution is undefined. If a money market security is available for investment, then this constrained portfolio selection problem must be solved from scratch and this is done in Section 2.3.4. The only way the weight of a money market security can be explicitly constrained is to choose its weight say $\pi_0(t)$ and ensure that the weights of the risky securities $\mathbf{S}$ sum to $\Upsilon(t) = 1 - \pi_0(t) = \sum_{i=1}^{N_S} \pi_i(t)$. Since portfolio security weights must sum to Υ, the equality constraint (2.5) must always be active. This is why the function Υ and not the value 1 is the right hand side of (2.5). For asset managers there are prudential guidelines which require minimum investment in cash. It is thus important for them to be able to constrain their money market security weight. For hedge fund managers however there is less restriction on the percentage investment in cash. Managers of these funds have more freedom in the bets they take. Thus it may be more appropriate to model their money market account as a catchall security. Dealing with their money market account in this way doesn't make it possible to explicitly constrain investment in this security. (As will be seen, the fuss we make about a money security is not unwarranted. The optimality equations (2.41) and (2.66) for example, are not special cases of each other. These are respectively constrained optimal investments where a money market security is unavailable and available for investment.) For the rest of this thesis, if no *inequality* constraints (2.6) are active, then we shall refer to the resulting portfolio as an *unconstrained portfolio*. A portfolio will be referred to as *constrained* only if at least one inequality constraint in (2.6) is active.

2.3 Constrained portfolio selection problem

Let $n \in \mathbb{N}$, $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{h} := (h_1, \ldots, h_m), m \in \mathbb{N}$ and $h_i : \mathbb{R}^n \to \mathbb{R}$ for all $i = 1, \ldots, m$. Then consider the constrained optimisation problem

$$\begin{aligned} & \max_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{h}(\mathbf{x}) \leq \mathbf{0}. \end{aligned} \tag{2.7}$$

From [94] an inequality constraint $h_i(\mathbf{x}) \leq 0$ is said to be *active* (*inactive*) at a feasible point $\mathbf{x}$ if $h_i(\mathbf{x}) = 0$ ($h_i(\mathbf{x}) < 0$). The constraints that are active at a feasible point $\mathbf{x}$ restrict the domain of feasibility in neighbourhoods of $\mathbf{x}$, while the inactive constraints have no influence in the neighbourhoods of $\mathbf{x}$. So if we know *a priori* which constraints in (2.7) are active, then the resulting solution is a local maximum point of $f(\mathbf{x})$ determined by ignoring the inactive constraints and treating all other constraints as equality constraints. In other words constrained optimisation problems subject to inequality constraints are solved by solving a family of constrained optimisation problems subject to equality constraints, where the equality constraints are different combinations of active inequality constraints. This insight is crucial when dealing with inequality-constrained optimisation problems.

In this section we solve two optimisation problems in which we maximise the expected utility of terminal wealth over a finite time horizon subject to the constraints (2.5)-(2.6). We consider a money market security being unavailable and available for investment and these are Sections 2.3.3 and 2.3.4 respectively. In Section 2.3.3 it is assumed that a money market security is not available for investment and we now define the investor's set of admissible portfolios in this case.

Definition 1 (Admissible portfolios) *A set of control processes* $\boldsymbol{\pi}$ *where* $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ *for all* $0 \leq t \leq T$, *is said to be* **admissible** *(or an* **admissible portfolio***) if (2.4) has a unique solution, (2.5)-(2.6) hold and the resulting wealth process* W *is positive almost surely. We denote by* $\mathcal{P}_1$ *the set of all admissible portfolios.* ♦

We want to solve the constrained optimisation problem

$$\sup_{\boldsymbol{\pi} \in \mathcal{P}_1} E_0[U(T, W(T))], \tag{2.8}$$

where $U : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ is a predefined utility function not necessarily concave which is assumed to best describe the investor's investment preferences and $\mathbb{E}_t[\cdot]$ is a conditional expectation operator defined as

$$\mathbb{E}_{t,W,\mathbf{S}}[\cdot] = \mathbb{E}[\cdot | W(t) = W, \mathbf{S}(t) = \mathbf{S}].$$

We also require that $U(t,W) := -\infty$ for $W \le 0$ which effectively imposes a positivity of wealth constraint.

2.3.1 The Value Functional

Solving (2.8) is not trivial since we are trying to maximise over entire functions and not simply point estimates of the controls $\boldsymbol{\pi}$. Using the *Bellman Principle of Optimality* [15], the problem of finding optimal control functions over the entire period $[0,T]$ can be reduced to a general subproblem of finding optimal control functions only over the period $[t,T]$ for each $t \in [0,T]$. Then using dynamic programming which specialises in solving sequential decision problems [131], the sequence of subproblems can be solved. For $\boldsymbol{\pi} \in \mathcal{P}_1$ let

$$J^{\boldsymbol{\pi}} \;=\; J^{\boldsymbol{\pi}}(t,W,\mathbf{S}) \;:=\; \mathbb{E}_{t,W,\mathbf{S}}[U(T,W(T))] \tag{2.9}$$

where W evolves according to (2.4) using $\boldsymbol{\pi}$. Then from (2.8)-(2.9) and the Bellman Principle of Optimality [15] we define the time-t value functional as

$$J = J(t,W,\mathbf{S}) = \sup_{\boldsymbol{\pi}\in\mathcal{P}_1} J^{\boldsymbol{\pi}}(t,W,\mathbf{S}). \tag{2.10}$$

2.3.2 Properties of the value functional

First, from (2.10), if U is bounded, then J is bounded. Next we show that the value functional J is concave in its second argument if U is concave in its second argument.

Proposition 1 *$J(t,W,\mathbf{S})$ is concave in W if $U(t,x)$ is concave in its second argument and the constraints in (2.5)-(2.6) are linear in $\boldsymbol{\pi}$.*

Proof: Let $t \in [0,T],\ i \in \{1,2\}$ and let $\boldsymbol{\pi}^i := (\pi^i_1,\ldots,\pi^i_{N_S})$ be a constrained optimal portfolio over $[t,T]$ if the time-t wealth value is W^i and the time-t security prices are $\mathbf{S}$. Let $0 \le \theta \le 1$ and for all $0 \le t \le T$ let

$$W^3 := \theta W^1 + (1-\theta)W^2.$$

Consider a portfolio $\boldsymbol{\pi}$ with a time-t wealth value of W and for all $t \le u \le T$ let its time-u value be denoted by $V_u(W,\boldsymbol{\pi}) = V_u(W,\mathbf{S},\boldsymbol{\pi})$. For all $t \le u \le T, i \in \{1,2\}, k \in \mathcal{N}_S$ let

$$N^i_k(u) \;:=\; \frac{\pi^i_k(u)V_u(W^i,\boldsymbol{\pi}^i)}{S_k(u)} \tag{2.11}$$

be the optimal units variable sample paths over $[t,T]$ corresponding to the constrained optimal portfolio $\boldsymbol{\pi}^i$. Note that given (2.11), for all $t \le u \le T, k \in \mathcal{N}_S, i \in \{1,2\}$

$$\pi^i_k(u) \;=\; \frac{N^i_k(u)S_k(u)}{V_u(W^i,\boldsymbol{\pi}^i)} \;=\; \frac{N^i_k(u)S_k(u)}{\sum_{j=1}^{N_S} N^i_j(u)S_j(u)}. \tag{2.12}$$

Next, for all $t \leq u \leq T, k \in \mathcal{N}_S$ let

$$\bar{N}_k(u) \quad := \quad \theta N_k^1(u) + (1-\theta)N_k^2(u) \tag{2.13}$$

Then we have from (2.13) that

$$V_u(W^3, \bar{\boldsymbol{\pi}}) \quad = \quad \theta V_u(W^1, \boldsymbol{\pi}^1) + (1-\theta)V_u(W^2, \boldsymbol{\pi}^2), \tag{2.14}$$

where $\bar{\boldsymbol{\pi}} := (\bar{\pi}_1, \ldots, \bar{\pi}_{N_S})$ and for all $t \leq u \leq T, k \in \mathcal{N}_S$ we have (as in (2.12)) that $\bar{\pi}_k(u) = \frac{\bar{N}_k(u)S_k(u)}{V_u(W^3, \bar{\boldsymbol{\pi}})}$. Since U is concave in its second argument we have from (2.14) with $u = T$ that

$$U(T, V_T(W^3, \bar{\boldsymbol{\pi}})) \;\geq\; \theta U(T, V_T(W^1, \boldsymbol{\pi}^1)) + (1-\theta)U(T, V_T(W^2, \boldsymbol{\pi}^2)). \tag{2.15}$$

Taking time-t expectations in (2.15) and using the fact that $\boldsymbol{\pi}^i, i \in \{1,2\}$ are constrained optimal portfolios, we get that

$$\mathbb{E}_t[U(T, V_T(W^3, \bar{\boldsymbol{\pi}}))] \;\geq\; \theta J(t, W^1, \mathbf{S}) + (1-\theta)J(t, W^2, \mathbf{S}). \tag{2.16}$$

Since $J(t, W^3, \mathbf{S}) \geq \mathbb{E}_t[U(T, V_T(W^3, \bar{\boldsymbol{\pi}}))]$ inequality (2.16) reduces to

$$J(t, W^3, \mathbf{S}) \quad \geq \quad \theta J(t, W^1, \mathbf{S}) + (1-\theta)J(t, W^2, \mathbf{S})$$

which proves the proposition. ■

The following corollary follows from the above proposition.

Corollary 1 *For all $0 \leq t \leq T$ the second partial derivative $\frac{\partial^2}{\partial W^2}J(t, W, \mathbf{S})$ is nonpositive almost surely where it is defined if U is concave in its second argument.* ■

We now show that $J(t, W, \mathbf{S})$ is concave in $\mathbf{S}$ if U is concave in its second argument.

Proposition 2 *$J(t, W, \mathbf{S})$ is concave in $\mathbf{S}$ if U is concave in its second argument and the constraints in (2.5)-(2.6) are linear in $\boldsymbol{\pi}$.*

Proof: Let $t \in [0, T]$, $i \in \{1, 2\}$ and let $\boldsymbol{\pi}^i := (\pi_1^i, \ldots, \pi_{N_S}^i)$ be a constrained optimal portfolio over $[t, T]$ if the time-t prices of $\mathbf{S}$ are $\mathbf{S}^i := (S_1^i, \ldots, S_{N_S}^i)$ and the time-t wealth value is W. If the time-t prices of $\mathbf{S}$ are $\mathbf{S}^i$, then for all $t \leq u \leq T$ let the resulting prices of $\mathbf{S}$ be denoted by $\mathbf{S}^i(u)$. (So in particular we have that $\mathbf{S}^i(t) = \mathbf{S}^i$.) Let $0 \leq \theta \leq 1$ and for all $t \leq u \leq T$ let

$$\mathbf{S}^3(u) := \theta \mathbf{S}^1(u) + (1-\theta)\mathbf{S}^2(u).$$

Consider a portfolio $\boldsymbol{\pi}$ with a time-t wealth value of W, time-t security prices of $\mathbf{S}$ and for all $t \leq u \leq T$ let its time-u value be denoted by $V_u(W, \mathbf{S}, \boldsymbol{\pi})$. For all $t \leq u \leq T, i \in \{1, 2\}, k \in \mathcal{N}_S$ let

$$N_k^i(u) \quad := \quad \frac{\pi_k^i(u)V_u(W, \mathbf{S}^i, \boldsymbol{\pi}^i)}{S_k^i(u)} \tag{2.17}$$

be the optimal units variable sample paths over $[t, T]$ corresponding to the constrained optimal portfolio $\boldsymbol{\pi}^i$. Note that given (2.17), for all $t \leq u \leq T, k \in \mathcal{N}_S, i \in \{1, 2\}$

$$\pi_k^i(u) \;=\; \frac{N_k^i(u) S_k^i(u)}{V_u(W, \mathbf{S}^i, \boldsymbol{\pi}^i)} \;=\; \frac{N_k^i(u) S_k^i(u)}{\sum_{j=1}^{N_S} N_j^i(u) S_j^i(u)}.$$

Next, for all $t \leq u \leq T$ let

$$\bar{N}_k(u) \;=\; \frac{\theta N_k^1(u) S_k^1(u) + (1-\theta) N_k^2(u) S_k^2(u)}{S_k^3(u)}. \tag{2.18}$$

Then we have from (2.18) that for all $t \leq u \leq T$

$$V_u(W, \mathbf{S}^3, \bar{\boldsymbol{\pi}}) \;=\; \theta V_u(W, \mathbf{S}^1, \boldsymbol{\pi}^1) + (1-\theta) V_u(W, \mathbf{S}^2, \boldsymbol{\pi}^2), \tag{2.19}$$

where $\bar{\boldsymbol{\pi}} := (\bar{\pi}_1, \ldots, \bar{\pi}_{N_S})$ and for all $t \leq u \leq T, k \in \mathcal{N}_S$ we have that $\bar{\pi}_k(u) = \frac{\bar{N}_k(u) S_k(u)}{V_u(W, \mathbf{S}^3, \bar{\boldsymbol{\pi}})}$. Since U is concave in its second argument we have from (2.19) with $u = T$ that

$$U(T, V_T(W, \mathbf{S}^3, \bar{\boldsymbol{\pi}})) \;\geq\; \theta U(T, V_T(W, \mathbf{S}^1, \boldsymbol{\pi}^1)) + (1-\theta) U(T, V_T(W, \mathbf{S}^2, \boldsymbol{\pi}^2)). \tag{2.20}$$

Taking time-t expectations in (2.20) and using the fact that $\boldsymbol{\pi}^i, j \in \{1, 2\}$ are constrained optimal portfolios, we get that

$$\mathbb{E}_t[U(T, V_T(W, \mathbf{S}^3, \bar{\boldsymbol{\pi}}))] \;\geq\; \theta J(t, W, \mathbf{S}^1, \boldsymbol{\pi}^1) + (1-\theta) J(t, W, \mathbf{S}^2, \boldsymbol{\pi}^2). \tag{2.21}$$

Since $J(t, W, \mathbf{S}^3) \geq \mathbb{E}_t[U(T, V_T(W, \mathbf{S}^3, \bar{\boldsymbol{\pi}}))]$ inequality (2.21) reduces to

$$J(t, W, \mathbf{S}^3) \;\;\geq\;\; \theta J(t, W, \mathbf{S}^1) + (1-\theta) J(t, W, \mathbf{S}^2)$$

which proves the proposition. ■

The following corollary follows from the above proposition.

Corollary 2 *For each $i \in \mathcal{N}_S$ the second partial derivative $\frac{\partial^2}{\partial S_i^2} J(t, W, \mathbf{S})$, $0 \leq t \leq T$ is nonpositive almost surely where it is defined if U is concave in its second argument.* ■

The following corollary follows from Propositions 1 and 2.

Corollary 3 *For fixed $t \in [0, T]$, $J(t, W, \mathbf{S})$ is continuous in W and $\mathbf{S}$.* ■

We now show that $J(t, W, \mathbf{S})$ satisfies a *Homothetic Property*.

Theorem 1 (The Homothetic Property) *Let $i \in \{1,2\}$, $\rho, x \in \mathbb{R}^+$ and let $k(t), C_i(t,x)$ be real-valued functions. Then if U satisfies*

$$U(t, k(t)x) \quad = \quad C_1(t, k(t)) + C_2(t, k(t))U(t,x) \quad \text{for all } 0 \leq t \leq T, \tag{2.22}$$

then J satisfies the homothetic property

$$J(t, \rho W, \rho \mathbf{S}) = C_1(T,\rho) + C_2(T,\rho) J(t, W, \mathbf{S}).$$

Proof: For all $0 \leq t \leq T$ let $\bar{\mathbf{S}}(t) := \rho \mathbf{S}(t)$ and $\bar{W}(t) := \rho W(t)$. Then from (2.10) and (2.22) we have that

$$\begin{aligned} J(t, \bar{W}, \bar{\mathbf{S}}) &= \sup_{\boldsymbol{\pi} \in \mathcal{P}_1} \mathbb{E}_t[U(T, \bar{W}(T)) \mid \bar{W}(t) = \bar{W}, \bar{\mathbf{S}}(t) = \bar{\mathbf{S}}] \\ &= \sup_{\boldsymbol{\pi} \in \mathcal{P}_1} \mathbb{E}_t[U(T, \rho W(T)) \mid W(t) = W, \mathbf{S}(t) = \mathbf{S}] \\ &= C_1(T,\rho) + C_2(T,\rho) J(t, W, \mathbf{S}). \end{aligned}$$

■

If $U(t,x) = \frac{x^\gamma}{\gamma},\ x, \gamma \in \mathbb{R}^+$, then J satisfies the homothetic property

$$J(t, \rho W, \rho \mathbf{S}) \quad = \quad \rho^\gamma J(t, W, \mathbf{S}).$$

If $U(t,x) = \log x,\ x \in \mathbb{R}^+$, then J satisfies the homothetic property

$$J(t, \rho W, \rho \mathbf{S}) \quad = \quad \log \rho + J(t, W, \mathbf{S}).$$

As in [101] the constrained portfolio selection problems where a (riskless) money market security is unavailable and available for investment, must be solved in slightly different ways. The reasons for this is that (i) the invertibility of the covariance matrices $\bar{\boldsymbol{\sigma}}(t), 0 \leq t \leq T$ (defined in (2.2)) is required for the tractability of the solution methodology and (ii) a money market security cannot be constrained if it is included in the investor's opportunity set. A money market security has zero volatility and this will result in the covariance matrices being singular. Thus we first solve the constrained portfolio selection problem where a money market security is unavailable for investment (Section 2.3.3) and then we solve the constrained portfolio selection problem where it is available for investment (Section 2.3.4).

2.3.3 Solving the constrained optimisation problem (2.8): No investment in a money market security

Let $\boldsymbol{\pi} \in \mathcal{P}_1$ and let the operator

$$\mathcal{L}_1[\cdot; t, W, \mathbf{S}] \quad := \quad \frac{\partial}{\partial t} + \sum_{i=1}^{N_S} \xi_i \pi_i W \frac{\partial}{\partial W} + \sum_{i=1}^{N_S} \xi_i S_i \frac{\partial}{\partial S_i} + \frac{1}{2} \sum_{i=1}^{N_S} \sum_{j=1}^{N_S} \bar{\sigma}_{ij} \pi_i \pi_j W^2 \frac{\partial^2}{\partial W^2}$$

$$+\frac{1}{2}\sum_{i=1}^{N_S}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}S_iS_j\frac{\partial^2}{\partial S_i\partial S_j}+\sum_{i=1}^{N_S}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}S_i\pi_jW\frac{\partial^2}{\partial S_i\partial W} \tag{2.23}$$

be the *differential generator [82] of the processes* W and $\mathbf{S}$. Let

$$\phi_1(\boldsymbol{\pi};t,W,\mathbf{S}):=\mathcal{L}_1[J^{\boldsymbol{\pi}};t,W,\mathbf{S}] \tag{2.24}$$

where $J^{\boldsymbol{\pi}}$ is defined in (2.9). From the theory of stochastic dynamic programming the following theorem provides a method for deriving constrained optimal controls (in this case portfolios) $\boldsymbol{\pi}$.

Theorem 2 *If the variables* $(W,\mathbf{S})$ *are Markov processes, then there exists a set of constrained optimal controls* $\boldsymbol{\pi}^*$ *satisfying*

$$0 \;=\; \phi_1(\boldsymbol{\pi}^*;t,W,\mathbf{S}) \;\geq\; \phi_1(\boldsymbol{\pi};t,W,\mathbf{S}) \quad \text{for all } \; 0\leq t\leq T.$$

Proof: See ([82], Chapter 4, Theorem 5) or ([7], Theorem 3.5.2). ■

In (2.10) it was important to define correctly the arguments of J. If we posited $J=J(t,W)$, then if we derived the corresponding HJB equation for J, then we would find that this equation contains more variables other than simply t and W. The variables $\mathbf{S}$ enter into the HJB equation for J since W is dependent on the drifts and volatilities of $\mathbf{S}$ which are functions of $\mathbf{S}$. This would imply that J is not a function of only t and W but $J=J(t,W,\mathbf{S})$. This is why $\mathcal{L}_1$ has the form in (2.23) - it includes partial differentiation with respect to $\mathbf{S}$. From Theorem 2 we have the HJB equation

$$\max_{\boldsymbol{\pi}\in\mathcal{P}_1}\mathcal{L}_1[J^{\boldsymbol{\pi}}] \quad = \quad 0 \tag{2.25}$$

almost surely which is equivalent to the problem

$$\max_{\boldsymbol{\pi}}\phi_1(\boldsymbol{\pi}) \tag{2.26}$$

$$\text{subject to} \qquad \sum_{i=1}^{N_S}\pi_i(t)=\Upsilon(t) \tag{2.27}$$

$$\mathbf{a}(t)\leq\boldsymbol{\pi}(t)\leq\mathbf{b}(t) \tag{2.28}$$

such that $\max_{\boldsymbol{\pi}}\phi_1(\boldsymbol{\pi})=0$ almost surely where $\boldsymbol{\pi}(t)\in\mathbb{R}^{N_S}$ for all $0\leq t\leq T$. From (2.23) the functional ϕ_1 is quadratic in $\boldsymbol{\pi}$, thus since the covariance matrix $\bar{\boldsymbol{\sigma}}$ is positive definite (from (2.2)), a sufficient condition for a unique global and local maximum of ϕ_1 is that $J_{WW}<0$. We want to find a local maximum of ϕ_1 satisfying the constraints (2.27)-(2.28) and we use the *Karush-Kuhn-Tucker conditions* (Appendix A) to do this. Thus suppose $\boldsymbol{\pi}^*$ is an optimal solution of (2.26)-(2.28). Then from [149] there must exist multipliers $\lambda\in\mathbb{R}^+$, $\boldsymbol{\mu}:=(\mu_1,\ldots,\mu_{N_S})$ and $\bar{\boldsymbol{\mu}}:=(\bar{\mu}_1,\ldots,\bar{\mu}_{N_S}),\mu_i,\bar{\mu}_i\in\mathbb{R}^+$ for all $i\in\mathcal{N}_S$ such that from

(2.26)-(2.28) we have that

$$
\begin{aligned}
\frac{\partial \phi_1(\boldsymbol{\pi}^*)}{\partial \pi_k} - \lambda \frac{\partial}{\partial \pi_k}\left(\sum_{i=1}^{N_S} \pi_i^* - \Upsilon\right) & \\
- \sum_{i=1}^{N_S} \mu_i \frac{\partial}{\partial \pi_k}(\pi_i^* - b_i) - \sum_{j=1}^{N_S} \bar{\mu}_j \frac{\partial}{\partial \pi_k}(-\pi_j^* + a_j) & = 0, \qquad k = 1, \ldots, N_S, \\
\lambda\left(\Upsilon - \sum_{i=1}^{N_S} \pi_i^*\right) & = 0, \\
\mu_i \left[\pi_i^* - b_i\right] & = 0, \qquad i = 1, \ldots, N_S \\
\bar{\mu}_j \left[-\pi_j^* + a_j\right] & = 0, \qquad j = 1, \ldots, N_S \\
(\lambda, \boldsymbol{\mu}, \bar{\boldsymbol{\mu}}) & \geq \mathbf{0},
\end{aligned} \tag{2.29}
$$

where we have suppressed the dependence on time for simplicity. Now $\phi_1(\boldsymbol{\pi}^*) = \mathcal{L}_1[J]$ so the system (2.29) reduces to

$$
\begin{aligned}
\xi_k W J_W + W^2 J_{WW} \sum_{m=1}^{N_S} \bar{\sigma}_{km} \pi_m^* & \\
+ W \sum_{i=1}^{N_S} \bar{\sigma}_{ki} S_i J_{iW} - \lambda - \mu_k + \bar{\mu}_k &= 0, \quad k = 1, \ldots, N_S, \qquad (2.30) \\
\lambda\left(\Upsilon - \sum_{i=1}^{N_S} \pi_i^*\right) &= 0, \qquad (2.31) \\
\mu_i[\pi_i^* - b_i] &= 0, \quad i = 1, \ldots, N_S, \qquad (2.32) \\
\bar{\mu}_j[-\pi_j^* + a_j] &= 0, \quad j = 1, \ldots, N_S, \qquad (2.33) \\
(\lambda, \boldsymbol{\mu}, \bar{\boldsymbol{\mu}}) &\geq \mathbf{0}, \qquad (2.34)
\end{aligned}
$$

where $J_W := \frac{\partial J}{\partial W}$ and $J_i := \frac{\partial J}{\partial S_i}, i \in \mathcal{N}_S$. Solving for π_i^* in (2.30), substituting π_i^* into (2.31), solving for λ, then substituting λ into (2.30), results in[1]

$$
\begin{aligned}
\pi_i^* \quad = \quad & \frac{\Upsilon}{\Gamma} \sum_{k=1}^{N_S} \nu_{ki} + \sum_{k=1}^{N_S} \nu_{ki} \left[\frac{1}{W^2 J_{WW}} \left(\mu_k - \bar{\mu}_k - \frac{1}{\Gamma} \sum_{m=1}^{N_S} \sum_{n=1}^{N_S} \nu_{nm}(\mu_n - \bar{\mu}_n) \right) \right. \\
& \left. + \frac{J_W}{W J_{WW}} \left(\frac{1}{\Gamma} \sum_{m=1}^{N_S} \sum_{n=1}^{N_S} \xi_n \nu_{nm} - \xi_k \right) + \frac{1}{W J_{WW}} \left(\frac{1}{\Gamma} \sum_{p=1}^{N_S} M_p - \sum_{j=1}^{N_S} \bar{\sigma}_{kj} M_j \right) \right],
\end{aligned} \tag{2.35}
$$

where $\boldsymbol{\nu} \equiv [\nu_{ij}] := \bar{\boldsymbol{\sigma}}^{-1}$, $\Gamma := \sum_{a=1}^{N_S} \sum_{b=1}^{N_S} \nu_{ab}$ and $M_i := J_{iW} S_i$. To determine the values of $\boldsymbol{\pi}^*$ in (2.35) we need to solve for the values of the unobservable multipliers $\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$.

[1]See Appendix B for the details.

Recall from the introduction of Section 2.3 that an inequality-constrained optimisation problem is solved by solving a family of equality-constrained optimisation problems (where the equality constraints are different combinations of active inequality constraints). Thus we do the following. Now from (2.6) we have that $a_i(t) < b_i(t)$ for all $0 \leq t \leq T, i \in \mathcal{N}_S$. Thus for any portfolio weight $\pi_l, l \in \mathcal{N}_S$, the expression $\mu_l - \bar{\mu}_l$ always reduces to μ_l^* where

$$\mu_l - \bar{\mu}_l =: \mu_l^* := \begin{cases} \mu_l & \text{if } \pi_l^* \leq b_l(t) \text{ is active (and } -\pi_l^* \leq -a_l(t) \text{ inactive),} \\ -\bar{\mu}_l & \text{if } -\pi_l^* \leq -a_l(t) \text{ is active (and } \pi_l^* \leq b_l(t) \text{ inactive),} \\ 0 & \text{otherwise.} \end{cases} \tag{2.36}$$

So (2.35) can be rewriten as

$$\pi_i^* = \hat{G}_i + \frac{1}{W^2 J_{WW}} \left(\sum_{k \in \mathcal{C}^*} \nu_{ki} \mu_k^* - \hat{G}_i \sum_{n \in \mathcal{C}^*} \nu_n \mu_n^* \right) + \hat{M}_i \tag{2.37}$$

where

$$\begin{aligned} \nu_i &:= \sum_{k=1}^{N_S} \nu_{ki}, \\ \hat{G}_i &:= \Upsilon \frac{\nu_i}{\Gamma} \quad \text{and} \\ \hat{M}_i &:= \frac{J_W}{W J_{WW}} \left(\frac{\nu_i}{\Gamma} \sum_{n=1}^{N_S} \xi_n \nu_n - \sum_{k=1}^{N_S} \xi_k \nu_{ki} \right) + \frac{1}{W J_{WW}} \left(\frac{\nu_i}{\Gamma} \sum_{p=1}^{N_S} M_p - M_i \right). \end{aligned}$$

In (2.37) the set $\mathcal{C}$ ($\bar{\mathcal{C}}$) is the index set of active upper (lower) bound constraints and $\mathcal{C} \cup \bar{\mathcal{C}} =: \mathcal{C}^* := \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, where m is the number of active inequality constraints. Each number $\alpha_j \in \mathcal{C}^*$ denotes that an inequality constraint of security S_{α_j} is active. So if we have a universe of 5 securities and only the upper (lower) bound constraint of π_1 and the lower (upper) bound constraint of π_4 are active, then $\mathcal{C}^* = \{1, 4\}$ in both cases. If $\sum_{a \in \mathcal{C}^*} B_a$ is defined to be zero for $\mathcal{C}^*$ empty and some expression B_a, then our derived equations reduce to the corresponding equations in [101]. Also if for all $i \in \mathcal{N}_S$ the bounds a_i and b_i tend to $-\infty$ and $+\infty$ respectively, then the number of elements in $\mathcal{C}^*$ decreases since fewer control variables $\boldsymbol{\pi}$ will hit the boundaries $\mathbf{a}$ and $\mathbf{b}$. Thus if for all $i \in \mathcal{N}_S$ the bounds a_i and b_i tend to $-\infty$ and $+\infty$ respectively, then optimal solutions of the constrained optimisation problem (2.26)-(2.28) tend toward optimal solutions of the unconstrained optimisation problem (2.26)-(2.27). From (2.6), since $\mathbf{a} < \mathbf{b}$ the upper and lower weight constraints of no security can be active at the same time.

For each $i \in \mathcal{N}_S$ we represent the two inequality constraints $\pi_i^* \leq b_i$ and $-\pi_i^* \leq -a_i$ in (2.28) in a more compact form as

$$(-1)^{d_i} \pi_i^* \leq c_i, \quad i = 1, \ldots, N_S, \tag{2.38}$$

where

$$\left\{ \begin{array}{l} c_i = b_i \\ d_i = 0 \end{array} \right\} \qquad \text{for the constraint } \pi_i^* \le b_i \quad \text{and}$$

$$\left\{ \begin{array}{l} c_i = -a_i \\ d_i = 1 \end{array} \right\} \qquad \text{for the constraint } -\pi_i^* \le -a_i.$$

Note that we have still have two constraints in (2.38), but we represent these as such because of the observation in (2.36). Thus we can rewrite (2.32)-(2.33) in a more compact form as

$$\mu_i^*[(-1)^{d_i}\pi_i^* - c_i] = 0, \qquad i = 1, \dots, N_S. \tag{2.39}$$

Substituting (2.37) into (2.39) and solving for the nonzero multipliers $\boldsymbol{\mu}^*$, we find that[2]

$$\boldsymbol{\mu}^* = W^2 J_{WW} \boldsymbol{\Psi}^{-1} \left[\bar{\mathbf{c}} - \hat{\mathbf{G}} - \hat{\mathbf{M}} \right], \tag{2.40}$$

where all vectors in (2.40) are of length m (the number of active inequality constraints), $\boldsymbol{\Psi}$ is an $m \times m$ matrix,

$$\begin{array}{lllll} \boldsymbol{\Psi} & \equiv & [\Psi_{jk}] & := & \left[\nu_{\alpha_k \alpha_j} - \frac{\nu_{\alpha_k} \nu_{\alpha_j}}{\Gamma} \right], \qquad \bar{c}_i \;:=\; (-1)^{-d_i} c_i, \\ \hat{\mathbf{G}} & \equiv & [\hat{G}_{\alpha_j}], & & \qquad \hat{\mathbf{M}} \;\equiv\; [\hat{M}_{\alpha_j}] \end{array}$$

and the invertibility of $\boldsymbol{\Psi}$ can be verified before projecting the model from the current time to the next. Since $\boldsymbol{\Psi}$ is not the covariance matrix, it is not necessarily invertible.

For all $i, j \in \{1, \dots, m\}$ let ς_{ij} denote the elements of $\boldsymbol{\Psi}^{-1}$. Continuing with our main derivation, if we substitute (2.40) into (2.37), then we get that[3]

$$\pi_i^* = C_i + E_i, \tag{2.41}$$

where $C_i := \hat{G}_i + \hat{M}_i$ and

$$\begin{aligned} R_i &:= \sum_{k,c \in \mathcal{C}^*} \nu_{ki} \varsigma_{kc} (\bar{c}_c - \hat{G}_c), \\ \bar{R} &:= \sum_{n,d \in \mathcal{C}^*} \nu_n \varsigma_{nd} (\bar{c}_d - \hat{G}_d), \\ N_i &:= \sum_{k,c \in \mathcal{C}^*} \nu_{ki} \varsigma_{kc} \left(\frac{\nu_c}{\Gamma} \sum_{n=1}^{N_S} \xi_n \nu_n - \sum_{m=1}^{N_S} \xi_m \nu_{mc} \right), \\ O_i &:= \sum_{k,c \in \mathcal{C}^*} \nu_{ki} \varsigma_{kc} \left(\frac{\nu_c}{\Gamma} \sum_{p=1}^{N_S} M_p - M_c \right), \\ \bar{N} &:= \sum_{n,d \in \mathcal{C}^*} \nu_n \varsigma_{nd} \left(\frac{\nu_d}{\Gamma} \sum_{b=1}^{N_S} \xi_b \nu_b - \sum_{q=1}^{N_S} \xi_q \nu_{qd} \right), \end{aligned}$$

[2]See Appendix C for the details.
[3]See Appendix D for the details.

$$
\begin{aligned}
\bar{O} &:= \sum_{n,d\in\mathcal{C}^*} \nu_n \varsigma_{nd} \left(\frac{\nu_d}{\Gamma} \sum_{e=1}^{N_S} M_e - M_d \right), \\
\hat{R}_i &:= R_i - \frac{\nu_i}{\Gamma}\bar{R}, \\
\hat{N}_i &:= N_i - \frac{\nu_i}{\Gamma}\bar{N}, \\
\hat{O}_i &:= O_i - \frac{\nu_i}{\Gamma}\bar{O} \quad \text{and} \\
E_i &:= \hat{R}_i - \frac{J_W \hat{N}_i}{W J_{WW}} - \frac{\hat{O}_i}{W J_{WW}}.
\end{aligned} \tag{2.42}
$$

From (2.41) we see that constrained optimal portfolios $\boldsymbol{\pi}^*$ are given in feedback form in terms of the solution J of the HJB equation (2.25) and its partial derivatives. Also a constrained optimal portfolio $\boldsymbol{\pi}$ of problem (2.8) is the unconstrained portfolio $C_i(t), i \in \mathcal{N}_S$ plus the terms present if $\boldsymbol{\pi}(t)$ are constrained. By inspection of E_i, if $\mathcal{C}^*$ is empty (in other words no inequality constraints (2.28) are active), then E_i is zero for all $i = 1, \ldots, N_S$ and (2.41) is exactly ([101], equation (25)). Substituting (2.41) into (2.25) we obtain the HJB equation

$$
0 = M^e + \sum_{i=1}^{N_S} \xi_i E_i W J_W + \frac{1}{2} \sum_{i=1}^{N_S} \sum_{j=1}^{N_S} \bar{\sigma}_{ij} (E_i C_j + C_i E_j + E_i E_j) W^2 J_{WW} + \sum_{i=1}^{N_S} \sum_{j=1}^{N_S} \bar{\sigma}_{ij} S_i E_j W J_{iW}, \tag{2.43}
$$

where

$$
\begin{aligned}
M^e \quad := \quad & J_t + \sum_{i=1}^{N_S} \xi_i S_i J_i + \frac{1}{2} \sum_{i,j=1}^{N_S} \bar{\sigma}_{ij} S_i S_j J_{ij} + \frac{W}{\Gamma} \sum_{j=1}^{N_S} M_j + W J_W \sum_{k=1}^{N_S} \xi_k \hat{G}_k \\
& + \frac{W^2 J_{WW}}{2\Gamma} - \frac{1}{2J_{WW}} \left(\sum_{j,m=1}^{N_S} \bar{\sigma}_{jm} M_j M_m - \Gamma \left(\sum_{i=1}^{N_S} M_i \right)^2 \right) \\
& - \frac{J_W}{J_{WW}} \left(\sum_{k=1}^{N_S} \xi_k M_k - \sum_{j,l=1}^{N_S} M_j \xi_l \hat{G}_l \right) - \frac{J_W^2}{2J_{WW}} \left(\sum_{k,l=1}^{N_S} \xi_k \xi_l \nu_{kl} - \Gamma \left(\sum_{l=1}^{N_S} \xi_l \nu_l \right)^2 \right).
\end{aligned}
$$

Substituting for C_k and E_k (defined in (2.41) and (2.42) respectively) in (2.43) it becomes[4]

$$
\begin{aligned}
0 \quad = \quad & M^e + W \sum_{i,j=1}^{N_S} \bar{\sigma}_{ij} \left(S_i \hat{R}_j J_{iW} + \frac{1}{2} Q_{ij}^O \right) + W J_W \sum_{i=1}^{N_S} \left(\xi_i \hat{R}_i + \frac{1}{2} \sum_{j=1}^{N_S} \bar{\sigma}_{ij} Q_{ij}^N \right) \\
& - \frac{1}{J_{WW}} \sum_{i,j=1}^{N_S} \bar{\sigma}_{ij} \left(S_i \hat{O}_j J_{iW} - \frac{1}{2} P_{ij}^O \right) - \frac{J_W^2}{J_{WW}} \sum_{i=1}^{N_S} \left(\xi_i \hat{N}_i - \frac{1}{2} \sum_{j=1}^{N_S} \bar{\sigma}_{ij} P_{ij}^N \right)
\end{aligned}
$$

[4]See Appendix E for the details.

$$-\frac{J_W}{J_{WW}}\sum_{i=1}^{N_S}\left(\xi_i\hat{O}_i+\sum_{j=1}^{N_S}\bar{\sigma}_{ij}\left(S_i\hat{N}_jJ_{iW}-\frac{1}{2}N_{ij}^O\right)\right)+\frac{1}{2}W^2J_{WW}\sum_{i,j=1}^{N_S}\bar{\sigma}_{ij}G_{ij}^R, \tag{2.44}$$

subject to the boundary condition $J(t, W, \mathbf{S}) = U(T, W(T))$. In (2.44) for $i = 1, \dots, N_S$ the variables

$$\begin{aligned}
\tilde{N}_i &:= \frac{\nu_i}{\Gamma}\sum_{n=1}^{N_S}\xi_n\nu_n-\sum_{k=1}^{N_S}\xi_k\nu_{ki},\\
\tilde{O}_i &:= \frac{\nu_i}{\Gamma}\sum_{p=1}^{N_S}M_p-M_i,\\
G_{ij}^R &:= \left(\hat{R}_i\hat{G}_j+\hat{G}_i\hat{R}_j+\hat{R}_i\hat{R}_j\right),\\
P_{ij}^V &:= \left(-\hat{V}_i\tilde{V}_j-\tilde{V}_i\hat{V}_j+\hat{V}_i\hat{V}_j\right)\\
Q_{ij}^V &:= \left(\hat{R}_i\tilde{V}_j-\hat{V}_i\hat{G}_j-\hat{G}_i\hat{V}_j+\tilde{V}_i\hat{R}_j-\hat{R}_i\hat{V}_j-\hat{V}_i\hat{R}_j\right) \text{ and}\\
N_{ij}^O &:= \left(-\hat{N}_i\tilde{O}_j-\hat{O}_i\tilde{N}_j-\tilde{N}_i\hat{O}_j-\tilde{O}_i\hat{N}_j+\hat{N}_i\hat{O}_j+\hat{O}_i\hat{N}_j\right),
\end{aligned}$$

where V is a variable taking values in the set of letters $\{N, O\}$. If no inequality constraints in (2.28) are active, then (2.44) reduces to

$$0 = M^e. \tag{2.45}$$

Suppose some subset $\mathcal{C}^*$ of the inequality constraints (2.28) are active. From (2.10) J is explicitly dependent on the wealth W, and in (2.4) we see that the wealth is explicitly dependent on the drifts and volatilities of the securities $\mathbf{S}$. If however the expected returns $\boldsymbol{\xi}$ and volatilities $\boldsymbol{\sigma}$ of all securities $\mathbf{S}$ are not dependent on (the security prices) $\mathbf{S}$, then the variables $\mathbf{S}$ do not appear in the HJB equation (2.25) for J. In this case $J = J(t, W)$ so (2.44) reduces to

$$\begin{aligned}
0 &= J_t+WJ_W\left[\sum_{k=1}^{N_S}\xi_k\hat{G}_k+\sum_{i=1}^{N_S}\left(\xi_i\hat{R}_i+\frac{1}{2}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}Q_{ij}^N\right)\right]\\
&\quad-\frac{J_W^2}{J_{WW}}\left[\frac{1}{2}\left(\sum_{k,l=1}^{N_S}\xi_k\xi_l\nu_{kl}-\Gamma\left(\sum_{l=1}^{N_S}\xi_l\nu_l\right)^2\right)+\sum_{i=1}^{N_S}\left(\xi_i\hat{N}_i-\frac{1}{2}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}P_{ij}^N\right)\right]\\
&\quad+\frac{1}{2}W^2J_{WW}\left(\frac{1}{\Gamma}+\sum_{i,j=1}^{N_S}\bar{\sigma}_{ij}G_{ij}^R\right)
\end{aligned} \tag{2.46}$$

and (2.41) reduces to

$$\pi_i^* = \hat{G}_i+\frac{J_W}{WJ_{WW}}\left(\frac{\nu_i}{\Gamma}\sum_{n=1}^{N_S}\xi_n\nu_n-\sum_{k=1}^{N_S}\xi_k\nu_{ki}\right)+\hat{R}_i-\frac{J_W\hat{N}_i}{WJ_{WW}}. \tag{2.47}$$

What we have shown in (2.46)-(2.47) are the forms of the HJB and optimality equations if the securities $\mathbf{S}$ are lognormal. In the next section we solve a constrained portfolio selection problem where it is assumed that a money market security is available for investment.

2.3.4 Solving the constrained optimisation problem (2.8): Investment in a money market security

Suppose S_0 is a money market security with evolution

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad 0 \leq t < T \tag{2.48}$$

where $S_0(0) := 1$ and $r = r(t) \in \mathbb{R}$ for all $0 \leq t \leq T$ is a stochastic, continuously compounded risk-free interest rate the same for both borrowing and lending. In Section 2.3.3 if at least one of the securities in the investor's universe is riskless (in other words its volatility is identically zero over $[0, T]$), then we cannot simply apply the analysis in Section 2.3.3 using a zero volatility for this security. Doing this will result in each covariance matrix $\bar{\boldsymbol{\sigma}}(t)$ being singular, and an optimal solution of the portfolio selection problem (2.8) then cannot be determined. Even if, in Section 2.3.3, we tried to let the volatility of say S_1 tend to zero in (2.41) (to mimic the dynamics of a riskless money market security), the magnitude of the portfolio $\boldsymbol{\pi}$ obtained from (2.41) will tend to infinity since the magnitude of the inverse of the covariance matrix $\bar{\boldsymbol{\sigma}}(t)$ will tend towards infinity. (In particular see the second term in $\hat{M}_i$ defined in (2.37).) Thus if a money market security is available for investment, then the constrained portfolio selection problem must be solved from scratch.

Suppose that the investor desires an optimal asset allocation for a universe comprised of risky securities $\mathbf{S}$ defined in (2.1) and a (riskless) money market security S_0. With the investor's wealth process defined as $W(t) = \sum_{i=0}^{N_S} N_i(t)S_i(t), 0 \leq t \leq T$ we have that if W is self-financing, then its evolution is given by

$$dW = \sum_{i=0}^{N_S} \xi_i \pi_i W dt + \sum_{i=0}^{N_S} \sigma_i \pi_i W dB_i = \sum_{i=1}^{N_S} (\xi_i - r)\pi_i W dt + \sum_{i=1}^{N_S} \sigma_i \pi_i W dB_i. \tag{2.49}$$

In (2.49) we require that for all $0 \leq t \leq T$

$$\sum_{i=0}^{N_S} \pi_i(t) = 1 \tag{2.50}$$

and

$$\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t) \tag{2.51}$$

are satisfied almost surely, where $\boldsymbol{\pi}$ is still defined as $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_{N_S})$. It is due to the second equation in (2.49) that we can solve a constrained portfolio selection problem if a money market security is available for investment. Essentially the money market security is treated as a catchall security - first optimal allocations to the risky securities $\mathbf{S}$ are determined, then $\pi_0(t)$ is defined so that (2.50) is satisfied. Also most importantly is that, to determine optimal allocations to $\mathbf{S}$, only their nonzero volatilities are taken into account (and not the identically zero money market security volatility). We now define the investor's set of admissible portfolios and then state the constrained portfolio selection problem we wish to solve.

Definition 2 (Admissible portfolios) *A set of control processes* $\boldsymbol{\pi}$ *where* $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ *for all* $0 \leq t \leq T$*, is said to be* **admissible** *(or an* **admissible portfolio***) if (2.50) and (2.51) hold almost surely, (2.49) has a unique solution and the resulting wealth process* W *is positive almost surely. We denote by* $\mathcal{P}_2$ *the set of all admissible portfolios.* ♦

The constrained portfolio selection problem we wish to solve is

$$\sup_{\boldsymbol{\pi} \in \mathcal{P}_2} E_0[U(T, W(T))]. \tag{2.52}$$

For $\boldsymbol{\pi} \in \mathcal{P}_2$ let

$$J^{\boldsymbol{\pi}} \; = \; J^{\boldsymbol{\pi}}(t, W, \mathbf{S}) \; := \; \mathbb{E}_{t,W,\mathbf{S}}[U(T, W(T))] \tag{2.53}$$

where W evolves according to (2.49) using $\boldsymbol{\pi}$. We define the time-t value functional as

$$J = J(t, W, \mathbf{S}) = \sup_{\boldsymbol{\pi} \in \mathcal{P}_2} J^{\boldsymbol{\pi}}(t, W, \mathbf{S}). \tag{2.54}$$

We could have defined J to be dependent on S_0, but we have not since the state variables W and $\mathbf{S}$ account for all variables that occur in the HJB equation for J. As in (2.24) we define $\phi_2(\boldsymbol{\pi}; t, W, \mathbf{S}) = \mathcal{L}_2[J^{\boldsymbol{\pi}}; t, W, \mathbf{S}]$ where $\mathcal{L}_2$ is defined by

$$\begin{aligned}\mathcal{L}_2[\cdot; t, W, \mathbf{S}] \;\; = \;\; & \frac{\partial}{\partial t} + \left(\sum_{i=1}^{N_S} (\xi_i - r)\pi_i + r \right) W \frac{\partial}{\partial W} + \frac{1}{2} \sum_{i=1}^{N_S} \sum_{j=1}^{N_S} \bar{\sigma}_{ij} \pi_i \pi_j W^2 \frac{\partial^2}{\partial W^2} \\ & + \sum_{i=1}^{N_S} \sum_{j=1}^{N_S} \bar{\sigma}_{ij} S_i \pi_j W \frac{\partial^2}{\partial S_i \partial W} + \sum_{i=1}^{N_S} \xi_i S_i \frac{\partial}{\partial S_i} + \frac{1}{2} \sum_{i=1}^{N_S} \sum_{j=1}^{N_S} S_i S_j \bar{\sigma}_{ij} \frac{\partial^2}{\partial S_i \partial S_j}.\end{aligned} \tag{2.55}$$

From Theorem 2 we obtain the HJB equation

$$\max_{\boldsymbol{\pi} \in \mathcal{P}_2} \mathcal{L}_2[J^{\boldsymbol{\pi}}] \;\; = \;\; 0 \tag{2.56}$$

which is equivalent to the problem

$$\max_{\boldsymbol{\pi}} \phi_2(\boldsymbol{\pi}) \tag{2.57}$$

$$\text{subject to} \qquad \mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t) \tag{2.58}$$

such that $\max_{\boldsymbol{\pi}} \phi_2(\boldsymbol{\pi}) = 0$ almost surely where $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ for all $0 \le t \le T$. The disadvantage of S_0 not featuring explicitly in the constrained portfolio selection problem (2.57)-(2.58) is that it is now a catchall security. (First all the constrained optimal security weights $\pi_1^*, \ldots, \pi_{N_S}^*$ are determined and then the portfolio weight π_0 is obtained via the unity weight constraint (2.50).) Thus it is not possible to explicitly constrain π_0. A necessary condition though which π_0 must satisfy for all $0 \le t \le T$ is that almost surely

$$1 - \sum_{i=1}^{N_S} b_i(t) \;\le\; \pi_0(t) \;\le\; 1 - \sum_{i=1}^{N_S} a_i(t)$$

since if all the upper (lower) bound constraints are active, then π_0 will have its minimum (maximum) value. Recall that this (modelling of S_0 as a catchall security) could be appropriate for hedge fund managers for example who have greater flexibility in the bets they take. Explicit constraining of a money market security portfolio weight can be done using the methodology in Section 2.3.3 - the function Υ.

The function $\phi_2(\boldsymbol{\pi})$ in (2.55) is quadratic in $\boldsymbol{\pi}$, thus since $\bar{\boldsymbol{\sigma}}$ is positive definite, a sufficient condition for a unique global and local maximum of ϕ_2 is that $J_{WW} < 0$. We want to find a local maximum of ϕ_2 satisfying the constraints (2.58). Suppose $\boldsymbol{\pi}^*$ is an optimal solution of (2.57)-(2.58). Then there must exist multipliers $\boldsymbol{\mu}, \bar{\boldsymbol{\mu}} \in \left(\mathbb{R}^{N_S}\right)^+$ such that the Karush-Kuhn-Tucker conditions hold viz

$$\begin{aligned} (\xi_k - r)WJ_W + W^2 J_{WW} \sum_{m=1}^{N_S} \bar{\sigma}_{km}\pi_m^* & \\ + W \sum_{i=1}^{N_S} \bar{\sigma}_{ki} S_i J_{iW} - \mu_k + \bar{\mu}_k &= 0, \quad k = 1, \ldots, N_S, \qquad (2.59) \\ \mu_i[\pi_i^* - b_i] &= 0, \quad i = 1, \ldots, N_S, \qquad (2.60) \\ \bar{\mu}_j[-\pi_j^* + a_j] &= 0, \quad j = 1, \ldots, N_S, \qquad (2.61) \\ (\boldsymbol{\mu}, \bar{\boldsymbol{\mu}}) &\ge \mathbf{0}, \qquad (2.62) \end{aligned}$$

where we have suppressed the dependence on time for notational simplicity. Solving for π_i^* in (2.59) we find that for all $i \in \mathcal{N}_S$

$$\pi_i^* = \frac{1}{WJ_{WW}} \sum_{k=1}^{N_S} \nu_{ki} \left(\frac{\mu_k^*}{W} - J_W(\xi_k - r) - \sum_{j=1}^{N_S} \bar{\sigma}_{kj} M_j \right), \qquad (2.63)$$

where μ_k^* is defined in (2.36) in other words

$$\mu_l - \bar{\mu}_l =: \mu_l^* := \begin{cases} \mu_l & \text{if } \pi_l^* \le b_l(t) \text{ is active (and } -\pi_l^* \le -a_l(t) \text{ inactive)}, \\ -\bar{\mu}_l & \text{if } -\pi_l^* \le -a_l(t) \text{ is active (and } \pi_l^* \le b_l(t) \text{ inactive)}, \\ 0 & \text{otherwise.} \end{cases}$$

The constraints (2.58) can also be rewritten as (2.38) and (2.39) can then also be derived viz

$$\mu_i^*[(-1)^{d_i}\pi_i^* - c_i] = 0, \qquad i = 1, \dots, N_S. \tag{2.64}$$

So substituting (2.63) into the Karush-Kuhn-Tucker conditions (2.64) we find that for each $k \in \mathcal{C}^*$

$$\mu_k^* \; = \; W^2 J_{WW} \sum_{j\in\mathcal{C}^*} \bar{\varsigma}_{kj}[\bar{c}_j + F_j], \tag{2.65}$$

where

$$\begin{aligned} \bar{\Psi}_{ij} &:= \nu_{\alpha_i\alpha_j}, \;\; \alpha_k \in \mathcal{C}^*, \\ \bar{\mathbf{\Psi}}^{-1} &\equiv [\bar{\varsigma}_{ij}] \quad \text{and} \\ F_i &:= \frac{1}{WJ_{WW}} \sum_{k=1}^{N_S} \nu_{ki} \left(J_W(\xi_k - r) + \sum_{j=1}^{N_S} \bar{\sigma}_{kj} M_j \right), \quad i = 1, \dots, N_S. \end{aligned}$$

Substituting (2.65) into (2.63) we find that for all $i \in \mathcal{N}_S$

$$\pi_i^* = -F_i + \sum_{k,j\in\mathcal{C}^*} \nu_{ki}\bar{\varsigma}_{kj}[\bar{c}_j - F_j]. \tag{2.66}$$

From (2.66) we see that a constrained optimal portfolio $\boldsymbol{\pi}$ of problem (2.52) is the unconstrained portfolio $-F_i(t), i \in \mathcal{N}_S$ plus the terms present if $\boldsymbol{\pi}(t)$ are constrained. Substituting (2.66) into the HJB equation (2.56) it reduces to

$$\begin{aligned} 0 \;\; &= \;\; \bar{M}^e - W \sum_{k\in\mathcal{C}^*} \nu_{kk} Z_k Y_k - W J_W \sum_{k\in\mathcal{C}^*} \nu_{kk} Z_k X_k + \frac{1}{2} W^2 J_{WW} \sum_{k\in\mathcal{C}^*} \nu_{kk} Z_k^2 \\ &\quad + \frac{1}{2J_{WW}} \sum_{k\in\mathcal{C}^*} \nu_{kk} Y_k^2 + \frac{J_W}{J_{WW}} \sum_{k\in\mathcal{C}^*} \nu_{kk} X_k Y_k + \frac{J_W^2}{2J_{WW}} \sum_{k\in\mathcal{C}^*} \nu_{kk} X_k^2 \end{aligned} \tag{2.67}$$

subject to the boundary condition $J(t, W, \mathbf{S}) = U(T, W(T))$ where

$$\begin{aligned} \bar{M}^e \;\; &= \;\; J_t + \sum_{i=1}^{N_S} \xi_i S_i J_i + \frac{1}{2} \sum_{i,j=1}^{N_S} \bar{\sigma}_{ij} S_i S_j J_{ij} + rWJ_W - \frac{1}{2J_{WW}} \sum_{j,m=1}^{N_S} \bar{\sigma}_{jm} M_j M_m \\ &\quad - \frac{J_W}{J_{WW}} \sum_{k=1}^{N_S} (\xi_k - r) M_k - \frac{J_W^2}{2J_{WW}} \sum_{k,l=1}^{N_S} (\xi_k - r)(\xi_l - r)\nu_{kl} \end{aligned} \tag{2.68}$$

and

$$Z_k := \sum_{c\in\mathcal{C}^*} \bar{\varsigma}_{kc}\bar{c}_c, \quad Y_k := \sum_{c\in\mathcal{C}^*} \bar{\varsigma}_{kc} M_c \quad \text{and} \;\; X_k := \sum_{c\in\mathcal{C}^*} \bar{\varsigma}_{kc} \sum_{b=1}^{N_S} \nu_{bc}(\xi_b - r).$$

If no inequality constraints in (2.58) are active, then (2.67) reduces to

$$0 = \bar{M}^e. \tag{2.69}$$

Suppose some subset $\mathcal{C}^*$ of the inequality constraints (2.58) are active. From (2.54) J is explicitly dependent on the wealth W, and in (2.49) we see that the wealth is explicitly dependent on the drifts and volatilities of the securities $\mathbf{S}$. If however the expected returns $\boldsymbol{\xi}$ and volatilities $\boldsymbol{\sigma}$ of all securities $\mathbf{S}$ are not dependent on (the security prices) $\mathbf{S}$, then the variables $\mathbf{S}$ do not appear in the HJB equation (2.56) for J. In this case $J = J(t, W)$ so (2.67) reduces to

$$\begin{aligned} 0 &= -J_t - rWJ_W + \frac{J_W^2}{2J_{WW}} \sum_{k,l=1}^{N_S} (\xi_k - r)(\xi_l - r)\nu_{kl} \\ &\quad + WJ_W \sum_{k \in \mathcal{C}^*} \nu_{kk} Z_k X_k - \frac{1}{2} W^2 J_{WW} \sum_{k \in \mathcal{C}^*} \nu_{kk} Z_k^2 - \frac{J_W^2}{2J_{WW}} \sum_{k \in \mathcal{C}^*} \nu_{kk} X_k^2 \end{aligned} \tag{2.70}$$

and (2.66) reduces to

$$\pi_i^* = -\frac{J_W}{WJ_{WW}} \sum_{k=1}^{N_S} \nu_{ki}(\xi_k - r) + \sum_{k,j \in \mathcal{C}^*} \nu_{ki} \bar{\varsigma}_{kj} \left[\bar{c}_j - \frac{J_W}{WJ_{WW}} \sum_{l=1}^{N_S} \nu_{lj}(\xi_l - r) \right]. \tag{2.71}$$

What we have shown in (2.70)-(2.71) are the forms of the HJB and optimality equations if the securities $\mathbf{S}$ are lognormal. We now provide a procedure for calculating constrained optimal portfolios for problems (2.8) and (2.52).

2.4 Procedure for calculating constrained optimal portfolios

In this section we provide a procedure for calculating constrained optimal portfolios for problem 2.8 (equivalently 2.52) using the methodology in Section 2.3.3 (equivalently Section 2.3.4 if a money market security is available for investment). First we solve for an unconstrained portfolio $\boldsymbol{\pi}(t)$ and the corresponding functional value $J(t, W, \mathbf{S})$. If $\boldsymbol{\pi}(t)$ satisfies the constraints (2.28) (in other words $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$), then the unconstrained portfolio $\boldsymbol{\pi}(t)$ is in fact also the time-t constrained optimal portfolio. Otherwise, which is the drawback of using the Karush-Kuhn-Tucker conditions to solve a constrained optimisation problem, we have to consider many different combinations of active and inactive constraints (2.28) (equivalently (2.58)), where each multiplier $\mu_i(t), i \in \mathcal{N}_S$ and $\bar{\mu}_j(t), j \in \mathcal{N}_S$ is zero and nonzero. In fact we have to consider at most $\bar{N}$ combinations of active and inactive inequality constraints (2.28) where

$$\bar{N} := 1 + \sum_{k=2}^{2N_S} \left[\binom{2(2N_S)}{k} - 2N_S \binom{2(2N_S)}{k-2} \right] \tag{2.72}$$

and $\binom{k}{0} := 1, k \in \mathbb{N}$. So if $N_S = 2, 3, 4, \ldots$, then one has to calculate at least one and at most $5, 19, 65, \ldots$ values respectively of $J(t, W, \mathbf{S})$ to find a time-t constrained optimal portfolio $\boldsymbol{\pi}^*(t)$. In (2.72) recall that

- the unity weight constraint is always active,
- the upper and lower bound constraint on some security weight cannot be active at the same time,
- if a money market security is (not) available for investment, then at most N_S ($N_S - 1$) inequality constraints can be active at the same time.

From Theorem 2 a time-t constrained optimal portfolio $\boldsymbol{\pi}^*(t)$ is that which satisfies (2.28) and has the largest value objective functional value $J(t, W, \mathbf{S})$. (Recall that we actually choose $\boldsymbol{\pi}^*(t)$ as that portfolio which maximises $\phi_1(\boldsymbol{\pi})$ (equivalently $\phi_2(\boldsymbol{\pi})$) such that $\max_{\boldsymbol{\pi} \in \mathcal{P}_1} \phi_1(\boldsymbol{\pi}) = 0$. By Theorem 2 this is equivalent to finding $\boldsymbol{\pi}^*(t)$ which - satisfies (2.28) and - maximises $J(t, W, \mathbf{S})$.)

Suppose we are at time t, the investor's wealth value is $W(t)$ and there are N_S risky securities in the investor's opportunity set. Then constrained optimal portfolios are determined as follows:

(i) Solve the unconstrained portfolio selection problem (2.26)-(2.27) (equivalently (2.57)) by solving the HJB equation (2.45) (equivalently (2.69)) for $J(t, W, \mathbf{S})$. Calculate the unconstrained portfolio $\boldsymbol{\pi}(t)$ via (2.41) (equivalently (2.66)).

(ii) If $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$, then the unconstrained portfolio $\boldsymbol{\pi}(t)$ is optimal for time t and we do not need to solve the constrained optimisation problem (2.26)-(2.28) (equivalently (2.57)-(2.58)) for a constrained optimal portfolio. Save this portfolio and its corresponding objective functional value. Proceed to step (i) above, increment time and calculate the next constrained optimal portfolio.

(iii) If in contrast to (ii) above $\pi_i(t) < a_i(t)$ or $b_i(t) < \pi_i(t)$ for some $i \in \mathcal{N}_S$, then we need to solve the constrained optimisation problem (2.26)-(2.28) to find a time-t constrained optimal portfolio. Proceed to step (iv) below.

(iv) Consider all possible ways of constraining the security weights $\boldsymbol{\pi}(t)$, in other words all possible ways of setting active the upper and lower inequality portfolio weight constraints (2.28). In each case save the portfolio $\boldsymbol{\pi}(t)$ and its corresponding objective functional value $J(t, W, \mathbf{S})$. From the set of constrained portfolios, the time-t constrained optimal portfolio is that which satisfies (2.28) and has the largest objective functional value $J(t, W, \mathbf{S})$. The time-t set of constrained portfolios (from which the time-t constrained optimal portfolio is chosen) is determined as follows. Start with setting only one constraint active.

   (a) Set active only the lower bound constraint on $\pi_1(t)$. Then the set $\mathcal{C}^*(t) = \{1\}$ and in (2.40) (equivalently (2.65)), $\bar{c}_1(t) = a_1(t)$ and $\bar{c}_i(t) = 0$ for all $i \in \mathcal{N}_S \backslash \{1\}$. Using these values of $\mathcal{C}^*(t)$ and $\bar{\mathbf{c}}(t)$, solve the HJB equation (2.44) (equivalently (2.67)) for $J(t, W, \mathbf{S})$ and calculate the optimal values of $\pi_2(t), \ldots, \pi_{N_S}(t)$ using (2.41).

(b) Set active only the upper bound constraint on $\pi_1(t)$. Then the set $\mathcal{C}^*(t) = \{1\}$ and in (2.40), $\bar{c}_1(t) = b_1(t)$ and $\bar{c}_i(t) = 0$ for all $i \in \mathcal{N}_S \backslash \{1\}$. Using these values of $\mathcal{C}^*(t)$ and $\bar{\mathbf{c}}(t)$, solve the HJB equation (2.44) for $J(t, W, \mathbf{S})$ and calculate the optimal values of $\pi_2(t), \ldots, \pi_{N_S}(t)$ using (2.41).

(c) Repeat steps (a) and (b) for all securities $\mathbf{S}$.

(d) If there are some portfolios in (a)-(c) which satisfy $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$, then the time-$t$ constrained optimal portfolio is that with the largest objective functional value $J(t, W, \mathbf{S})$. Proceed to step (i) above, increment time and calculate the next constrained optimal portfolio. Otherwise proceed to step (e).

(e) Set active only the lower bound constraints on $\pi_1(t)$ and $\pi_2(t)$. Then the set $\mathcal{C}^*(t) = \{1, 2\}$ and in (2.40), $\bar{c}_1(t) = a_1(t), \bar{c}_2(t) = a_2(t)$ and $\bar{c}_i(t) = 0$ for all $i \in \mathcal{N}_S \backslash \{1, 2\}$. Using these values of $\mathcal{C}^*(t)$ and $\bar{\mathbf{c}}(t)$, solve the HJB equation (2.44) for $J(t, W, \mathbf{S})$ and calculate the optimal values of $\pi_3(t), \ldots, \pi_{N_S}(t)$ using (2.41).

(f) Set active only the lower bound constraints on $\pi_1(t)$ and $\pi_3(t)$. Then the set $\mathcal{C}^*(t) = \{1, 3\}$ and in (2.40), $\bar{c}_1(t) = a_1(t), \bar{c}_3(t) = a_3(t)$ and $\bar{c}_i(t) = 0$ for all $i \in \mathcal{N}_S \backslash \{1, 3\}$. Using these values of $\mathcal{C}^*(t)$ and $\bar{\mathbf{c}}(t)$, solve the HJB equation (2.44) for $J(t, W, \mathbf{S})$ and calculate the optimal values of $\pi_2(t), \pi_4(t), \ldots, \pi_{N_S}(t)$ using (2.41).

(g) Repeat (d) and (e) for all security pairs and all pairs of active upper and lower inequality portfolio weight constraints.

(h) If there are some portfolios in (e)-(g) which satisfy $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$, then the time-$t$ constrained optimal portfolio is that with the largest objective functional value $J(t, W, \mathbf{S})$. Proceed to step (i) above, increment time and calculate the next constrained optimal portfolio. Otherwise proceed to step (i).

(i) Find constrained portfolios $\boldsymbol{\pi}(t)$ for different sets of active upper and lower portfolio weight inequality constraints.

(j) The constrained optimal portfolio $\boldsymbol{\pi}(t)$ is that which satisfies (2.28) and has the largest objective functional value $J(t, W, \mathbf{S})$.

(k) Save this time-t constrained optimal portfolio and its corresponding objective functional value. Increment time and calculate the next constrained optimal portfolio.

Remark 1 *We make the following two remarks.*

- *A grid of constrained optimal portfolios corresponding to different values of t, W and $\mathbf{S}$ should be produced. Then we start at time 0 and let the stochastic differential equations (2.1) evolve. From the grid produced, for each time t, we simply read off the constrained optimal portfolio (for that sample point $\omega \in \Omega$).*

- *Recall that (2.44) is a final value problem. Thus we use the time-t values $J(t, W, \mathbf{S})$ in the calculation of constrained optimal portfolios at time $t - \Delta t, 0 < \Delta t \ll 1$.*

We now solve an example.

2.5 Example

Suppose we want to solve the constrained optimisation problem (2.8) with the following inputs:

(i) The constant relative risk-averse (CRRA) investor's opportunity set is comprised of two securities S_1 and S_2 with evolutions

$$\frac{dS_i(t)}{S_i(t^-)} = \xi_i(t)dt + \sigma_i(t)dB_i(t),$$

where for $i = 1, 2$, ξ_i and σ_i are stochastic processes with values given in Table 2.1 and these are independent of the security prices S_1 and S_2. Thus to solve (2.8) we have to solve the HJB equation (2.46) subject to the boundary condition

$$J(t, W) = U(T, W(T)) = (p(T)x + q(T))^\gamma, \quad 0 < \gamma < 1, \tag{2.73}$$

where in this example the functions p and q are deterministic and of the form

$$p(T) = \exp(T) \quad \text{and} \quad q(T) = \ln(1 + T).$$

The relative risk aversion coefficient $0 < \gamma < 1$ because we require J to be strictly concave in $W(t)$ - see the paragraphs below (2.28) and above (2.59).

(ii) The other financial market parameters are assumed to have values:

$$\begin{aligned} N_S &= 2, \quad T = 1 \text{ year}, \quad \Delta t = \tfrac{1}{12} \text{ years}, \\ \gamma &= 0.5, \quad \Upsilon \equiv 1 \end{aligned}$$

and the values given in Tables 2.1 and 2.2.

We solve (2.46) discretely for constrained optimal portfolios at times $\frac{8}{12}, \frac{9}{12}, \frac{10}{12}, \frac{11}{12}$ years and create a grid of constrained optimal portfolios as depicted in Table 2.3. For this example this grid varies only in two dimensions viz time t and wealth level $W(t)$. (If any of the variables $\boldsymbol{\xi}, \boldsymbol{\sigma}, \mathbf{a}, \mathbf{b}$ are dependent on $\mathbf{S}$, then this grid will vary in more dimensions and consequently computational time will increase.) To find a constrained optimal portfolio $\boldsymbol{\pi}(t)$ the following 5 cases need to be considered:

t	$\frac{8}{12}$	$\frac{9}{12}$	$\frac{10}{12}$	$\frac{11}{12}$	$\frac{12}{12}$
a_1	0.40	0.40	0.40	0.40	0.40
a_2	0.40	0.40	0.40	0.40	0.40
b_1	0.80	0.80	0.80	0.80	0.80
b_2	0.80	0.80	0.80	0.80	0.80
ξ_1	-0.0204	0.1232	0.0494	0.1058	0.0002
ξ_2	0.1199	-0.0345	0.0853	0.1455	0.0533
σ_1	0.8449	0.3659	0.4538	0.7287	0.4808
σ_2	0.6405	0.4829	0.7016	0.7354	0.8173

Table 2.1: Time-t values of financial market parameters for securities S_1 and S_2 in Example 2.5.

t	$\bar{\boldsymbol{\sigma}}$
$\frac{8}{12}$	$\begin{pmatrix} 0.7139 & 0.1369 \\ 0.1369 & 0.4103 \end{pmatrix}$
$\frac{9}{12}$	$\begin{pmatrix} 0.1339 & -0.0238 \\ -0.0238 & 0.2332 \end{pmatrix}$
$\frac{10}{12}$	$\begin{pmatrix} 0.2059 & 0.0789 \\ 0.0789 & 0.4923 \end{pmatrix}$
$\frac{11}{12}$	$\begin{pmatrix} 0.5310 & -0.0784 \\ -0.0784 & 0.5408 \end{pmatrix}$
$\frac{12}{12}$	$\begin{pmatrix} 0.2312 & -0.0134 \\ -0.0134 & 0.6680 \end{pmatrix}$

Table 2.2: Time-t covariance matrices of returns of securities S_1 and S_2 in Example 2.5.

Case 1 $\mu_1^*(t) = 0, \mu_2^*(t) = 0$. Here $\mathcal{C}^*(t)$ is empty so (2.46) reduces to

$$0 \;=\; J_t + WJ_W \sum_{k=1}^{2} \xi_k \hat{G}_k + \frac{W^2 J_{WW}}{2\Gamma} - \frac{J_W^2}{2J_{WW}} \left(\sum_{k,l=1}^{2} \xi_k \xi_l \nu_{kl} - \Gamma \left(\sum_{l=1}^{2} \xi_l \nu_l \right)^2 \right) \tag{2.74}$$

and (2.47) reduces to

$$\pi_i \;=\; \hat{G}_i + \frac{J_W}{WJ_{WW}} \left(\frac{\nu_i}{\Gamma} \sum_{n=1}^{2} \xi_n \nu_n - \sum_{k=1}^{2} \xi_k \nu_{ki} \right). \tag{2.75}$$

We then have to solve (2.74) for $J(t, W)$ and then calculate $\pi_1(t)$ and $\pi_2(t)$ via (2.75). (Actually we only need to calculate either $\pi_1(t)$ or $\pi_2(t)$ from (2.75) and calculate the other via the unity weight constraint (2.5).) If we find that $\mathbf{a}(t) \le \boldsymbol{\pi}(t) \le \mathbf{b}(t)$, then we do not consider cases 2-5 below since the unconstrained portfolio $\boldsymbol{\pi}(t)$ is in fact also the constrained optimal portfolio for time t and wealth level $W(t)$. Otherwise do all of Cases 2-5 below.

Case 2 $\mu_1(t) > 0, \bar{\mu}_1(t) = 0, \mu_2(t) = 0, \bar{\mu}_2(t) = 0$. Here $\mathcal{C}^*(t) = \{1\}$ and $\bar{c}_1(t) = b_1(t)$. We then have to solve the resulting HJB equation (2.46) for $J(t, W)$ and calculate $\pi_1(t)$ and $\pi_2(t)$ from (2.47).

Case 3 $\mu_1(t) = 0, \bar{\mu}_1(t) > 0, \mu_2(t) = 0, \bar{\mu}_2(t) = 0$. Here $\mathcal{C}^*(t) = \{1\}$ and $\bar{c}_1(t) = a_1(t)$. We then have to solve the resulting HJB equation (2.46) for $J(t, W)$ and calculate $\pi_1(t)$ and $\pi_2(t)$ from (2.47).

Case 4 $\mu_1(t) = 0, \bar{\mu}_1(t) = 0, \mu_2(t) > 0, \bar{\mu}_2(t) = 0$. Here $\mathcal{C}^*(t) = \{2\}$ and $\bar{c}_2(t) = b_2(t)$. We then have to solve the resulting HJB equation (2.46) for $J(t, W)$ and calculate $\pi_1(t)$ and $\pi_2(t)$ from (2.47).

Case 5 $\mu_1(t) = 0, \bar{\mu}_1(t) = 0, \mu_2(t) = 0, \bar{\mu}_2(t) > 0$. Here $\mathcal{C}^*(t) = \{2\}$ and $\bar{c}_2(t) = a_2(t)$. We then have to solve the resulting HJB equation (2.46) for $J(t, W)$ and calculate $\pi_1(t)$ and $\pi_2(t)$ from (2.47).

The constrained optimal portfolio for time t and wealth level $W(t)$ is that which satisfies the inequality constraints (2.28) and has the largest value of $J(t, W)$. The actual numerical values obtained are given in Table 2.3. The results show constrained optimal portfolios for specific values of t and $W(t)$. So at time $t = 11/12$ years, if the investor's wealth value is $W = 6076076$, then the constrained optimal investments in securities S_1 and S_2 are 43.6% and 56.4% respectively. We now show why the constrained optimal portfolio for $t = 10/12$ and $W = 7577578 =: \tilde{W}$ is $(0.6; 0.4)$. Let $t_{12} := 1, t_{11} := 11/12, t_{10} := 10/12$ and $t_9 := 9/12$ years. To find $\boldsymbol{\pi}^*(t_{10})$ we need to choose from a set of constrained portfolios. Now from (2.47), to calculate one constrained portfolio $\boldsymbol{\pi}(t_{10})$, we need to know $J(t_{10}, W)$. So since the problem has been discretised, we actually compare values of J at time t_9 to determine which constrained portfolio is

t = 8/12	t = 9/12	t = 10/12	t = 11/12	t = 1
			(0.4; 0.6)	W = 70072
		(0.6; 0.4)	(0.430; 0.570)	W = 1571573
	(0.8; 0.2)	(0.6; 0.4)	(0.432; 0.568)	W = 3073074
(0.2; 0.8)	(0.8; 0.2)	(0.6; 0.4)	(0.434; 0.566)	W = 4574575
	(0.8; 0.2)	(0.6; 0.4)	(0.436; 0.564)	W = 6076076
		(0.6; 0.4)	(0.438; 0.562)	W = 7577578
			(0.439; 0.561)	W = 9079079

Table 2.3: Grid of constrained optimal portfolios for Example 2.5.

in fact the time-t_{10} constrained optimal portfolio. So suppose we have calculated the constrained optimal portfolio $\boldsymbol{\pi}(t_{11})$ and the functionals $J(t_{11}, W)$ and $J(t_{10}, W)$. We discretise (2.46) in the time variable t and cases 1-5 mentioned above then have the following form:

Case 1 From Tables 2.1 and 2.2 $J(t_9, W)$ is calculated via the difference equation

$$J(t_9, W) \quad = \quad J(t_{10}, W) + \Delta t\left(0.06WJ_W(t_{10}, W) + 0.09W^2J_{WW}(t_{10}, W) + 0.3\frac{J_W^2(t_{10}, W)}{J_{WW}(t_{10}, W)}\right),$$

where $\hat{\mathbf{G}}(t_{10}) = \begin{pmatrix} 0.77 \\ 0.24 \end{pmatrix}$ and $\Gamma(t_{10}) = 5.68$. The functional $J(t_{10}, W)$ is known and we found that $J(t_9, \tilde{W}) = 4446.53$. Equation (2.75) reduces to

$$\boldsymbol{\pi}^1(t_{10}) \quad = \quad \begin{pmatrix} 0.77 \\ 0.24 \end{pmatrix} + \begin{pmatrix} 0.07 \\ -0.07 \end{pmatrix} \frac{J_W(t_{10}, \tilde{W})}{\tilde{W}J_{WW}(t_{10}, \tilde{W})} \quad = \quad \begin{pmatrix} 0.63 \\ 0.37 \end{pmatrix}.$$

Since the unconstrained portfolio $\pi_2^1(t_{10}) < a_2(t_{10}) = 0.4$ we have to consider the other four cases to find a constrained optimal portfolio for time t_{10} and wealth level $\tilde{W}$.

Case 2 $\mu_1(t_{10}) > 0, \bar{\mu}_1(t_{10}) = 0, \mu_2(t_{10}) = 0, \bar{\mu}_2(t_{10}) = 0$. Here $\mathcal{C}^*(t_{10}) = \{1\}$ and $\bar{c}_1(t_{10}) = b_1(t_{10}) = 0.8$. This constrained portfolio will not be optimal since $\pi_1^2(t_{10}) = 0.8$ implies that $\pi_2^2(t_{10}) = 0.2 < a_2(t_{10}) = 0.4$, however we still go through the calculations for expositional purposes. From Tables 2.1 and 2.2 equation (2.46) reduces to

$$J(t_9, W) \quad = \quad J(t_{10}, W) + \Delta t\left(0.07WJ_W(t_{10}, W) + 0.08W^2J_{WW}(t_{10}, W) + 0.30\frac{J_W^2(t_{10}, W)}{J_{WW}(t_{10}, W)}\right),$$

where $\Psi = 1.85, \bar{R} = 0.08, \bar{N} = 0.16$ and

$$\begin{aligned}
\mathbf{R} &:= \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 0.10 \\ -0.02 \end{pmatrix}, \\
\hat{\mathbf{R}} &:= \begin{pmatrix} \hat{R}_1 \\ \hat{R}_2 \end{pmatrix} = \begin{pmatrix} 0.04 \\ -0.04 \end{pmatrix}, \\
\mathbf{N} &:= \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} 0.19 \\ -0.03 \end{pmatrix}, \\
\hat{\mathbf{N}} &:= \begin{pmatrix} \hat{N}_1 \\ \hat{N}_2 \end{pmatrix} = \begin{pmatrix} 0.07 \\ -0.07 \end{pmatrix}, \\
\mathbf{Q}^N &:= [Q_{ij}^N] = \begin{pmatrix} -0.10 & 0.04 \\ 0.04 & 0.03 \end{pmatrix}, \\
\mathbf{G}^R &:= [G_{ij}^R] = \begin{pmatrix} 0.06 & -0.02 \\ -0.02 & -0.02 \end{pmatrix} \quad \text{and} \\
\mathbf{P}^N &:= [P_{ij}^N] = \begin{pmatrix} -0.0044 & 0.0044 \\ 0.0044 & -0.0044 \end{pmatrix}.
\end{aligned}$$

Since the functional $J(t_{10}, W)$ is known we found that $J(t_9, \tilde{W}) = 4445.83$ and equation (2.47) reduces to

$$\boldsymbol{\pi}^2(t_{10}) = \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}.$$

Case 3 $\mu_1(t_{10}) = 0, \bar{\mu}_1(t_{10}) > 0, \mu_2(t_{10}) = 0, \bar{\mu}_2(t_{10}) = 0$. Here $\mathcal{C}^*(t_{10}) = \{1\}$ and $\bar{c}_1(t_{10}) = a_1(t_{10}) = 0.4$. From Tables 2.1 and 2.2 equation (2.46) reduces to

$$J(t_9, W) = J(t_{10}, W) + \Delta t \left(0.08 W J_W(t_{10}, W) + 0.14 W^2 J_{WW}(t_{10}, W) + 0.30 \frac{J_W^2(t_{10}, W)}{J_{WW}(t_{10}, W)} \right).$$

Since the functional $J(t_{10}, W)$ is known we found that $J(t_9, \tilde{W}) = 4445.26$ and equation (2.47) reduces to

$$\boldsymbol{\pi}^3(t_{10}) = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}.$$

Case 4 $\mu_1(t_{10}) = 0, \bar{\mu}_1(t_{10}) = 0, \mu_2(t_{10}) > 0, \bar{\mu}_2(t_{10}) = 0$. Here $\mathcal{C}^*(t_{10}) = \{2\}$ and $\bar{c}_2(t_{10}) = b_2(t_{10}) = 0.8$. This constrained portfolio will not be optimal since $\pi_2^4(t_{10}) = 0.8$ implies that $\pi_1^4(t_{10}) = 0.2 < a_1(t_{10}) = 0.4$, however we still go through the calculations for expositional purposes. From Tables 2.1 and 2.2 equation (2.46) reduces to

$$J(t_9, W) = J(t_{10}, W) + \Delta t \left(0.09 W J_W(t_{10}, W) + 0.17 W^2 J_{WW}(t_{10}, W) + 0.30 \frac{J_W^2(t_{10}, W)}{J_{WW}(t_{10}, W)} \right).$$

We found that $J(t_9, \tilde{W}) = 4442.07$ and equation (2.47) reduces to

$$\boldsymbol{\pi}^4(t_{10}) = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}.$$

Case 5 $\mu_1(t_{10}) = 0, \bar{\mu}_1(t_{10}) = 0, \mu_2(t_{10}) = 0, \bar{\mu}_2(t_{10}) > 0$. Here $\mathcal{C}^*(t_{10}) = \{2\}$ and $\bar{c}_2(t_{10}) = a_2(t_{10}) = 0.4$. From Tables 2.1 and 2.2 equation (2.46) reduces to

$$J(t_9, W) = J(t_{10}, W) + \Delta t \left(0.07W J_W(t_{10}, W) + 0.11W^2 J_{WW}(t_{10}, W) + 0.30 \frac{J_W^2(t_{10}, W)}{J_{WW}(t_{10}, W)} \right).$$

We found that $J(t_9, \tilde{W}) = 4446.51$ and equation (2.47) reduces to

$$\boldsymbol{\pi}^5(t_{10}) = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}.$$

The result from Cases 1-5 is that, since $J(t_9, \tilde{W}, \boldsymbol{\pi}^3(t_{10})) < J(t_9, \tilde{W}, \boldsymbol{\pi}^5(t_{10}))$, the portfolio

$$\boldsymbol{\pi}(t_{10}) = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

is the constrained optimal portfolio for time t_{10} and wealth value $\tilde{W}$. Recall that the portfolios $\boldsymbol{\pi}^2(t_{10})$ and $\boldsymbol{\pi}^4(t_{10})$ are not considered because these do not satisfy the time-t_{10} inequality constraints (2.28) where $\mathbf{a}(t_{10})$ and $\mathbf{b}(t_{10})$ are defined in Table 2.1. To show that Theorem 2 holds in this case, a plot of $\phi_1(\boldsymbol{\pi}^*(t_{10}); \boldsymbol{\pi})$ versus $\boldsymbol{\pi}$ (subject to the unity weight constraint (2.5)) is given in Figure 2.1. The functional $\phi_1(\boldsymbol{\pi}^*(t_{10}); \boldsymbol{\pi})$ is evaluated as follows:

(i) Substitute the time-t_{10} constrained optimal portfolio $\boldsymbol{\pi}^*(t_{10})$ into the HJB equation (2.25).

(ii) In (i) above a particular HJB equation is derived which is determined by the values of $\boldsymbol{\pi}^*(t_{10})$. Solve for the value functional $J(t_{10}, W, \mathbf{S})$ in this particular HJB equation.

(iii) Using $J(t_{10}, W, \mathbf{S})$ calculated in (ii) above, evaluate its partial derivatives and substitute these into $\phi_1(\boldsymbol{\pi}^*(t_{10}); \boldsymbol{\pi})$ defined in (2.24).

(iv) Plot $\phi_1(\boldsymbol{\pi}^*(t_{10}); \boldsymbol{\pi})$ by using the partial derivatives of $J(t_{10}, W, \mathbf{S})$ as coefficients and $\boldsymbol{\pi}$ as independent variables. Consequently $\phi_1(\boldsymbol{\pi}^*(t_{10}); \boldsymbol{\pi})$ is quadratic in $\boldsymbol{\pi}(t_{10})$.

We see that $\phi_{1,5}(\boldsymbol{\pi}(t_{10}); \boldsymbol{\pi}) := \phi_1(\boldsymbol{\pi}_5(t_{10}); \boldsymbol{\pi})$ has a maximum value of zero. This is exactly the function ϕ_1 that is referred to in Theorem 2. Recall from Theorem 2 that to find a constrained optimal portfolio of the problem (2.8) the following two approaches are equivalent:

(i) Choosing a constrained portfolio $\tilde{\boldsymbol{\pi}}(t_{10})$ with $\max_{\boldsymbol{\pi} \in \mathcal{P}_1} \phi_1(\tilde{\boldsymbol{\pi}}(t_{10}); \boldsymbol{\pi}) = 0$.

(ii) Choosing a constrained portfolio $\tilde{\boldsymbol{\pi}}(t_{10})$ which has the largest objective functional value $J(t_{10}, W, \mathbf{S})$ (defined in (2.10)).

In the next section the no-constraining (NC) region of a portfolio is analysed.

Figure 2.1: Plot of functions $\phi_{1,i}(\boldsymbol{\pi}(t_{10}); \boldsymbol{\pi}), i = 2, 3, 4, 5$ for different active and inactive constraints.

2.6 The No-constraining (NC) Region

In this section we show how to describe the NC region NC_i of each security S_i assuming a money market security is not available for investment. (The analysis is similar if a money market security is available for investment.) The point of the NC region analysis is, at each time t, to determine beforehand whether or not the time intensive process of constraining portfolio weights (to find a constrained optimal portfolio) will have to be done. The NC region analysis allows us to determine beforehand whether or not, at time t, we shall have to go through the process of constraining the portfolio weights until we find that portfolio which satisfies the inequality constraints (2.28) (equivalently (2.58)) and has the largest objective functional value J.

Now the portfolio NC region NC_W is defined as $\bigcap_{i=1}^{N_S} NC_i$. Analogous to the portfolio NT (no-transaction) region derived in ([38], [54]), the portfolio NC region NC_W derived in this section is a region defined only in terms of the state variables $(W, \mathbf{S})$. Secondly it is a region such that, if the state variables are inside it, then it is guaranteed that an unconstrained portfolio $\boldsymbol{\pi}$ is in fact a constrained optimal portfolio and so will not need to be constrained to some subset of the bounds $\mathbf{a}$ and $\mathbf{b}$. So we are interested in the calculation of the boundaries of NC_W. Intuitively, the way NC_W will be obtained, is by letting the weights $\boldsymbol{\pi}^*$ be unconstrained and roam free. When the weights hit their boundaries $\mathbf{a}$ and $\mathbf{b}$, record the values of the state variables $(W, \mathbf{S})$. One can then plot these values of the state variables in the space with axes $(W, \mathbf{S})$ and this will be the boundaries of NC_W. The NC region NC_i of security S_i is obtained by letting the weight π_i^* roam free and noting the values of the state variables $(W, \mathbf{S})$ when π_i^* hits its boundaries a_i and b_i. Essentially, what we are doing when trying to determine NC_i, is to set $\mathcal{C}^*$ empty and $\pi_i^* = a_i$ in (2.41) and obtaining a relationship between the state variables $(W, \mathbf{S})$ at the lower boundary a_i. Then we set $\mathcal{C}^*$ empty and $\pi_i^* = b_i$ in (2.41) and obtain a relationship between the state variables $(W, \mathbf{S})$ at the upper boundary b_i. We now do this. Recalling that for all $i \in \mathcal{N}_S$

$$\hat{M}_i = \frac{J_W}{WJ_{WW}} p_i(t, \mathbf{S}) + \frac{1}{WJ_{WW}} \left(\hat{G}_i \sum_{p=1}^{N_S} S_p J_{pW} - S_i J_{iW} \right)$$

where

$$p_i(t, \mathbf{S}) := \frac{\nu_i}{\Gamma} \sum_{n=1}^{N_S} \xi_n \nu_n - \sum_{k=1}^{N_S} \xi_k \nu_{ki}, \tag{2.76}$$

defining the set $\mathcal{C}^*$ to be empty, (2.41) can be written as

$$\pi_i^* = \hat{G}_i + \frac{J_W}{WJ_{WW}} p_i(t, \mathbf{S}) + \frac{1}{WJ_{WW}} \left(\hat{G}_i(t, \mathbf{S}) \sum_{p=1}^{N_S} S_p J_{pW} - J_{iW} S_i \right). \tag{2.77}$$

We want to find a relationship between the state variables $(W, \mathbf{S})$ if $\pi_i^* = b_i$ and if $\pi_i^* = a_i$ in (2.77). These relationships define the boundary ∂NC_i of NC_i. To obtain an analytical relationship between the state variables on ∂NC_i from the nonlinear equation (2.77) is unlikely, so these relationships will most probably be obtained numerically. In simple cases however, closed-form relationships between the state variables on ∂NC_i can be obtained, which is what we now do. We make the following two assumptions which allow us to obtain an analytical relationship between the state variables on ∂NC_i, viz for all $i \in \mathcal{N}_S$

$$\begin{array}{ll} \text{(i)} & a_i(t, W, \mathbf{S}) = \hat{a}_i(t, \mathbf{S})W + \bar{a}_i(t, \mathbf{S}) \quad \text{and} \quad b_i(t, W, \mathbf{S}) = \hat{b}_i(t, \mathbf{S})W + \bar{b}_i(t, \mathbf{S}), \\ \text{(ii)} & \hat{a}_i \equiv 0 \text{ and } \hat{b}_i \equiv 0, \end{array} \tag{2.78}$$

where $\hat{a}_i(t, \mathbf{S}), \bar{a}_i(t, \mathbf{S}), \hat{b}_i(t, \mathbf{S})$ and $\bar{b}_i(t, \mathbf{S})$ are functions of t and $\mathbf{S}$. In (2.78)(i) the linearity of a_i and b_i in W is arbitrary and is chosen so that we can determine analytical relationships between the state variables on ∂NC_i. (In practice portfolio weight bounds $\mathbf{a}$ and $\mathbf{b}$ are almost always constant and not stochastic, so (2.78)(i) is more than adequate for modelling real world situations.) From (2.78)(i) with $\pi_i^* = b_i$, equation (2.77) reduces to

$$(b_i - \hat{G}_i)WJ_{WW} = p_i J_W + \hat{G}_i \sum_{p=1}^{N_S} S_p J_{pW} - J_{iW} S_i. \tag{2.79}$$

By integrating (2.79) with respect to W we get that

$$\hat{b}_i W^2 J_W + (\bar{b}_i - \hat{G}_i) W J_W - (2\hat{b}_i W + \bar{b}_i - \hat{G}_i - p_i) J + 2\hat{b}_i \int J dW = \hat{G}_i \sum_{p=1}^{N_S} J_p S_p - J_i S_i. \tag{2.80}$$

Analogous to the NT region derivation in ([38], [54]), to produce an analytical description of ∂NC_i, we want to obtain a first-order, linear, homogeneous partial differential equation involving only first partial derivatives of some function of $(W, \mathbf{S})$. We try the substitution

$$F(t, W, \mathbf{S}) := f(t, W, \mathbf{S}) J(t, W, \mathbf{S}), \tag{2.81}$$

where $f(t, W, \mathbf{S})$ is to be determined. Substituting (2.81) into (2.80) and multiplying by f we get that

$$\left(\hat{G}_i \left(1 + \sum_{p=1}^{N_S} \frac{f_p}{f} S_p \right) + \left(\hat{G}_i - \hat{b}_i W - \bar{b}_i \right) W \frac{f_W}{f} - 2\hat{b}_i \left(W - f \int f^{-1} dW \right) - p_i - \bar{b}_i - \frac{f_i}{f} S_i \right) F$$
$$= (-(\hat{b}_i W + \bar{b}_i) + \hat{G}_i) W F_W + \hat{G}_i \sum_{p=1}^{N_S} F_p S_p - F_i S_i. \tag{2.82}$$

We see that the right-hand side of (2.82) involves only first partial derivatives of F, so we want to make the left-hand side of (2.82) identically zero. The term $\int f^{-1} dW$ is not easy to deal with, so using (2.78)(ii) equation (2.82) reduces to

$$\left(\hat{G}_i \left(f + \sum_{p=1}^{N_S} f_p S_p \right) + \left(\hat{G}_i - \bar{b}_i \right) W f_W - (p_i + \bar{b}_i) f - f_i S_i \right) \frac{F}{f}$$
$$= (-\bar{b}_i + \hat{G}_i) W F_W + \hat{G}_i \sum_{p=1}^{N_S} F_p S_p - F_i S_i. \tag{2.83}$$

This is the relationship that the state variables $(W, \mathbf{S})$ must satisfy given assumptions 2.78 and if the portfolio weight π_i^* attains its maximum allowed value b_i. The lower boundary state variable relationship of NC_i is obtained by replacing b with a in (2.83). We now consider a particular example to continue the NC region analysis which becomes quite complicated even with the assumptions (2.78).

2.6.1 Example

In this example we shall make the following assumptions viz

(i) in (2.83) $\xi_i \equiv 0$ for all $i = 1, \ldots, N_S$,

(ii) $\sigma_1 = \sigma_1(t, S_1) = S_1^{-1}$ and $\sigma_i = \sigma_i(t), i = 2, \ldots, N_S$,

(iii) the Brownian motion correlation matrix $\boldsymbol{\rho} \equiv \mathbf{I}$ the identity matrix and

(iv) $\bar{a}_1 = \bar{a}_1(t)$ and $\bar{b}_1 = \bar{b}_1(t)$.

(2.84)

From 2.84(i), by inspection of (2.83) we see that the function $f(t, W, \mathbf{S}) = W^{-1}$ will result in a partial differential equation involving only first partial derivatives of F, viz

$$0 = (-\bar{b}_i + \hat{G}_i) W F_W + \hat{G}_i \sum_{p=1}^{N_S} F_p S_p - F_i S_i. \tag{2.85}$$

By inspection of (2.85) we see that NC_i is dependent on W and the prices of all securities $\mathbf{S}$. Also the coefficients $\bar{b}_i$ and $\hat{G}_i$ of the partial derivatives in (2.85) are functions of $(W, \mathbf{S})$. Thus it is not necessarily the case that a solution of (2.85) will always exist. Moreover, due to the many independent variables $(W, \mathbf{S})$ which will be the axes in the space in which NC_i will be plotted, it is not in general possible to visualise (in other words produce a three-dimensional view of) NC_i. To produce a three-dimensional view of NC_i we must have one or two *non*-lognormal securities in the investor's opportunity set. Thus from (2.84)(ii)-(iii), in other words S_1 is the only non-lognormal security and the Brownian motions $\mathbf{B}$ are uncorrelated, we have that

$$\bar{\boldsymbol{\sigma}} = \begin{pmatrix} S_1^{-2} & 0 & \dots & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \sigma_{N_S}^2 \end{pmatrix}, \quad \boldsymbol{\nu} = \begin{pmatrix} S_1^{2} & 0 & \dots & \dots & 0 \\ 0 & \sigma_2^{-2} & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \sigma_{N_S}^{-2} \end{pmatrix}$$

and

$$\Gamma = S_1^2 + \sum_{i=2}^{N_S} \sigma_i^{-2}, \quad \hat{G}_1 = \frac{S_1^2}{S_1^2 + \sum_{i=2}^{N_S} \sigma_i^{-2}}, \quad \hat{G}_i = \frac{\bar{\sigma}_i^{-2}}{S_1^2 + \sum_{i=2}^{N_S} \sigma_i^{-2}}, \ i = 2, \dots, N_S.$$

Since S_1 is the only security with its volatility dependent on its price we have that $F = F(t, W, S_1)$. Assuming $F_W \neq 0$ in (2.85) it reduces to

$$0 = (-\bar{b}_1 + \hat{G}_1) W + (\hat{G}_1 - 1) S_1 \frac{dW}{dS_1}. \tag{2.86}$$

Using (2.84)(iv) (in other words $\bar{b}_1$ is independent of $\mathbf{S}$), the unique solution of (2.86) is

$$W = \exp\left(\frac{(1 - \bar{b}_1(t)) S_1^2}{2 \sum_{i=2}^{N_S} \sigma_i^{-2}} \right) S_1^{-\bar{b}_1(t)}.$$

This is the relationship that W and S_1 must satisfy at the upper boundary of NC_i (in the presence of Assumptions 2.78 and 2.84). The upper boundary value

Figure 2.2: Plot of the no-constraining region NC_1 of risky security S_1 discussed in Example 2.6.1, where $0.05 \leq \pi_1(t) \leq 0.85$ for all $0 \leq t \leq T$.

$\bar{b}_1$ was arbitrary thus the relationship that W and S_1 must satisfy at the lower boundary of NC_i is

$$W \quad = \quad \exp\left(\frac{(1-\bar{a}_1(t))S_1^2}{2\sum_{i=2}^{N_S}\sigma_i^{-2}}\right) S_1^{-\bar{a}_1(t)}.$$

With $N_S = 3, \bar{a}_1 = 0.05, \bar{b}_1 = 0.85$ and $\sigma_i = 0.3, i = 2, 3$, for all $0 \leq t \leq T$ the region NC_1 has the form given in Figure 2.2. NT_W is then the intersection of $NC_i, i = 1, 2, 3$. In the next chapter a constrained portfolio selection problem is solved with the following relaxations relative to Chapter 2 viz

(i) the risky securities are modelled as being non-Markov processes,

(ii) the logarithmic returns of the risky securities are allowed to exhibit jumps,

(iii) the hypothetical investor is modelled as having more information available to him other than that generated by the financial market - the investor is assumed to be an insider, and

(iv) Particular types of investment strategy of the insider are penalised. This is accomplished by including *penalty functions* in the objective functional of the insider portfolio selection problem solved.

Chapter 3

Constrained portfolio selection with non-Markov processes and Insiders (I)

A financial market comprised of non-Markov securities driven by Lévy processes is considered. The securities are non-Markov since the expected returns, volatilities and jump coefficients of the securities are path dependent. A constrained portfolio selection problem for an insider with a strictly increasing, concave and at least once-differentiable utility function is solved. An insider is an investor who has more information available about the disturbances in a financial market than an honest investor. The models of ([18], [42], [62]) are closely followed and generalised to a multidimensional setting and this immediately introduces (amongst other things) an explicit unity weight constraint on the portfolio security weights which is not present in these papers. Inequality constraints on the insider's portfolio security weights are also imposed and the resulting constrained portfolio selection problems are solved via the method of calculus of variations. This chapter contains only theoretical results. In Chapter 4 some analytical solutions are derived and some problems are solved numerically.

3.1 Introduction

In Chapter 2 constrained optimal portfolios comprised of a money market security and diffusions were derived. Since the method of dynamic programming was employed to solve those optimisation problems, we were restricted to including (in the opportunity set of the honest investor) only state variables which are Markov processes. (This is why only diffusions, which are Markov processes, were assumed to be available for investment by the honest investor.) The other drawback of the dynamic programming framework is the *Curse of Dimensionality* [131]. With the use of dynamic programming it is not advisable to include too many state variables in the (dynamic programming) problem. The reason for this is that since the dimensionality of the dynamic programming problem increases disproportionately with the number of state variables, it becomes very difficult to solve a practical dynamic programming problem, in particular a portfolio selection problem. In this chapter the state variables (in this case the security prices) are allowed to be non-Markov. Further complicating the portfolio selection problem is that it is also assumed that the hypothetical investor is an *insider* defined in the next paragraph. The *calculus of variations* is employed to solve the constrained portfolio selection problems in this case.

From [62], by an *insider* in a financial market we mean an investor who possesses more information than the information generated by the disturbances in the financial market itself. An insider may be for example an executive or simply an employee of a company. [62] An *honest* investor can only use the filtration (or information) generated by the market itself if making an investment decision. An insider has a larger filtration available to him and uses this to make investment decisions. In reality insiders do not trade in the absence of market inefficiencies. Considered in this chapter and the next is the market inefficiency of portfolio security weight inequality constraints since an insider may not be able to trade unconstrained monetary amounts of some security. An application of this work is to improve the detection of insider trading. This work can also be extended to the pricing of contingent claims in the presence of investment constraints and where the amount of information agents have is important - for example *partial equilibrium* and *general equilibrium* [97] models.

The fundamental difficulty associated with solving a portfolio selection problem for an insider in continuous-time in the presence of non-deterministic disturbances, is how to interpret the resulting integrals, which in general are no longer stochastic integrals. For example, from [18], let B be a standard Brownian motion on the complete filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ is the natural filtration of B. Consider an insider who has access to a filtration $\mathbb{H} := \{\mathcal{H}_t\}_{0\leq t\leq T}$ which is larger than $\mathbb{F}$, in other words $\mathcal{F}_t \subseteq \mathcal{H}_t$ for all $0 \leq t \leq T$. Let $f = f(t)$ be a process describing the insider's investment behaviour. Then f is not $\mathbb{F}$-predictable but $\mathbb{H}$-predictable. We are interested in the interpretation of the integral of f with respect to B. (These types of integral arise in this chapter since the insider's portfolio is predictable with respect to a larger filtration than that generated by the financial market disturbances - see section 3.4 below.) This integral is denoted by the object

$$"\int_0^T f(t,\omega)d^- B(t)", \quad \omega \in \Omega. \tag{3.1}$$

Since the most well-known form of stochastic integration requires the integrand to be predictable with respect to the filtration of the integrator [117], from [18] a natural (and the most common) approach is to assume that $\mathbb{H}$ is such that B

is an $(\mathbb{H}, \mathbb{P})$-semimartingale. In this case from [117] we can write

$$B(t) = \hat{B}(t) + H(t), \quad 0 \le t \le T, \tag{3.2}$$

where $\hat{B}$ is an $(\mathbb{H}, \mathbb{P})$-Brownian motion and H is a continuous, $\mathbb{H}$-predictable, bounded variation process. One can then define

$$\int_0^T f(t,\omega)d^- B(t) = \int_0^T f(t,\omega)d\hat{B}(t) + \int_0^T f(t,\omega)dH(t).$$

Quoting from ([18], Section 1), approach (3.2) generates the following questions viz (i) How do we know if (3.2) is possible?, (ii) If (3.2) is possible, then how do we find H? and (iii) What do we do if (3.2) is not possible? In this chapter we follow and generalise the models in ([18], [42], [62]) and solve a constrained portfolio selection problem for an insider without assuming that (3.2) holds. As in ([18], [42], [62]) we show that if a constrained optimal insider portfolio $\boldsymbol{\pi} := (\pi_1, \ldots, \pi_{N_S})$ exists, then in fact (3.2) holds with (the bounded variation process) H closely related to $\boldsymbol{\pi}$. Since inequality constraints are imposed on $\boldsymbol{\pi}$, if the bounds are large enough, then our results will reduce to those in ([18], [42], [62]).

Now the portfolio $\boldsymbol{\pi}$ of an insider is $\mathbb{H}$-predictable since the insider makes his investment decisions based on the information $\mathbb{H}$ available to him. Thus, as will be seen in equations (3.19) and (3.21), the resulting integrals in the stochastic differential equation of the wealth process of the insider are no longer stochastic integrals (since it will not necessarily be the case that the integrand is predictable with respect to the filtration of the integrator). Now if the insider is in fact honest, then we want optimal insider portfolios to reduce to optimal honest investor portfolios. Thus in this chapter, as in ([18], [42], [62]), we choose to model these integrals (in the differential equation of the insider wealth process) as *forward stochastic integrals* (defined in Section 3.3 below). We do this since forward integrals

(i) are objects where the integrand is predictable with respect to a larger filtration than that of the integrator,

(ii) are also defined as a limit of Riemann sums and

(iii) reduce to stochastic integrals if the integrand is predictable with respect to the filtration of the integrator.

This is why forward stochastic integration is such an important part of the portfolio selection theory in this chapter.

In ([62], [116]) optimal insider portfolios comprised of a money market security and a diffusion are derived by maximising the expected logarithmic utility of terminal wealth of the insider. In [62] penalty functions are included in the objective functional so that optimal insider portfolios are not conspicuous (relative to an optimal honest investor portfolio) and so that the objective functional maximised in [62] is finite. In [116] the authors assume a particular form for the insider's filtration and moreover assume that the Brownian motion disturbance (employed in [116]) is in fact a semimartingale with respect to the insider's filtration. In [31] optimal insider portfolios comprised of a money market security and a diffusion are derived by maximising the expected difference between the logarithmic utility of the terminal wealth of the insider and the logarithmic utility of the terminal wealth of an honest investor. In ([18], [79]) optimal insider portfolios comprised of a money market security and a diffusion are derived by maximising the expected utility of the insider's terminal wealth, where the utility function need only be concave and at least once differentiable. In [79] however the coefficients in the stochastic differential equation of the diffusion are modelled as being anticipative. In ([51], [57]) optimal portfolios comprised of a money market security and risky securities driven by independent Brownian motions and (compound) Poisson processes are derived by maximising the sum of the expected utility of intertemporal consumption and the expected utility of terminal wealth of the insider. In ([42], [78], [110]) optimal insider portfolios comprised of a money market security and a risky security driven by a Lévy process are derived by maximising the expected logarithmic utility of terminal wealth of the insider. In [110] however the coefficients in the stochastic differential equation of the risky security are assumed to be anticipative. The portfolio selection model considered in this chapter differs from those mentioned above in the following ways:

(i) We allow an arbitrary, finite number of securities in an insider's portfolio.

(ii) A riskless (money market) security may or may not be available for investment by the insider.

(iii) The risky securities are driven by Lévy processes and can be non-Markov.

(iv) Drifts, volatilities and jump coefficients of the risky securities can be non-Markov processes but these must not be anticipative.

(v) The expected utility of terminal wealth over a finite time horizon is maximised (rather than the expected utility of consumption over an infinite time horizon).

(vi) Penalty functions are introduced into the objective functionals so that optimal insider portfolios are not conspicuous (relative to optimal honest investor portfolios) and so that the objective functionals maximised are finite.

(vii) In particular penalty functions are included in the case where the insider has logarithmic utility and the securities are driven by Lévy processes (with jumps).

(viii) The insider portfolio selection problem is solved for general utility where the securities are driven by Lévy processes (with jumps).

(ix) Both buying and short-selling constraints are included in the portfolio selection models.

(x) A positivity of wealth constraint is imposed.

With respect to (v) above, as in Chapter 2, this is done since the future liabilities (or consumption) of an institutional portfolio, for example a portfolio managed by an asset manager, are known *a priori*, hence an optimum consumption rule need not be determined. (For example the fee agreement between the asset manager and client is agreed upon upfront.) The inclusion of intertemporal consumption better describes the investment behaviour of an individual - see [22] and [105].

The rest of this chapter is organised as follows: Section 3.2 includes preliminaries; in Section 3.3 forward stochastic integration is defined; in Section 3.4 the financial market model is defined; in Section 3.5 Itô's formula for functionals of forward Lévy processes is stated and the forward stochastic differential equations of the insider wealth process are solved; in Section 3.6 the optimisation problems to be solved are stated; and in Sections 3.7 and 3.8 the insider's portfolio selection problems, where the securities are driven by diffusions and Lévy processes with jumps respectively, are solved.

3.2 Preliminaries

In this section definitions and results required for the rest of this thesis are stated. This section was summarised from [7]. First a *Lévy process* is defined.

Definition 3 (Lévy process) *Let* $(\Omega, \mathbb{F}, \mathbb{P})$ *be a complete filtered probability space and let* $X = (X(t), 0 \leq t)$ *be a real-valued stochastic process defined on* $(\Omega, \mathbb{F}, \mathbb{P})$*. Then we say that* X *has* independent increments *if for each* $n \in \mathbb{N}$ *and each* $0 \leq t_1 < t_2 < \cdots < t_{n+1} < \infty$*, the random variables* $(X(t_{j+1}) - X(t_j), 1 \leq j \leq n)$ *are independent. We say that* X *has* stationary increments *if each random variable* $X(t_{j+1}) - X(t_j)$ *has the same distribution as the random variable* $X(t_{j+1} - t_j) - X(0)$*. We say that* X *is a Lévy process if:*

(L1) $X(0) = 0$ *almost surely.*

(L2) X *has independent and stationary increments.*

(L3) X *is stochastically continuous, in other words for all* $z \in \mathbb{R}^+$ *and for all* $0 \leq s, t$

$$\lim_{t \to s} \mathbb{P}(z < |X(t) - X(s)|) = 0.$$

♦

From [117] one can show that every Lévy process is càdlàg and for the rest of this thesis it will be assumed that all Lévy processes considered are càdlàg. See [7] for more information on Lévy processes. We now define *Poisson random measures.*

Definition 4 (Poisson random measures) *Let* $(S, \mathcal{G})$ *be a measurable space. Then a* random measure q *on* $(S, \mathcal{G})$ *is a collection of random variables* $(q(B), B \in \mathcal{G})$ *such that:*

(i) $q(\emptyset) = 0$.

(ii) *Let* $(A_n \in \mathcal{G}, n \in \mathbb{N})$ *be a sequence of mutually disjoint sets. Then it is required that*

$$q\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} q(A_n).$$

(iii) *For each disjoint family* $(B_1, \ldots, B_n)$ *in* $\mathcal{G}$*, the random variables* $q(B_1), \ldots, q(B_n)$ *are independent.*

If each random variable $q(B)$ *has a Poisson distribution whenever* $q(B) < \infty$*, then* q *is called a* Poisson random measure*.* ♦

We now define a *compensated Poisson random measure* for the special case where the set S in Definition 4 is the product space $[0,T] \times \mathbb{R}^N, N \in \mathbb{N}$. First, from ([7], pp87), a set $A \in \mathcal{B}(\mathbb{R}^N), N \in \mathbb{N}$ is said to be *bounded below* if $\mathbf{0} \notin \bar{A}$, where $\bar{A}$ is the closure of the set A.

Definition 5 *Let the set* $A \in \mathcal{B}(\mathbb{R}^N)$ *be bounded below and for all* $0 \leq t \leq T$ *let*

$$q(t, A) := q([0,t] \times A).$$

Suppose the set S *in Definition 4 is the product space* $[0,T] \times \mathbb{R}^N, N \in \mathbb{N}$ *and define the* intensity measure $\nu(A) = \mathbb{E}[q(1, A)]$*. Then for all* $0 \leq t \leq T$ *the* compensated Poisson random measure $\tilde{q}$ *is defined as*

$$\tilde{q}(t, A) = q(t, A) - t\nu(A). \tag{3.3}$$

♦

We now define a *Lévy measure.*

Definition 6 (Lévy measure) *Let* $\bar{S} \subseteq \mathbb{R}$*. Then a* Borel measure *is any measure on the space* $(\bar{S}, \mathcal{B}(\bar{S}))$*. Let* ν *be a Borel measure defined on* $\mathbb{R}\backslash\{0\}$*. Then we say that* ν *is a Lévy measure if*

$$\int_{\mathbb{R}\backslash\{0\}} (y^2 \wedge 1)\nu(dy) \ < \ \infty.$$

♦

For notational simplicity, for the rest of this thesis if we integrate over $\mathbb{R}^N, N \in \mathbb{N}$ and it should be bounded below, then it should be assumed that it is. (This convention permits us to write $\int_{\mathbb{R}^N} (\cdots)\nu(d\mathbf{z})$ instead of $\int_{\mathbb{R}^N\backslash\{\mathbf{0}\}} (\cdots)\nu(d\mathbf{z})$ each time.) For some set $A \in \mathcal{B}(\mathbb{R}^N)$ however it will clearly be stated whether or not this set must be bounded below. The following results are also required.

Theorem 3 (Girsanov-Meyer) *Let* $(\Omega, \mathbb{F}, \mathbb{P})$ *be a complete filtered probability space. Let* $\mathbb{P}$ *and* $\mathbb{Q}$ *be equivalent probability measures on* $(\Omega, \mathbb{F})$ *with* $d\mathbb{P} = F(T)d\mathbb{Q}$ *and* $F(t) := \mathbb{E}[F(T)|\mathcal{F}_t]$ *for all* $0 \le t \le T$*, where* $F(T)$ *is a (nonnegative)* $\mathcal{F}_T$*-measurable random variable. Let* M *be an* $(\mathbb{F}, \mathbb{P})$*-semimartingale with decomposition* $M = M_1 + M_2$*, where* M_1 *is an* $(\mathbb{F}, \mathbb{P})$*-local martingale and* M_2 *is a process of bounded variation. Then* M *is also an* $(\mathbb{F}, \mathbb{Q})$*-semimartingale and has decomposition* $M = M_3 + M_4$*, where*

$$M_3(t) := M_1(t) - \int_0^t \frac{1}{F(s)} d[F, M_1](s), \quad 0 \le t \le T$$

is an $(\mathbb{F}, \mathbb{Q})$*-local martingale and* $M_4 := M - M_3$ *is a* $\mathbb{Q}$ *bounded variation process.*

Proof: See ([117], Theorem III.35). ■

Corollary 4 *Let* $\mathbb{P}$ *and* $\mathbb{Q}$ *be equivalent probability measures on* $(\Omega, \mathbb{F})$ *with* $d\mathbb{P} = F(T)d\mathbb{Q}$ *and* $F(t) := \mathbb{E}[F(T)|\mathcal{F}_t]$ *for all* $0 \le t \le T$*. Let* M *be an* $(\mathbb{F}, \mathbb{P})$*-martingale. Then*

$$M_3(t) := M(t) - \int_0^t \frac{1}{F(s)} d[F, M](s), \quad 0 \le t \le T$$

is an $(\mathbb{F}, \mathbb{Q})$*-local martingale.*

Proof: Let $M_2 \equiv 0$ and $M_1 \equiv M$ in Theorem 3. ■

Theorem 4 *Let* M *be an* $(\mathbb{F}, \mathbb{P})$*-local martingale. Then* M *is an* $(\mathbb{F}, \mathbb{P})$*-martingale with* $\mathbb{E}[M^2(t)] < \infty$ *for all* $0 \le t$*, if and only if* $\mathbb{E}[M, M](t) < \infty$ *for all* $0 \le t$*.*

Proof: See ([117], Corollary II.3). ■

Theorem 5 *Let* $X = (X(t), 0 \le t \le T)$ *be an* $(\mathbb{F}, \mathbb{P})$*-semimartingale and let* $f = f(t)$ *be integrable with respect to* X*. Then we have almost surely that for all* $0 \le t \le T$

$$\left[\int_0^t f(s)dX(s), \int_0^t f(s)dX(s)\right] = \int_0^t f^2(s)d[X, X](s).$$

Proof: See ([117], Theorem II.29). ■

We now define forward stochastic integration.

3.3 Forward stochastic integration

In this section forward (stochastic) integration is defined. Forward integration is required in this chapter since the integrals encountered (in the stochastic differential equation of the insider wealth process) have the integrand predictable with respect to a larger filtration than that of the integrator. Forward integrals occurring in this chapter is a direct consequence of solving a constrained portfolio selection problem for an insider who has more information available to him than that of an honest investor. We now define this type of integration.

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space. Then from [117] the most well-known stochastic integrals that can be defined are of the form

$$\int_0^t f(s,\omega)dZ(s,\omega),$$

where $(Z(t), 0 \leq t \leq T)$ is an $(\mathbb{F}, \mathbb{P})$-semimartingale and $f = f(t,\omega)$ is an $\mathbb{F}$-predictable integrable process. Let $\mathbb{H} \supseteq \mathbb{F}$ be another filtration and let $g = g(t,\omega)$ be an $\mathbb{H}$-predictable integrable process. Then the integral of g with respect to Z is called a *forward stochastic integral* and it is denoted by the object

$$\text{"}\int_0^t g(s,\omega)d^- Z(s,\omega)\text{"}. \tag{3.4}$$

The stochastic process in (3.4) includes stochastic integrals as a special case, thus it does not always satisfy all properties of a stochastic integral. If the process g is in fact $\mathbb{F}$-predictable, then (3.4) reduces to a stochastic integral.

Let $\epsilon \in \mathbb{R}$ and $(X(t), 0 \leq t \leq T)$ and $(Y(t), 0 \leq t \leq T)$ be two stochastic processes continuous at 0 and T, where X is an $(\mathbb{H}, \mathbb{P})$-semimartingale and Y an $(\mathbb{F}, \mathbb{P})$-semimartingale. Then from [123] the *forward stochastic integral* $\int_0^T X d^- Y$ of X with respect to Y is

$$\int_0^T X(t)d^- Y(t) \;:=\; \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^T X(t)(Y((t+\epsilon) \wedge T) - Y(t))dt, \tag{3.5}$$

where the limit in (3.5) is taken in probability. If the limit in (3.5) exists, then X is said to be *forward integrable with respect to* Y. In the next two Sections 3.3.1 and 3.3.2, the *forward diffusion integral* and the *forward Poisson integral* are defined.

3.3.1 Forward diffusion integration

A forward diffusion integral is a special case of (3.5) where the processes X and Y are continuous over the whole interval $[0, T]$. As mentioned in Section 3.1, we model the insider's wealth process as a forward integral since we want at least insider constrained optimal portfolios to reduce to honest investor constrained optimal portfolios if the insider is in fact honest. We show in Proposition 3 below when a forward diffusion integral reduces to an Itô diffusion integral.

First however, from [123], for a locally integrable function f on $\mathbb{R}^+$, we define $\mathcal{Z}(f)$ to be the set of all $0 < t$ such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t} f(s)ds \neq f(t). \tag{3.6}$$

Next, let $(X(t), 0 \leq t \leq T)$ be a stochastic process. Then for every $t \in [0, T]$ let the process

$$X^t(u) := X(u)\chi_{[0,t]}(u), \quad 0 \leq u \leq T,$$

where for $A \subseteq \mathbb{R}$ χ_A is the indicator function. From [123] we state the following proposition in which it is shown that if a process f is predictable with respect to the filtration of the Brownian motion integrator, then the forward diffusion integral reduces to an Itô diffusion integral.

Proposition 3 *Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space, let $\mathbb{H} \supseteq \mathbb{F}$ be a larger filtration and let B be an $(\mathbb{F}, \mathbb{P})$-Brownian motion. Suppose $B(t) = \hat{B}(t) + H(t), 0 \leq t \leq T$ where $\hat{B}$ is an $(\mathbb{H}, \mathbb{P})$-Brownian motion and H is a continuous, $\mathbb{H}$-predictable, bounded variation process. Suppose f is an $\mathbb{F}$-predictable and bounded process such that*

$$\int_0^T \chi_{\{s \in \mathcal{Z}(f)\}}(d|H|(s) + ds) = 0 \quad \text{almost everywhere.} \tag{3.7}$$

Then for every $t \in [0, T]$ we have almost surely that

$$\int_0^T f^t(s)d^-B(s) = \int_0^T f^t(s)dB(s).$$

Proof: See ([123], Proposition 1.1). ■

3.3.2 Forward Poisson integration

First *Poisson integration* and then *forward Poisson integration* is defined.

Poisson integration

Suppose[1] $f : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function. For each sample point $\omega \in \Omega$ we are interested in the integral of f with respect to a Poisson random measure q over the set $[0, T] \times A$, where $A \in \mathcal{B}(\mathbb{R}^N)$ is bounded below. This integral is denoted by the object

$$\text{"} \int_0^T \int_A f(t, \mathbf{z})q(dt, d\mathbf{z}) \text{"} \tag{3.8}$$

[1] This subsection was summarised from [7].

which is simply a Lebesgue integral. From [7] define the vector pure jump process associated with q as

$$\mathbf{Y} = \mathbf{Y}(t) = \int_A \mathbf{z} q(t, d\mathbf{z}), \quad 0 \leq t \leq T.$$

Then $\mathbf{Y}$ has the form

$$\mathbf{Y}(t) = \mathbf{Y}(t^-) + \Delta\mathbf{Y}(t), \quad 0 \leq t \leq T$$

where

$$\Delta\mathbf{Y}(t) := \begin{cases} \mathbf{0} & \text{if jump size } \mathbf{z} \text{ of } \mathbf{Y} \text{ at time } t \text{ is not in } A \\ \mathbf{z} & \text{if jump size } \mathbf{z} \text{ of } \mathbf{Y} \text{ at time } t \text{ is in } A. \end{cases}$$

From ([7], equation (4.4)), the integral of a predictable function $f = f(t, \mathbf{z})$ with respect to the (Poisson random measure) q is defined as the random finite sum

$$\int_0^T \int_A f(t, \mathbf{z}) q(dt, d\mathbf{z}) = \sum_{0 \leq t \leq T} f(t, \Delta\mathbf{Y}(t)) \chi_A(\Delta\mathbf{Y}(t)). \tag{3.9}$$

From ([7], Exercise 4.3.3) the predictability of f in (3.9) is required to ensure that it is at least a local martingale.

For integrals with respect to a compensated Poisson random measure $\tilde{q}$, we have from (3.3) and (3.9) that

$$\begin{aligned} \int_0^T \int_A f(t, \mathbf{z}) \tilde{q}(dt, d\mathbf{z}) &= \int_0^T \int_A f(t, \mathbf{z}) q(dt, d\mathbf{z}) - \int_0^T \int_A f(t, \mathbf{z}) \nu(d\mathbf{z}) dt \\ &= \sum_{0 \leq t \leq T} f(t, \Delta\mathbf{Y}(t)) \chi_A(\Delta\mathbf{Y}(t)) - \int_0^T \int_A f(t, \mathbf{z}) \nu(d\mathbf{z}) dt. \end{aligned} \tag{3.10}$$

The important difference between (3.9) and (3.10) is that, from ([117], Section I.4), the set A need not bounded below[2] in (3.10).

Forward Poisson integration

We now have the following definition taken from [41] for *forward Poisson integration.*

Definition 7 *The* forward Poisson integral *of an $\mathbb{H}$-predictable process $f = f(t, \mathbf{z})$ with respect to the compensated Poisson random measure $\tilde{q}$ is denoted by the object*

$$\text{"}\int_0^T \int_A f(t, \mathbf{z}) \tilde{q}(d^-t, d\mathbf{z})\text{"}, \tag{3.11}$$

[2]From ([117], Section I.4), if integrating with respect to a *compensated* Poisson random measure, then the spatial set need not be bounded below.

where $A \in \mathcal{B}(\mathbb{R}^N)$ need not be bounded below. From ([42], Definition 4) the object in (3.11) is defined as

$$\int_0^T \int_A f(t,\mathbf{z})\tilde{q}(d^-t, d\mathbf{z}) \quad = \quad \lim_{m\to\infty} \int_0^T \int_A f(t,\mathbf{z})\chi_{U_m}(\mathbf{z})\tilde{q}(dt, d\mathbf{z}) \qquad (3.12)$$

if the limit in (3.12) exists in $L^2(\mathbb{P})$. In (3.12) we have that the $\{U_m\}_{m\in\mathbb{N}}$ is an increasing sequence of compact sets such that $U_m \subseteq A$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} U_m = A$. If the limit in (3.12) exists, then the function f is said to be forward integrable with respect to *(the compensated Poisson random measure) $\tilde{q}$ or simply* forward Poisson integrable. ♦

In Definition 7 the set A need not be bounded below since the forward Poisson integral (3.12) is a limit of compensated Poisson integrals. See ([41], Proposition 3.1) which shows that, if the integrand of a forward Poisson integral is predictable with respect to the filtration of (the pure jump processes associated with) the integrator $\tilde{q}$, then the forward Poisson integral reduces to a compensated Poisson integral.

3.4 The Financial Market Model

The financial market model is comprised of the following:

- Let $\mathbf{q} := (q_1, \ldots, q_{N_q}), q_j = q_j(dt, d\mathbf{z}), j \in \mathcal{N}_q$ be Poisson random measures such that the vector pure jump processes associated with $\mathbf{q}$ are independent.
- Let $\boldsymbol{\nu}^{\mathbb{F}} := (\nu_1^{\mathbb{F}}, \ldots, \nu_{N_q}^{\mathbb{F}}), \nu_j^{\mathbb{F}} = \nu_j^{\mathbb{F}}(d\mathbf{z}), j \in \mathcal{N}_q$ be the corresponding intensity measures of $\mathbf{q}$. Denote the jth compensated Poisson random measure $q_j(dt, d\mathbf{z}) - \nu_j^{\mathbb{F}}(d\mathbf{z})dt$ by $\tilde{q}_j = \tilde{q}_j(dt, d\mathbf{z})$ and define
$$\tilde{\mathbf{q}} = (\tilde{q}_1, \ldots, \tilde{q}_{N_q}). \qquad (3.13)$$
- Let $\mathbf{B} := (B_1, \ldots, B_{N_B})$ be independent standard Brownian motions (which are independent of the pure jump processes associated with $\tilde{\mathbf{q}}$.)
- Let $\mathbb{F}$ be the natural filtration of $\mathbf{B}$ and (the pure jump processes associated with) $\tilde{\mathbf{q}}$.
- Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions [56].

There is an insider who invests in a financial market comprised of risky securities[3] S_0 and $\mathbf{S} := (S_1, \ldots, S_{N_S})$ assumed to evolve as

$$\frac{dS_0(t)}{S_0(t^-)} \quad = \quad \xi_0(t)dt + \sum_{j=1}^{N_B} \sigma_{0,j}(t)dB_j(t) + \sum_{j=1}^{N_q} \int_{\mathbb{R}^N} g_{0,j}(t,\mathbf{z})\tilde{q}_j(dt, d\mathbf{z}) \qquad (3.14)$$

[3]Note that derivative securities may also be included in the insider's opportunity set.

$$\frac{dS_i(t)}{S_i(t^-)} = \xi_i(t)dt + \sum_{j=1}^{N_B} \sigma_{ij}(t)dB_j(t) + \sum_{j=1}^{N_q} \int_{\mathbb{R}^N} g_{ij}(t,\mathbf{z})\tilde{q}_j(dt,d\mathbf{z}). \qquad (3.15)$$

Equations (3.14) and (3.15) are written separately because in (3.14) any or all of the processes $\sigma_{0,j}, j \in \mathcal{N}_B$ are allowed to be identically zero. In (3.15) however, for each $i \in \mathcal{N}_S$, we do not allow all the processes $\sigma_{i,j}, j \in \mathcal{N}_B$ to be identically zero. More specifically these restrictions relate to the requirement of the invertibility of the covariance matrices defined in (3.17) below. To ensure that the insider's portfolio selection problem can be solved the financial market model (3.14)-(3.15) must satisfy the following requirements:

- If S_0 is a money market security, then for all $0 \leq t \leq T, j \in \mathcal{N}_B, k \in \mathcal{N}_q$ we must have that $\xi_0(t) = r(t)$ the stochastic continuously compounded interest rate assumed to be the same for both borrowing and lending, and $\sigma_{0,j}(t) = 0$ and $g_{0,k}(t) = 0$.

- For all $i \in \mathcal{N}_S \cup \{0\}, j \in \mathcal{N}_B, k \in \mathcal{N}_q$, we must have that the stochastic continuously compounded interest rate r, the expected security returns ξ_i, volatilities σ_{ij} and jump coefficients g_{ik} must be $\mathbb{F}$-predictable integrable processes satisfying

$$\infty > \mathbb{E}\left[\int_0^T \left(|r(t)| + \sum_{i=0}^{N_S} \left\{ |\xi_i(t)| + \sum_{j=0}^{N_B} \sigma_{ij}^2(t) + \sum_{j=0}^{N_q} \int_{\mathbb{R}^N} g_{ij}^2(t,\mathbf{z})\nu_j^{\mathbb{F}}(d\mathbf{z}) \right\} \right) dt \right]. \qquad (3.16)$$

- Moreover for all $i \in \mathcal{N}_S \cup \{0\}, j \in \mathcal{N}_B, k \in \mathcal{N}_q$ it is required that r, ξ_i, σ_{ij} and g_{ik} must satisfy certain regularity conditions ([117], Chapter V) which ensure the existence of a strictly positive solution of (3.14)-(3.15). In particular for all $0 \leq t \leq T, \mathbf{z} \in \mathbb{R}^N, i \in \mathcal{N}_S \cup \{0\}$ we require that $-1 < \sum_{j=1}^{N_q} g_{ij}(t,\mathbf{z})$ almost surely.

- For all $0 \leq t \leq T, i,j \in \mathcal{N}_S, k \in \mathcal{N}_B$ let

$$\hat{\sigma}_{ik}(t) := \sigma_{ik}(t) - \sigma_{0,k}(t) \quad \text{and} \quad \bar{\sigma}_{ij}(t) := \sum_{k=1}^{N_B} \hat{\sigma}_{ik}(t)\hat{\sigma}_{jk}(t). \qquad (3.17)$$

  Then we require for all $0 \leq t \leq T$ that the covariance matrix $\bar{\boldsymbol{\sigma}}(t) \equiv [\bar{\sigma}_{ij}(t)]$ is nonsingular. The processes $\hat{\sigma}_{ik}$ in (3.17) arise in Sections 3.7.1 and 3.8.2 where we convert a constrained optimisation problem to an unconstrained optimisation problem. We do this by eliminating one of the securities S_0 from the optimisation problem. If a money market security is not available for investment, then S_0 should be regarded as any other risky security.

- We also require that the shares of the securities are infinitely divisible, short sales are permitted with full use of the proceeds, taxes on capital gains are zero and transaction costs are zero.

Note that in (3.14)-(3.15) for each $i \in \mathcal{N}_S \cup \{0\}, j \in \mathcal{N}_B, k \in \mathcal{N}_q$ we have from [117] that the processes

$$\int_0^t \sigma_{ij}(s)dB_j(s) \quad \text{and} \quad \int_0^t \int_{\mathbb{R}^N} g_{ik}(s,\mathbf{z})\tilde{q}_k(ds,d\mathbf{z}) \tag{3.18}$$

are $(\mathbb{F},\mathbb{P})$-local martingales. As in (Chapter 2) Section 2.3 we solve the insider portfolio selection problems assuming a money market security is unavailable and available for investment, thus we now define two sets of *admissible portfolio*. Recall that the reason for doing this is that a money market security has zero volatility and this will result in the covariance matrices $\bar{\boldsymbol{\sigma}}$ being singular. One then cannot simply extend the analysis where all securities are risky, to the case where some security is riskless, to solve that constrained portfolio selection problem. (As in Section 2.3 the constrained optimal portfolios derived in each case are not special cases of each other. Compare for example the formulae in (4.26) and (4.41)).

For all $0 \le t \le T$ let $\boldsymbol{\pi}(t) := (\pi_1(t), \ldots, \pi_{N_S}(t))$, where the variable $\pi_i(t)$ denotes the time-t portfolio weight of security S_i. We now define the set of admissible portfolios assuming a money market security is not available for investment (by the insider).

Definition 8 (Admissible portfolios - money market security not available for investment) *A set of control processes* $\boldsymbol{\pi}$ *where* $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ *for all* $0 \le t \le T$, *is said to be* **admissible** *(or an* **admissible portfolio***) if the following are satisfied:*

(i) $\boldsymbol{\pi}$ *is* $\mathbb{H}$*-predictable and bounded. (Recall that* $\boldsymbol{\pi}$ *is* $\mathbb{H}$*-predictable since the insider implements his portfolio allocation* $\boldsymbol{\pi}$ *based on his filtration* $\mathbb{H}$*.)*

(ii) Let $W(t)$ *denote the insider's time-t wealth value. Then we require for all* $0 \le s \le t \le T, i \in \mathcal{N}_S, j \in \mathcal{N}_B, k \in \mathcal{N}_q$ *that the following forward integrals exist, viz*

$$\int_s^t \sigma_{ij}(u)\pi_i(u)W(u^-)d^-B_j(u) \quad \text{and} \quad \int_s^t \int_{\mathbb{R}^N} g_{ik}(u,\mathbf{z})\pi_i(u)W(u^-)\tilde{q}_k(d^-u,d\mathbf{z})$$
$$\int_s^t \sigma_{ij}(u)\pi_i(u)d^-B_j(u) \quad \text{and} \quad \int_s^t \int_{\mathbb{R}^N} g_{ik}(u,\mathbf{z})\pi_i(u)\tilde{q}_k(d^-u,d\mathbf{z}).$$

(iii) In [57] the insider wealth process W *is shown to evolve according to the forward stochastic differential equation*

$$\frac{d^-W(t)}{W(t^-)} = \sum_{i=1}^{N_S} \xi_i(t)\pi_i(t)dt + \sum_{j=1}^{N_B}\sum_{i=1}^{N_S} \sigma_{ij}(t)\pi_i(t)d^-B_j(t)$$

$$+\sum_{j=1}^{N_q}\sum_{i=1}^{N_S}\int_{\mathbb{R}^N} g_{ij}(t,\mathbf{z})\pi_i(t)\tilde{q}_j(d^-t,d\mathbf{z}). \tag{3.19}$$

We require $\boldsymbol{\pi}$ to be such that (3.19) has a unique solution for W and $0 < W(t)$ almost surely for all $0 \le t \le T$. In particular for all $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N$ we require that $-1 < \sum_{j=1}^{N_q}\sum_{i=1}^{N_S} g_{ij}(t,\mathbf{z})\pi_i(t)$ almost surely. (Forward stochastic integrals are present in (3.19) since the insider portfolio $\boldsymbol{\pi}$ is $\mathbb{H}$-predictable and the Brownian motions $\mathbf{B}$ and (the pure jump processes associated with) $\tilde{\mathbf{q}}$ are $\mathbb{F}$-adapted.)

(iv) For all $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, j \in \mathcal{N}_q$ we must have that $1+\sum_{i=1}^{N_S} g_{ij}(t,\mathbf{z})\pi_i(t) > 0$ almost surely since this expression will be the argument of the natural logarithm in (3.30) below.

♦

In Definition 8 $\boldsymbol{\pi}$ is not constrained to sum to unity because the portfolio selection problems in Section 3.6 are solved subject to constraints of which the unity weight constraint is one - see the discussion after Definition 9. We now define the set of admissible portfolios assuming a money market security is available for investment (by the insider).

Definition 9 (Admissible portfolios - money market security available for investment) *A set of control processes $\boldsymbol{\pi}$ where $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ for all $0 \le t \le T$, is said to be* **admissible** *(or an* **admissible portfolio***) if the following are satisfied:*

(i) $\boldsymbol{\pi}$ is $\mathbb{H}$-predictable and bounded.

(ii) Let $W = W(t)$ denote the insider's time-t wealth value. For all $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, i, j \in \mathcal{N}_S, k \in \mathcal{N}_B, m \in \mathcal{N}_q$ let

$$\begin{array}{rclcrcl} \hat{\xi}_i(t) & := & \xi_i(t) - \xi_0(t), & & \hat{\sigma}_{ik}(t) & := & \sigma_{ik}(t) - \sigma_{0,k}(t), \\ \hat{g}_{im}(t,\mathbf{z}) & := & g_{im}(t,\mathbf{z}) - g_{0,m}(t,\mathbf{z}) & \text{and} & \bar{\sigma}_{ij}(t) & := & \sum_{k=1}^{N_B}\hat{\sigma}_{ik}(t)\hat{\sigma}_{jk}(t). \end{array} \tag{3.20}$$

Then we require for all $0 \le s \le t \le T, i \in \mathcal{N}_S, j \in \mathcal{N}_B, k \in \mathcal{N}_q$ that the following forward integrals exist, viz

$$\int_s^t \hat{\sigma}_{ij}(u)\pi_i(u)W(u^-)d^-B_j(u) \quad \text{and} \quad \int_s^t\int_{\mathbb{R}^N} \hat{g}_{ik}(u,\mathbf{z})\pi_i(u)W(u^-)\tilde{q}_k(d^-u,d\mathbf{z})$$
$$\int_s^t \hat{\sigma}_{ij}(u)\pi_i(u)d^-B_j(u) \quad \text{and} \quad \int_s^t\int_{\mathbb{R}^N} \hat{g}_{ik}(u,\mathbf{z})\pi_i(u)\tilde{q}_k(d^-u,d\mathbf{z}).$$

(iii) If a money market security is available for investment, then from [57] the evolution of the insider wealth process is given by the forward stochastic

differential equation

$$\frac{d^- W(t)}{W(t^-)} = \sum_{i=0}^{N_S} \xi_i(t)\pi_i(t)dt + \sum_{j=1}^{N_B}\sum_{i=0}^{N_S} \sigma_{ij}(t)\pi_i(t)d^- B_j(t) + \sum_{j=1}^{N_q}\sum_{i=0}^{N_S}\int_{\mathbb{R}^N} g_{ij}(t,\mathbf{z})\pi_i(t)\tilde{q}_j(d^- t, d\mathbf{z}). \tag{3.21}$$

We require $\boldsymbol{\pi}$ *to be such that (3.21) has a unique solution for* W *and* $0 < W(t)$ *almost surely for all* $0 \leq t \leq T$*. In particular for all* $0 \leq t \leq T, \mathbf{z} \in \mathbb{R}^N$ *we require that* $-1 < \sum_{j=1}^{N_q}\sum_{i=1}^{N_S} \hat{g}_{ij}(t,\mathbf{z})\pi_i(t)$ *almost surely.*

(iv) For all $0 \leq t \leq T, \mathbf{z} \in \mathbb{R}^N, j \in \mathcal{N}_q$ *let*

$$G_j(t,\mathbf{z}) := g_{0,j}(t,\mathbf{z}) + \sum_{i=1}^{N_S} \hat{g}_{ij}(t,\mathbf{z})\pi_i(t). \tag{3.22}$$

Then we require that $1 + G_j(t,\mathbf{z}) > 0$ *almost surely since this expression will be the argument of the natural logarithm in (3.32) below.*

♦

We now discuss the equality and inequality constraints which admissible portfolios must satisfy. First the equality constraints. In Sections 3.6.1 and 3.6.3 the security S_0 (which could possibly be riskless) is eliminated from the optimisation problem to make it unconstrained. In this case we require that the portfolio $(\pi_0, \boldsymbol{\pi})$ satisfies

$$\sum_{i=0}^{N_S} \pi_i(t) = 1 \tag{3.23}$$

almost surely for all $0 \leq t \leq T$. In Section 3.6.2 no riskless securities are available for investment and in this case we require $\boldsymbol{\pi}$ to satisfy

$$\sum_{i=1}^{N_S} \pi_i(t) = \Upsilon(t) \in \mathbb{R} \tag{3.24}$$

almost surely for all $0 \leq t \leq T$. In (3.24), as in equation (2.5), the function Υ is almost always identically 1 requiring the portfolio weights $\boldsymbol{\pi}$ to sum to unity at each time t. The function Υ is included in (3.24) so that a money market security weight can be explicitly constrained. Suppose a money market security is not available for investment. At some point in finding a constrained optimal portfolio (for the insider), the inverse of the covariance matrix $\bar{\boldsymbol{\sigma}}(t)$ will be calculated. If however a money market security is available for investment, then one cannot simply extend the analysis where all securities are risky, to the case where some security is riskless (to derive insider constrained optimal portfolios). The reason for this is that the covariance matrices $\bar{\boldsymbol{\sigma}}$ will no longer be nonsingular. Thus, if a money market security is available for investment, then it is not possible to explicitly constrain the weight π_0 of S_0. If one wants to explicitly constrain investment in S_0, then decide on the desired money market security weight $\pi_0(t)$. Then ensure that the weights of the risky securities $\mathbf{S}$ sum to $\Upsilon(t) = 1 - \pi_0(t)$.

If however the insider is not concerned about explicitly constraining investment in a money market security, then via (3.23) eliminate π_0 from the portfolio optimisation and solve the resulting portfolio selection problem. Dealing with a money market security in this way doesn't make it possible to explicitly constrain investment in this security. We discuss this further in Section 3.6.3. As stated in Chapter 2 (Section 2.2), for asset managers there are prudential guidelines which require a minimum percentage investment in cash. Thus it is important for them to be able to explicitly constrain their money market security weight. For these managers the portfolio selection model in Section 3.6.2 is more appropriate. For hedge fund managers however there is less restriction on the percentage investment in cash. Managers of these funds have more freedom in the bets they take. Thus it may be more appropriate to model their money market account as a catchall security, which is the approach in Sections 3.6.1 and 3.6.3 (where π_0 is eliminated via (3.23)).

Next we discuss the inequality constraints which admissible portfolios must satisfy. In Section 3.6.1 an unconstrained portfolio selection problem is solved. In Sections 3.6.2 and 3.6.3 however constrained optimal portfolios $\boldsymbol{\pi}$ are found which must satisfy almost surely the inequality constraints

$$\mathbf{a}(t) \;\leq\; \boldsymbol{\pi}(t) \;\leq\; \mathbf{b}(t) \quad \text{for all } 0 \leq t \leq T. \tag{3.25}$$

In (3.25) we have that $\mathbf{a}(t) := (a_1(t), \ldots, a_{N_S}(t))$ and $\mathbf{b}(t) := (b_1(t), \ldots, b_{N_S}(t))$, where for all $0 \leq t \leq T, i \in \mathcal{N}_S, a_i(t) < b_i(t)$, are some bounded exogenously given bounds for the portfolio weights $\boldsymbol{\pi}(t)$. If a money market security is (not) available for investment, then the bounds $\mathbf{a}(t)$ and $\mathbf{b}(t)$ must be such that if any set of constraints (3.25) is active, then we must have that equation (3.23) (equation (3.24)) must be satisfied almost surely. To continue we require Itô's formula for functionals of forward Lévy processes.

3.5 Itô's formula for functionals of forward Lévy processes

Let $u = u(s, \omega)$ be Lebesgue integrable. For each $j \in \mathcal{N}_B$ let the process $v_j = v_j(t, \omega)$ be forward integrable with respect to B_j. For each $k \in \mathcal{N}_q$ let $A_k \in \mathcal{B}(\mathbb{R}^N)$ be bounded below and let each process $p_k = p_k(t, \mathbf{z}, \omega)$ be forward

Poisson integrable. Consider the forward Lévy process of the form

$$\begin{aligned} X(t) &= X(0) + \int_0^t \left(u(s) + \sum_{j=1}^{N_q} \int_{A_j} p_j(s,\mathbf{z}) \nu_j^{\mathbb{F}}(d\mathbf{z}) \right) ds \\ &\quad + \sum_{j=1}^{N_B} \int_0^t v_j(s) d^- B_j(s) + \sum_{j=1}^{N_q} \int_0^t \int_{A_j} p_j(s,\mathbf{z}) \tilde{q}_j(d^- s, d\mathbf{z}). \end{aligned} \tag{3.26}$$

From [42] we have the Itô formula for functionals of forward Lévy processes of the form (3.26). This formula has the same form as the usual Itô formula (for functionals of Lévy processes) in ([7], Lemma 4.4.6).

Theorem 6 *For each $j \in \mathcal{N}_q$ let $A_j \in \mathcal{B}(\mathbb{R}^N)$ be bounded below. If X is a forward Lévy process of the form (3.26), then for each $f \in C^2(\mathbb{R})$ and for each $0 \leq t$ we have almost surely that*

$$\begin{aligned} f(X(t)) - f(X(0)) &= \int_0^t \left(f'(X(s))u(s) + \frac{1}{2} f''(X(s)) \sum_{j=1}^{N_B} v_j^2(s) \right) ds \\ &\quad + \sum_{j=1}^{N_B} \int_0^t f'(X(s)) v_j(s) d^- B_j(s) \\ &\quad - \sum_{j=1}^{N_q} \int_0^t \int_{A_j} f'(X(s)) p_j(s,\mathbf{z}) \nu_j^{\mathbb{F}}(d\mathbf{z}) ds \\ &\quad + \sum_{j=1}^{N_q} \int_0^t \int_{A_j} \left[f\left(X(s^-) + p_j(s,\mathbf{z})\right) - f(X(s^-)) \right] \nu_j^{\mathbb{F}}(d\mathbf{z}) ds \\ &\quad + \sum_{j=1}^{N_q} \int_0^t \int_{A_j} \left[f\left(X(s^-) + p_j(s,\mathbf{z})\right) - f(X(s^-)) \right] \tilde{q}_j(d^- s, d\mathbf{z}). \end{aligned}$$

Proof: See ([42], Theorem 8). ■

Using Theorem 6 we now derive the form of the insider wealth process W if the securities $\mathbf{S}$ are respectively continuous and discontinuous.

3.5.1 Form of W if $\mathbf{S}$ is continuous

In this section the form of the insider wealth process W is derived assuming the securities $\mathbf{S}$ are continuous. Let the matrix of jump coefficients be denoted by $\mathbf{g} \equiv [g_{ij}], i \in \mathcal{N}_S, j \in \mathcal{N}_q$. Suppose $\mathbf{S}$ are driven by diffusions, in other words $\mathbf{g} \equiv \mathbf{0}$ almost surely in (3.19). For all $0 \leq t \leq T, \mathbf{z} \in \mathbb{R}^N, j \in \mathcal{N}_B, k \in \mathcal{N}_q$ let

$$\begin{aligned} f(x) &= \ln x, & u(t) &= W(t^-) \textstyle\sum_{i=1}^{N_S} \xi_i(t) \pi_i(t), \\ v_j(t) &= W(t^-) \textstyle\sum_{i=1}^{N_S} \sigma_{ij}(t) \pi_i(t) \quad \text{and} & p_k(t,\mathbf{z}) &= 0. \end{aligned}$$

Then from Definition 8 and Theorem 6 we have that

$$\begin{aligned} W(T) &= W(t)\exp\left(\int_t^T \left(\sum_{i=1}^{N_S}\xi_i(s)\pi_i(s) - \frac{1}{2}\sum_{j=1}^{N_B}\left(\sum_{i=1}^{N_S}\sigma_{ij}(s)\pi_i(s)\right)^2\right) ds \right.\\ &\quad \left. + \sum_{j=1}^{N_B}\sum_{i=1}^{N_S}\int_t^T \sigma_{ij}(s)\pi_i(s) d^- B_j(s)\right). \end{aligned} \tag{3.27}$$

In later sections we shall require the formula of the wealth process for the case where π_0 is eliminated from the portfolio optimisation via the equality constraint (3.23). Thus with $\mathbf{g} \equiv \mathbf{0}$ almost surely in (3.21) we have that

$$\frac{d^- W(t)}{W(t^-)} = \left(\xi_0(t) + \sum_{i=1}^{N_S}\hat{\xi}_i(t)\pi_i(t)\right) dt + \sum_{j=1}^{N_B}\left(\sigma_{0,j}(t) + \sum_{i=1}^{N_S}\hat{\sigma}_{ij}(t)\pi_i(t)\right) d^- B_j(t). \tag{3.28}$$

For all $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, j \in \mathcal{N}_B, k \in \mathcal{N}_q$ let

$$\begin{aligned} f(x) &= \ln x, \\ u(t) &= W(t^-)\left(\xi_0(t) + \sum_{i=1}^{N_S}\hat{\xi}_i(t)\pi_i(t)\right), \\ v_j(t) &= W(t^-)\left(\sigma_{0,j}(t) + \sum_{i=1}^{N_S}\hat{\sigma}_{ij}(t)\pi_i(t)\right) \quad \text{and} \\ p_k(t,\mathbf{z}) &= 0. \end{aligned}$$

Then from (3.28) and Theorem 6 we have that

$$\begin{aligned} W(T) &= W(t)\exp\left(\int_t^T \left(\xi_0(s) + \sum_{i=1}^{N_S}\hat{\xi}_i(s)\pi_i(s) \right.\right.\\ &\quad \left. - \frac{1}{2}\sum_{j=1}^{N_B}\left(\sigma_{0,j}(s) + \sum_{i=1}^{N_S}\hat{\sigma}_{ij}(s)\pi_i(s)\right)^2\right) ds \\ &\quad \left. + \sum_{j=1}^{N_B}\int_t^T \left(\sigma_{0,j}(s) + \sum_{i=1}^{N_S}\hat{\sigma}_{ij}(s)\pi_i(s)\right) d^- B_j(s)\right). \end{aligned} \tag{3.29}$$

3.5.2 Form of W if S is discontinuous

In this section the form of the insider wealth process W is derived assuming the securities $\mathbf{S}$ are discontinuous. For all $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, j \in \mathcal{N}_B, k \in \mathcal{N}_q$ let

$$\begin{aligned} f(x) &= \ln x, & u(t) &= W(t^-)\textstyle\sum_{i=1}^{N_S}\xi_i(t)\pi_i(t), \\ v_j(t) &= W(t^-)\textstyle\sum_{i=1}^{N_S}\sigma_{ij}(t)\pi_i(t) \quad \text{and} & p_k(t,\mathbf{z}) &= W(t^-)\textstyle\sum_{i=1}^{N_S}g_{ik}(t,\mathbf{z})\pi_i(t). \end{aligned}$$

Then from Definition 8 and Theorem 6 we have that

$$
\begin{aligned}
W(T) &= W(t)\exp\left(\int_t^T \left(\sum_{i=1}^{N_S}\xi_i(s)\pi_i(s) - \frac{1}{2}\sum_{j=1}^{N_B}\left(\sum_{i=1}^{N_S}\sigma_{ij}(s)\pi_i(s)\right)^2\right)ds\right.\\
&\quad + \sum_{j=1}^{N_B}\sum_{i=1}^{N_S}\int_t^T \sigma_{ij}(s)\pi_i(s)d^-B_j(s)\\
&\quad + \sum_{j=1}^{N_q}\int_t^T\int_{A_j}\ln\left(1+\sum_{i=1}^{N_S}g_{ij}(s,\mathbf{z})\pi_i(s)\right)\tilde{q}_j(d^-s,d\mathbf{z})\\
&\quad \left. + \sum_{j=1}^{N_q}\int_t^T\int_{A_j}\left(\ln\left(1+\sum_{i=1}^{N_S}g_{ij}(s,\mathbf{z})\pi_i(s)\right) - \sum_{i=1}^{N_S}g_{ij}(s,\mathbf{z})\pi_i(s)\right)\nu_j^{\mathbb{F}}(d\mathbf{z})ds\right).
\end{aligned}
\tag{3.30}
$$

If π_0 is eliminated from (3.21) using (3.23), then we have that

$$
\begin{aligned}
\frac{d^-W(t)}{W(t^-)} &= \left(\xi_0(t)+\sum_{i=1}^{N_S}\hat{\xi}_i(t)\pi_i(t)\right)dt + \sum_{j=1}^{N_B}\left(\sigma_{0,j}(t)+\sum_{i=1}^{N_S}\hat{\sigma}_{ij}(t)\pi_i(t)\right)d^-B_j(t)\\
&\quad + \sum_{j=1}^{N_q}\int_{\mathbb{R}^N}\left(g_{0,j}(t,\mathbf{z})+\sum_{i=1}^{N_S}\hat{g}_{ij}(t,\mathbf{z})\pi_i(t)\right)\tilde{q}_j(d^-t,d\mathbf{z}).
\end{aligned}
\tag{3.31}
$$

For all $0\le t\le T, \mathbf{z}\in\mathbb{R}^N, j\in\mathcal{N}_B, k\in\mathcal{N}_q$ let

$$
\begin{aligned}
f(x) &= \ln x,\\
u(t) &= W(t^-)\left(\xi_0(t)+\sum_{i=1}^{N_S}\hat{\xi}_i(t)\pi_i(t)\right),\\
v_j(t) &= W(t^-)\left(\sigma_{0,j}(t)+\sum_{i=1}^{N_S}\hat{\sigma}_{ij}(t)\pi_i(t)\right) \quad \text{and}\\
p_k(t,\mathbf{z}) &= W(t^-)\left(g_{0,k}(t,\mathbf{z})+\sum_{i=1}^{N_S}\hat{g}_{ik}(t,\mathbf{z})\pi_i(t)\right).
\end{aligned}
$$

Then from (3.31) and Theorem 6 we have that

$$
\begin{aligned}
W(T) &= W(t)\exp\left(\int_t^T\left(\xi_0(s)+\sum_{i=1}^{N_S}\hat{\xi}_i(s)\pi_i(s)\right.\right.\\
&\quad \left. -\frac{1}{2}\sum_{j=1}^{N_B}\left(\sigma_{0,j}(s)+\sum_{i=1}^{N_S}\hat{\sigma}_{ij}(s)\pi_i(s)\right)^2\right)ds\\
&\quad +\sum_{j=1}^{N_B}\int_t^T\left(\sigma_{0,j}(s)+\sum_{i=1}^{N_S}\hat{\sigma}_{ij}(s)\pi_i(s)\right)d^-B_j(s)
\end{aligned}
$$

$$+\sum_{j=1}^{N_q}\int_t^T\int_{A_j}\ln\left(1+G_j(s,\mathbf{z})\right)\tilde{q}_j(d^-s,d\mathbf{z})$$
$$\left.+\sum_{j=1}^{N_q}\int_t^T\int_{A_j}\left[\ln\left(1+G_j(s,\mathbf{z})\right)-G_j(s,\mathbf{z})\right]\nu_j^{\mathbb{F}}(d\mathbf{z})ds\right). \tag{3.32}$$

We now define the insider portfolio selection problems we wish to solve.

3.6 The Optimisation Problems

Let $U:[0,T]\times\mathbb{R}^+\to\mathbb{R}$ be a strictly increasing, concave and at least once differentiable utility function which is assumed to best describe the insider's investment preferences. We also require that $U(t,W):=-\infty$ for $W\leq 0$ which effectively imposes a positivity of wealth constraint. In this section we state the insider constrained portfolio selection problems we wish to solve. The constrained portfolio selection problems are multidimensional generalisations of those in ([18], [42], [62]) and this immediately introduces (amongst other things) an explicit unity weight constraint on the portfolio security weights which is not present in these papers. Inequality constraints on the insider's portfolio security weights are also imposed and the resulting constrained portfolio selection problems are solved via the method of *calculus of variations*. We employ the calculus of variations to show how at each point in the derivation of insider constrained optimal portfolios, our results reduce to those in ([18], [42], [62]) in the unconstrained one-dimensional case. We now define the insider portfolio selection problems we wish to solve.

3.6.1 Problem (P1)

The first portfolio selection problem we wish to solve is of the form

<u>**Problem (P1)** :</u>

$$\sup_{\boldsymbol{\pi}\in\mathcal{P}_1} J_1(\boldsymbol{\pi}) := \sup_{\boldsymbol{\pi}\in\mathcal{P}_1}\mathbb{E}[U(T,W(T))] \tag{3.33}$$

where $\mathcal{P}_1$ is the set of admissible portfolios (Definition 9) for problem **(P1)**. No explicit portfolio weight constraints or *penalty functions* [39] (defined in Section 3.6.2 below) are included in **(P1)**. The reason for this is that if the utility function in (3.33) is not logarithmic, then we cannot solve problem **(P1)** in the presence of explicit portfolio weight constraints and/or presence of penalty functions. See Section 3.7.2 specifically Remark 3 for further discussion of this. Note that the portfolio weights $\boldsymbol{\pi}$ must always sum to unity, so in the presence of the equality constraint (3.23), problem **(P1)** is actually a constrained optimisation problem. We convert **(P1)** to an unconstrained optimisation problem however by eliminating π_0 using (3.23).

3.6.2 Problem (P2)

From [62], considered in [116] is an insider portfolio selection problem. There the expected logarithmic utility of the insider's terminal wealth over a finite time horizon $[0, T]$ is maximised, in other words

$$V := \sup_{\pi} \mathbb{E}\left[\log W(T)\right], \tag{3.34}$$

where the supremum in (3.34) is taken over all admissible portfolios as defined in [116] and the insider's financial market is comprised of a money market security and a diffusion. In [116] it is assumed that, at time 0, the insider knows in addition the future value of the underlying Brownian motion B at time $T_0 \in \mathbb{R}^+$, where $T < T_0$. His filtration $\mathbb{H}$ is then given by $\mathcal{H}_t = \sigma(\mathcal{F}_t \cup \sigma(B(T_0)))$, the filtration generated by the Brownian motion up to time t and $B(T_0)$. In [116] it is proved that in this case the optimal insider portfolio is

$$\pi(t) = \frac{\xi(t) - r(t)}{\sigma^2(t)} + \frac{B(T_0) - B(t)}{\sigma(t)(T_0 - t)} \tag{3.35}$$

and that the corresponding maximal expected utility is

$$V \quad = \quad \mathbb{E}\left[\int_0^T \left(r(s) + \frac{1}{2}\frac{(\xi(s) - r(s))^2}{\sigma^2(s)} + \frac{1}{2(T_0 - s)}\right) ds\right]. \tag{3.36}$$

The portfolio (3.35) is unappealing for the insider since as $t \to T_0^-$, (3.35) becomes proportional to the derivative of B at $t = T_0^-$. But B is nowhere differentiable so this will result in the insider having to implement significantly different portfolio rebalances (infinitely often) and so draw much attention to his investment. Secondly, as $T \to T_0$ in (3.36), V becomes infinite. Due to the introduction of penalty functions in [62] the above undesired properties of (3.35) (the optimal insider portfolio derived in [116]) are mitigated. So from [62] let $\mathbb{L}$ be a linear operator of the form

$$\mathbb{L}(\pi(t)) = \kappa(t)\pi(t) \quad \text{or} \quad \mathbb{L}(\pi(t)) = \bar{\kappa}(t)\frac{d}{dt}\pi(t), \quad 0 \le t \le T \tag{3.37}$$

where κ is deterministic and $\bar{\kappa}$ at least once-differentiable such that $(\mathbb{L}(\pi(t)))^2$ is Lebesgue integrable. (For simplicity we do not assume that $\bar{\kappa}$ is *differentiable in distribution* as explained in [88].) For notational simplicity we write $\mathbb{L}(\pi(t))$ instead of $(\mathbb{L}(\pi))(t)$ (which emphasises that $\mathbb{L}$ acts on a function π and produces another function of time). In (3.37) the operators $\mathbb{L}$ measure respectively the size and fluctuations of the portfolio π. To prevent or suppress the undesired behaviour of a wildly fluctuating optimal insider portfolio (3.35), the following objective functional is posited in [62] viz

$$V := \sup_{\pi} \mathbb{E}\left[\log W(T) - \frac{1}{2}\int_0^T (\mathbb{L}(\pi(s)))^2 ds\right]. \tag{3.38}$$

Before we state the second insider constrained portfolio selection problem we wish to solve, we define the set of admissible portfolios.

Definition 10 (Admissible portfolios) *A set of control processes* $\boldsymbol{\pi}$ *where* $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ *for all* $0 \leq t \leq T$*, is said to be* **admissible** *(or an* **admissible portfolio***) for problem* **(P2)** *if the following hold:*

(i) $\boldsymbol{\pi}$ *satisfies Definition 8.*

(ii) For each $i, j \in \mathbb{N}_S$ *let*

$$\mathbb{L}_{ij} : L^2([0,T]) \to L^2([0,T])$$

be a linear operator. Let $\mathbb{L} \equiv [\mathbb{L}_{ij}]$*. Then for each* $\omega \in \Omega$ *the portfolio* $\boldsymbol{\pi}$ *must be in the domain of* $\mathbb{L}$*, the* N_S *processes* $\mathbb{L}(\boldsymbol{\pi}(t))$ *must be* $\mathbb{H}$*-adapted and using the Hilbert-Schmidt norm*[4] *the function* $\|\mathbb{L}(\boldsymbol{\pi}(t))\|^2$ *must be Lebesgue integrable.*

We denote by $\mathcal{P}_2$ *the set of all admissible portfolios for problem* **(P2)**. ♦

In Definition 10 the operator $\mathbb{L}$ is linear to ensure that $\|\mathbb{L}(\boldsymbol{\pi}(t))\|^2$ is convex in each $\pi_i(t), i \in \mathcal{N}_S$ so that the objective functional (3.43) defined below is concave in each $\pi_i(t), i \in \mathcal{N}_S$. Also particular clusters of securities **S** may be penalised differently which is why $\mathbb{L}$ is not simply a diagonal matrix operator. For each $i, j \in \mathcal{N}_S$ the penalty function $\mathbb{L}_{ij} : \mathcal{P}_2 \to \mathcal{P}_2$ measures for example the size and/or fluctuations of the portfolio weight π_j. As a multidimensional generalisation of the penalty functions discussed in [62] we could have for all $0 \leq t \leq T$ that

$$\mathbb{L}(\boldsymbol{\pi}(t)) = \boldsymbol{\kappa}(t)\boldsymbol{\pi}(t), \tag{3.39}$$

where the $N_S \times N_S$ matrix of real-valued functions $\boldsymbol{\kappa}$ are deterministic such that the function $\|\mathbb{L}(\boldsymbol{\pi}(t))\|^2$ is Lebesgue-integrable. The form of $\mathbb{L}$ in (3.39) models the situation where the insider is penalised for large volumes of trade in the securities **S**. Again generalising the penalty functions discussed in [62], an alternative choice could be

$$\mathbb{L}(\boldsymbol{\pi}(t)) = \bar{\boldsymbol{\kappa}}(t)\frac{d}{dt}\boldsymbol{\pi}(t), \tag{3.40}$$

where the $N_S \times N_S$ matrix of real-valued functions $\bar{\boldsymbol{\kappa}}$ are at least once-differentiable such that the function $\|\mathbb{L}(\boldsymbol{\pi}(t))\|^2$ is Lebesgue-integrable. In this case the insider is penalised for large trade fluctuations in the securities **S**. Other choices of $\mathbb{L}$ are possible, including combinations of (3.39) and (3.40).

In this section the objective functional of the portfolio selection problem we solve was inspired by (3.38), the objective functional posited in [62]. We now explain why the insider constrained portfolio selection problem **(P2)** has the

[4]From ([61], [144]) the Hilbert-Schmidt norm of a real-valued $N_S \times N_S$ matrix $\mathbf{K} \equiv [K_{ij}]$ is defined as

$$\|\mathbf{K}\| = \sqrt{\sum_{i,j=1}^{N_S} \mathbb{K}_{ij}^2}.$$

form stated in (3.43)-(3.45) below. Now we aim to derive constrained portfolios which provide for an insider the maximal expected utility of terminal wealth (over a finite time horizon). To explain the ideas we state one of the simpler insider constrained portfolio selection problems we wish to solve viz

$$\sup_{\boldsymbol{\pi} \in \mathcal{P}_2} \mathbb{E}[U(T, W(T))] \tag{3.41}$$

$$\text{subject to } (3.24) \text{ and } (3.25). \tag{3.42}$$

The difficulty with solving the constrained optimisation problem (3.41)-(3.42) is that the objective functional (3.41) is comprised not only of Lebesgue integrals but also more general stochastic integrals. If (3.41) was comprised only of Lebesgue integrals (in other words the financial market model (3.14)-(3.15) was deterministic), then it is very easy to solve a constrained optimisation problem subject to either algebraic or (Lebesgue) integral constraints. See [53] for how to do this. (Algebraic constraints are for example of the form (3.23)-(3.25).) From [53], a deterministic constrained *variational problem* (in other words a problem in *variational calculus* [125]) is solved by focusing on the integrand in the objective functional and forming the Lagrangian by incorporating the constraints. The (constrained) variational problem (3.41)-(3.42) is stochastic. Thus we cannot focus only on the integrand in (3.41) (and form the Lagrangian). From the form (3.30) of the insider wealth process W, the integrators $\mathbf{B}$ and $\mathbf{q}$ contain crucial information about the evolution of the financial market (3.14)-(3.15). We have not found any theory which solves stochastic variational problems subject to algebraic constraints where the important point is that forward stochastic integrals are present in the objective functional. The theory which comes closest to possibly solving (3.41)-(3.42) is *duality theory* discussed in ([72], Chapters 5-6). Duality theory allows one to solve stochastic constrained variational problems where (the objective functional is a stochastic integral and) the constraints are algebraic. Duality theory however cannot be applied to the insider problem (3.41)-(3.42) since duality theory relies heavily on sub- and super-martingale properties of the stochastic integrals comprising the investor's wealth process which is not present in our setting. (Recall that the disturbances $\mathbf{B}$ and $\mathbf{q}$ are $\mathbb{F}$-adapted whereas the insider portfolio process $\boldsymbol{\pi}$ is $\mathbb{H}$-predictable and $\mathbb{F} \subseteq \mathbb{H}$. This is why the insider wealth process W is comprised of forward stochastic integrals.) This is the first reason why the constrained portfolio selection problem **(P2)** has the alternative form defined in (3.43)-(3.45) below.

Next, an *isoperimetric problem* is a variational problem subject to integral constraints. There is standard theory in ([39], Section 3.6) and [45] for solving isoperimetric problems. We have not however been able to convert the problem (3.41)-(3.42) (subject to algebraic constraints) to an equivalent isoperimetric problem (subject to integral constraints). At this point however we recall the following from the introduction of Chapter 2, Section 2.3:

> *Consider a constrained optimisation problem with a concave objective function. The constraints that are active at a feasible portfolio* $\boldsymbol{\pi}$ *restrict the domain of feasibility in neighbourhoods of* $\boldsymbol{\pi}$*,*

while the inactive constraints have no influence in the neighbourhoods of $\boldsymbol{\pi}$*. So if we know a priori which constraints in (3.25) are active, then the resulting portfolio is a local maximum determined by ignoring the inactive constraints and* **treating all other constraints as equality constraints**. *We then solve each of a family of equality constrained optimisation problems from scratch to find a constrained optimal solution.*

So the way to solve a constrained optimisation problem *with a concave objective functional* is first to solve the unconstrained form of the problem. If the unconstrained solution satisfies the inequality constraints (3.25), then the unconstrained solution is in fact also the constrained optimal solution. If the unconstrained solution violates (3.25), then different combinations of inequality constraints need to be set active (at the times when the unconstrained solution violates the inequality constraints) and each equality constrained optimisation problem must be solved *from scratch*. A constrained optimal solution is that which satisfies the (equality and inequality) constraints in (3.42) and has the largest objective functional value (3.41). So in summary:

- We are interested in solving the constrained optimisation problem (3.41)-(3.42) which is comprised of an integral objective functional subject to algebraic constraints.
- We cannot simply focus on the integrand in (3.41) and form the Lagrangian (by incorporating the constraints (3.42)) since (3.41) is comprised of stochastic integrals. The integrators $\mathbf{B}$ and $\mathbf{q}$ provide crucial information about the state of the financial market (3.14)-(3.15).
- We have not found any theory which solves variational problems subject to algebraic constraints and where forward stochastic integrals are present in the objective functional.
- Theory for solving isoperimetric problems exist. We have not however been able to convert the problem (3.41)-(3.42) (subject to algebraic constraints) to an equivalent isoperimetric problem.
- Constrained optimisation problems are solved by setting active different combinations of inequality constraints when required and solving each equality constrained optimisation problem from scratch.
- Recalling how inequality constrained optimisation problems are actually solved, to solve (3.41)-(3.42) (subject to algebraic constraints), we solve an alternative isoperimetric problem (subject to integral constraints) in which we can set constraints active when required.
- We employ the standard theory in ([39], Section 3.6) and [45] (to solve isoperimetric problems) to solve the insider constrained portfolio selection problem **(P2)** defined below.

Thus the (alternative) insider constrained portfolio selection problem we solve is

Problem (P2) :

$$\bar{J}_2 := \sup_{\boldsymbol{\pi}\in\mathcal{P}_2} J_2(\boldsymbol{\pi}) := \sup_{\boldsymbol{\pi}\in\mathcal{P}_2} \mathbb{E}\left[U(T,W(T)) - \frac{1}{2}\int_0^T \|\mathbb{L}(\boldsymbol{\pi}(s))\|^2 ds\right] \quad (3.43)$$

$$\text{subject to } \mathbb{E}\left[\int_0^T k_0^1(t)\left|\Upsilon(t) - \sum_{i=1}^{N_S}\pi_i(t)\right| dt\right] = 0 \quad (3.44)$$

$$\mathbb{E}\left[\int_0^T k_j^1(t)\left|\sum_{i=1}^{N_S} h_{ij}(t)\pi_i(t) - \bar{h}_j(t)\right| dt\right] = 0, \ \ j = 1,\dots,M \in \mathbb{N}. (3.45)$$

For generality the function $k_0^1(t)$ in (3.44) is allowed to have the value 0 but in practice it is identically 1. Thus since the integrand in (3.44) is nonnegative it implies almost surely that (3.24) holds. Although (3.44) is a special case of (3.45), it is explicitly included to distinguish between the cases where a money market security is unavailable and available for investment by the insider. In (3.45) $M \in \mathbb{N}$ denotes the number of (linear) integral equality constraints. Let the set $\mathcal{N}_M := \{1,\dots,M\}$. Then for $j \in \mathcal{N}_M$ each $k_j^1 = k_j^1(t,\omega)$ is an *a priori* defined function which has either the value 0 or 1 and as a result defines over which time intervals in $[0,T]$ that integral equality constraint is active. For $i \in \mathcal{N}_S, j \in \mathcal{N}_M$ each function $h_{ij} = h_{ij}(t,\omega)$ and $\bar{h}_j = \bar{h}_j(t,\omega)$ must be such that each Lebesgue integral in (3.45) exists. The general integral constraints (3.44)-(3.45) will be used to impose (or set active) the algebraic constraints (3.24)-(3.25) when required. Other equality constraints which are also special cases of (3.45) include for example the following:

- The weighted sum of the expected returns of the securities **S** must equal some required portfolio expected return $\bar{\xi} = \bar{\xi}(t,\omega)$ (as in a mean-variance optimisation). This constraint is imposed via the integral constraint

$$\mathbb{E}\left[\int_0^T k_1^1(t)\left|\sum_{i=1}^{N_S}\xi_i(t)\pi_i(t) - \bar{\xi}(t)\right| dt\right] = 0 \quad (3.46)$$

  where the function k_1^1 is identically 1. Note that since the integrand in (3.46) is nonnegative, (3.46) implies almost surely that we must have

$$\sum_{i=1}^{N_S}\xi_i(t)\pi_i(t) = \bar{\xi}(t) \quad \text{for all } 0 \le t \le T.$$

  One could also impose that the weighted sum of the expected returns of some group of securities **S** must equal some required expected return.

- The sum of the weights of some group of securities must equal a desired value. For example

$$\mathbb{E}\left[\int_0^T k_2^1(t)\,|\pi_1(t)+\pi_4(t)+\pi_{17}(t)-\bar{\pi}(t)|\,dt\right]=0 \tag{3.47}$$

  where $\bar{\pi}=\bar{\pi}(t,\omega)$, and the function k_2^1 could have the value 1 only over some subset of $[0,T]$. Since the integrand in (3.47) is nonnegative it implies almost surely that we must have

$$\pi_1(t)+\pi_4(t)+\pi_{17}(t)=\bar{\pi}(t) \qquad \text{for all } 0\le t\le T.$$

- We could constrain the weight of some security to be a prespecified value for a prespecified period of time over $[0,T]$. This will be effected by the integral constraint

$$\mathbb{E}\left[\int_0^T k_3^1(t)\,|\pi_6(t)-b_6(t)|\,dt\right]=0 \tag{3.48}$$

  where the function $b_6=b_6(t,\omega)$ could be an *a priori* defined upper bound of π_6 and the function $k_3^1=k_3^1(t,\omega)$ could have the value 1 only over the time intervals $[0,1]$ and $[3,5]$ (with $T=10$ say) and zero otherwise.

So knowing how to solve constrained optimisation problems, we solve problem (3.41)-(3.42) (subject to algebraic constraints) by solving the alternative isoperimetric problem **(P2)**. Continuing, we use the theory of the calculus of variations ([39], [45]) in the presence of integral equality constraints, and form the Lagrangian

$$\begin{aligned} V_2(\boldsymbol{\pi}) \;:=\;& J_2(\boldsymbol{\pi})-\mathbb{E}\left[\int_0^T k_0^2k_0^1(s)\left|\Upsilon(s)-\sum_{i=1}^{N_S}\pi_i(s)\right|ds\right.\\ &\left.-\sum_{j=1}^{M}\int_0^T k_j^2k_j^1(s)\left|\sum_{i=1}^{N_S}h_{ij}(s)\pi_i(s)-\bar{h}_j(s)\right|ds\right] \end{aligned} \tag{3.49}$$

where the constants $k_0^2, k_j^2\in\mathbb{R}^+$ are the Lagrange multipliers associated with the integral constraints (3.44)-(3.45). At some point in trying to solve **(P2)** we shall have to differentiate (3.49). This will be done by considering the cases where the arguments of the absolute values in (3.49) are either positive or negative. The results of differentiating will differ only by a negative sign. Thus not to duplicate the operations we define for all $0\le t\le T, j\in\mathcal{N}_M$ the functions

$$\begin{aligned} \lambda_0(t) \;:=\;& \begin{cases} -k_0^2k_0^1(t) & \text{if } \Upsilon(t)-\sum_{i=1}^{N_S}\pi_i(t)\ge 0\\ k_0^2k_0^1(t) & \text{otherwise}\end{cases}\\ &\text{and}\\ \lambda_j(t) \;:=\;& \begin{cases} -k_j^2k_j^1(t) & \text{if } \sum_{i=1}^{N_S}h_{ij}(t)\pi_i(t)-\bar{h}_j(t)\ge 0\\ k_j^2k_j^1(t) & \text{otherwise.}\end{cases} \end{aligned} \tag{3.50}$$

We then rewrite the Lagrangian (3.49) as

$$\begin{aligned} V_2(\boldsymbol{\pi}) \quad := \quad & J_2(\boldsymbol{\pi}) + \mathbb{E}\left[\int_0^T \lambda_0(s)\left(\Upsilon(s) - \sum_{i=1}^{N_S}\pi_i(s)\right) ds \right. \\ & \left. + \sum_{j=1}^{M}\int_0^T \lambda_j(s)\left[\sum_{i=1}^{N_S} h_{ij}(s)\pi_i(s) - \bar{h}_j(s)\right] ds \right]. \end{aligned} \tag{3.51}$$

Thus by differentiating (3.51) once we produce the results of differentiating (3.49) in the two cases, by simply considering the positive and negative form of the multipliers $\lambda_j, j \in \mathcal{N}_M \cup \{0\}$. As is standard in the calculus of variations, by partially differentiating (3.51) with respect to the Lagrange multipliers k_0^2, k_j^2, we obtain the integral constraints (3.44)-(3.45). Thus a solution of **(P2)** is equivalent to finding a solution of the problem

$$\sup_{\boldsymbol{\pi} \in \mathcal{P}_2} V_2(\boldsymbol{\pi}) \text{ subject to (3.44)-(3.45).}$$

3.6.3 Problem (P3)

In the solution methodologies of both problems **(P1)** and **(P2)**, to obtain an explicit formula for an optimal portfolio $\boldsymbol{\pi}$, at some point in the analysis the inverse of a covariance matrix will need to be calculated. If a money market security S_0 (defined in (3.14)) is available for investment, then since its volatility is zero, the resulting covariance matrices will be singular. We circumvent this problem by eliminating π_0 via the unity weight constraint (3.23) and do not include the constraint (3.23) in the optimisation problem since it is then always satisfied. The difference between problems **(P2)** and **(P3)** is that in **(P2)** π_0 can be explicitly constrained to particular values whereas in **(P3)** the best one can do is ensure that for all $0 \le t \le T$ almost surely

$$1 - \sum_{i=1}^{N_S} b_i(t) \ \le \ \pi_0(t) \ \le \ 1 - \sum_{i=1}^{N_S} a_i(t),$$

where $\mathbf{a}$ and $\mathbf{b}$ are lower and upper bounds defined in (3.25). Essentially in **(P3)**, S_0 is a balancing security, which is the drawback of the approach in **(P3)**. Note however that in **(P2)** none of the characteristics of S_0 (such as its expected return $r(t)$ and its zero volatility) are taken into account in the portfolio optimisation. In **(P2)** the value of $\pi_0(t)$ is fixed irrespective of the relationship between the moments of the return distributions of S_0 and $\mathbf{S}$. This is the drawback of the approach in **(P2)**. In **(P3)** all the moments of the return distribution of S_0 are taken into account in the portfolio optimisation. In practice either one of the two approaches can be employed. Long-only managers may have to use the approach in **(P2)**, whereas hedge fund managers possibly have more flexibility and can employ the approach in **(P3)**. We now define the insider's set of admissible portfolios and then state the constrained portfolio selection problem we wish to solve.

Definition 11 (Admissible portfolios) *A set of control processes* $\boldsymbol{\pi}$ *where* $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ *for all* $0 \leq t \leq T$, *is said to be* **admissible** *(or an* **admissible portfolio***) for problem* **(P3)** *if the following hold:*

(i) $\boldsymbol{\pi}$ *satisfies Definition 9.*

(ii) Definition 10(ii).

We denote by $\mathcal{P}_3$ *the set of all admissible portfolios for problem* **(P3)***.* ♦

If a money market security is available for investment, then the insider constrained portfolio selection problem we want to solve is

Problem (P3) :

$$\bar{J}_3 := \sup_{\boldsymbol{\pi}\in\mathcal{P}_3} J_3(\boldsymbol{\pi}) := \sup_{\boldsymbol{\pi}\in\mathcal{P}_3} \mathbb{E}\left[U(T, W(T)) - \frac{1}{2}\int_0^T \|\mathbb{L}(\boldsymbol{\pi}(s))\|^2 ds\right] \quad (3.52)$$

$$\text{subject to} \quad (3.45). \quad (3.53)$$

As in Section 3.6.2 we can show that solving **(P3)** is equivalent to solving the constrained optimisation problem

$$\sup_{\boldsymbol{\pi}\in\mathcal{P}} V_3(\boldsymbol{\pi}) \text{ subject to } (3.45),$$

where the Lagrangian

$$V_3(\boldsymbol{\pi}) := J_3(\boldsymbol{\pi}) + \mathbb{E}\left[\sum_{j=1}^{M}\int_0^T \lambda_j(s)\left[\sum_{i=1}^{N_S} h_{ij}(s)\pi_i(s) - \bar{h}_j(s)\right] ds\right], \quad (3.54)$$

$J_3(\boldsymbol{\pi})$ is defined in (3.52), $\lambda_j, j \in \mathcal{N}_M$ are defined in (3.50) (and for all $0 \leq t \leq T$ we must have that (3.23) holds almost surely). We now consider different forms of the disturbances in the financial market (3.14)-(3.15) and solve the portfolio selection problems **(P1)-(P3)** in each case. We consider two types of financial market where the disturbances driving the evolution of the securities **S** are respectively diffusions and Lévy processes with jumps. For each portfolio selection problem and financial market there are additional requirements for a portfolio $\boldsymbol{\pi}$ to be admissible and these are listed in each section.

3.7 Market driven by Diffusions

In this section problems **(P1)-(P3)** are solved assuming the risky securities **S** are driven by diffusions. The financial market in this section is a special case of the financial market (3.14)-(3.15) viz with the jump coefficients $\mathbf{g} \equiv \mathbf{0}$ almost surely. For a general utility function U, we have only been able to solve the insider portfolio selection problem in the absence of explicit weight constraints and absence of penalty functions. We explain in Section 3.7.2, specifically Remark 3, why this is the case. In Section 3.7.1 we solve the insider portfolio selection problem for general utility - this is problem **(P1)** discussed in Section 3.6.1. In Sections 3.7.2 and 3.7.3 we solve constrained portfolio selection problems in the presence of penalty functions and explicit weight constraints and assuming the insider has logarithmic utility. We then solve this problem assuming a money market security is unavailable and available for investment and these are problems **(P2)** and **(P3)** respectively discussed in Sections 3.6.2 and 3.6.3.

3.7.1 General utility

In this section we solve problem **(P1)**. Here the insider is assumed to have general utility and no explicit portfolio weight constraints and no penalty functions are present in the portfolio selection problem. As mentioned in Section 3.6.1, the unity weight constraint (3.23) is used to eliminate π_0 from the optimisation problem and in so doing problem **(P1)** is made unconstrained. By eliminating π_0 the wealth process has the form (3.29). The main result in this section is (3.70) an optimality equation which an optimal portfolio $\boldsymbol{\pi}$ for problem **(P1)** must satisfy. For the rest of this thesis let $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_{N_S})$ be a portfolio. We now define the set of admissible portfolios $\mathcal{P}_{B1}$.

Definition 12 (Admissible portfolios) *A set of control processes* $\boldsymbol{\pi}$ *where* $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ *for all* $0 \leq t \leq T$, *is said to be* **admissible** *(or an* **admissible portfolio***) for problem* **(P1)** *if the following hold:*

(i) $\boldsymbol{\pi}$ *satisfies Definition 9.*

(ii) We require that $\mathbb{E}[U'(T, W(T))W(T)] < \infty$, *where* $U'(t,x) := \frac{\partial}{\partial x}U(t,x)$. *(This is required to ensure that the Radon-Nikodym derivative in (3.57) below is finite so that Bayes' Theorem can be applied. Recall that* $U'(t,x) > 0$ *since* U *is strictly concave, and* $W(T) > 0$ *almost surely.)*

(iii) Let $\boldsymbol{\pi}$ *and* $\boldsymbol{\theta}$ *be two portfolios which satisfy (i)-(ii) above. Recall the form (3.29) of the insider wealth process. For all* $0 \leq t \leq T, i \in \mathcal{N}_S$ *let*

$$M_i(t) := \int_0^t \left(\hat{\xi}_i(s) - \sum_{j=1}^{N_B} \sigma_{0,j}(s)\hat{\sigma}_{ij}(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) \right) ds + \sum_{j=1}^{N_B} \int_0^t \hat{\sigma}_{ij}(s) dB_j(s), \tag{3.55}$$

where from (3.20) we have that $\bar{\sigma}_{ij}(t) := \sum_{k=1}^{N_B} \hat{\sigma}_{jk}(t)\hat{\sigma}_{ik}(t)$. *Let* $\mathbf{y} := (y_1, \ldots, y_{N_S}) \in \mathbb{R}^{N_S}$. *Then there must exist a* $\delta > 0$ *such that for each* $i \in \mathcal{N}_S$ *we have that* $y_i \in (-\delta, \delta)$ *and the family*

$$\{U'(T, W(T, \boldsymbol{\pi}+diag(\mathbf{y})\boldsymbol{\theta}))W(T, \boldsymbol{\pi}+diag(\mathbf{y})\boldsymbol{\theta})|M_i(T, \boldsymbol{\pi}+diag(\mathbf{y})\boldsymbol{\theta})|\}_{\mathbf{0}\leq\mathbf{y}\leq\boldsymbol{\delta}}$$

is uniformly integrable, where $W(T) \equiv W(T, \boldsymbol{\pi})$ *and* $M_i(T) \equiv M_i(T, \boldsymbol{\pi})$ *and* $\boldsymbol{\delta}$ *is an* $N_S \times 1$ *matrix with all elements equal to* δ. *(This is required* *to ensure that the partial derivatives of the objective functional in (3.33) exist.)*

We denote by $\mathcal{P}_{B1}$ *the set of all admissible portfolios for problem* **(P1)**. ♦

In what follows, Theorems 7-9 are generalisations of ([18], Theorem 3.5(i), Theorem 3.6, Theorem 3.8) respectively. We now prove the following theorem in which we show that that the processes $\mathbf{M} := (M_1, \ldots, M_{N_S})$ defined in (3.55) are $(\mathbb{H}, \mathbb{Q})$-martingales.

Theorem 7 *Suppose* $\boldsymbol{\pi} \in \mathcal{P}_{B1}$ *is an optimal portfolio for problem* **(P1)**. *Then each* M_i *in (3.55) is an* $(\mathbb{H}, \mathbb{Q})$*-martingale where the probability measure* $\mathbb{Q}$ *is defined by*

$$d\mathbb{Q} = F(T)d\mathbb{P} \tag{3.56}$$

and

$$F(T) := (\mathbb{E}[U'(T, W(T))W(T)])^{-1}U'(T, W(T))W(T). \tag{3.57}$$

Proof: Recall (3.33) viz

$$\sup_{\boldsymbol{\pi} \in \mathcal{P}_{B1}} J_1(\boldsymbol{\pi}) \;:=\; \sup_{\boldsymbol{\pi} \in \mathcal{P}_{B1}} \mathbb{E}[U(T, W(T))].$$

Let $\boldsymbol{\theta} \in \mathcal{P}_{B1}$ be another admissible portfolio of the following form. Fix $t \in [0, T)$ and $h > 0$ such that $t + h \leq T$ and

$$\theta_i(s) = \theta_i^0(t)\chi_{[t,t+h]}(s) \qquad \text{for all } 0 \leq s \leq T, i \in \mathcal{N}_S, \tag{3.58}$$

where each random variable $\theta_i^0(t)$ is bounded and $\mathcal{H}_t$-measurable. Then, by assumption of the optimality of $\boldsymbol{\pi}$, for $\mathbf{y} \in (-\boldsymbol{\delta}, \boldsymbol{\delta})$ the function

$$f(\mathbf{y}) := J_1\left(\boldsymbol{\pi} + diag(\mathbf{y})\boldsymbol{\theta}\right) \tag{3.59}$$

is maximal for $\mathbf{y} = \mathbf{0}$, hence for each $i \in \mathcal{N}_S$

$$\begin{aligned} 0 \;&=\; \left[\frac{\partial}{\partial y_i} f(\mathbf{y})\right]_{\mathbf{y}=\mathbf{0}} \\ &=\; \mathbb{E}\left[U'(T, W(T))W(T)\left(\int_0^T \left(\hat{\xi}_i(s) - \sum_{j=1}^{N_B} \sigma_{0,j}(s)\hat{\sigma}_{ij}(s)\right.\right.\right. \\ &\qquad \left.\left.\left. - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s)\right)\theta_i(s)ds + \sum_{j=1}^{N_B}\int_0^T \hat{\sigma}_{ij}(s)\theta_i(s)d^- B_j(s)\right)\right]. \end{aligned} \tag{3.60}$$

Since the expected returns, volatilities and portfolios $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ are bounded, by the Lebesgue dominated convergence theorem the interchanging of classical differentiation and Lebesgue integration in (3.60) is justified. (See Remark 2 below.) Substituting the particular form (3.58) of $\boldsymbol{\theta}$ into (3.60) and multiplying by $(\mathbb{E}[U'(T,W(T))W(T)])^{-1}$ we get that

$$
\begin{aligned}
0 &= \mathbb{E}\left[F(T)\theta_i^0(t)\left(\int_t^{t+h}\left(\hat{\xi}_i(s)-\sum_{j=1}^{N_B}\sigma_{0,j}(s)\hat{\sigma}_{ij}(s)\right.\right.\right. \\
&\qquad \left.\left.\left.-\sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right)ds+\sum_{j=1}^{N_B}\int_t^{t+h}\hat{\sigma}_{ij}(s)dB_j(s)\right)\right], \qquad (3.61)
\end{aligned}
$$

where $F(T)$ is defined in (3.57). Note that there is no longer forward diffusion integration in (3.61) since $\hat{\boldsymbol{\sigma}}$ are $\mathbb{F}$-predictable and $\mathbf{B}$ are $(\mathbb{F},\mathbb{P})$-Brownian motions. Since (3.61) is valid for all bounded $\mathcal{H}_t$-measurable random variables $\theta_i^0(t), i\in\mathcal{N}_S$, we conclude that

$$
\begin{aligned}
0 &= \mathbb{E}\left[F(T)\left(\int_t^{t+h}\left(\hat{\xi}_i(s)-\sum_{j=1}^{N_B}\sigma_{0,j}(s)\hat{\sigma}_{ij}(s)\right.\right.\right. \\
&\qquad \left.\left.\left.-\sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right)ds+\sum_{j=1}^{N_B}\int_t^{t+h}\hat{\sigma}_{ij}(s)dB_j(s)\right)\middle|\mathcal{H}_t\right]. \qquad (3.62)
\end{aligned}
$$

For all $i\in\mathcal{N}_S$ let M_i be defined as in (3.55). Then recalling Bayes' Theorem[5] with $d\mathbb{Q}:=F(T)d\mathbb{P}$, we have from (3.62) that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[M_i(t+h)-M_i(t)|\mathcal{H}_t] &= (\mathbb{E}_{\mathbb{P}}[F(T)|\mathcal{H}_t])^{-1}\mathbb{E}_{\mathbb{P}}[F(T)(M_i(t+h)-M_i(t))|\mathcal{H}_t] \\
&= 0, \qquad (3.63)
\end{aligned}
$$

which implies that

$$\mathbb{E}_{\mathbb{Q}}[M_i(t+h)|\mathcal{H}_t]=M_i(t) \qquad (3.64)$$

since each $M_i(t)$ is $\mathcal{H}_t$-measurable. Since $t\in[0,T]$ and $h>0$ are arbitrary in (3.64), we conclude that each process M_i is an $(\mathbb{H},\mathbb{Q})$-martingale. ■

Remark 2 *We make the following remarks.*

(i) In the proof of Theorem 7 classical partial differentiation was employed. In [62] the theory of Malliavin calculus is discussed. It is also briefly discussed in this thesis in Section 3.8.1. In the derivation of ([62], equation (5.14)) however Malliavin calculus need not be employed. The reason is

[5] Bayes' Theorem: Let $\mathbb{Q}$ and $\mathbb{P}$ be two probability measures on the space $(\Omega,\mathcal{H}_T,\mathbb{H})$, let the random variable $K:=\frac{d\mathbb{Q}}{d\mathbb{P}}$ and let $\xi(t):=\mathbb{E}_{\mathbb{P}}[K|\mathcal{H}_t]$. Then for some random variable X

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{H}_t]=(\xi(t))^{-1}\mathbb{E}_{\mathbb{P}}[X\xi(T)|\mathcal{H}_t].$$

that the variational parameter y defined in ([62], equation (5.14)) is a constant and can be taken outside of the forward diffusion integral. This is why Malliavin calculus is not employed in the derivation of (3.60). Malliavin calculus will be required to find constrained optimal portfolios if the securities $\mathbf{S}$ *exhibit jumps. The reason for this is that in this case it is not possible to take the variational parameters* $\mathbf{y}$ *outside of the forward Poisson integrals and then classically differentiate with respect to* $\mathbf{y}$. *The variational parameters* $\mathbf{y}$ *cannot be taken outside of the forward Poisson integrals because they form part of the argument of the (nonlinear) natural logarithm found in the analytical forms of the wealth process (3.30) and (3.32).*

(ii) Since the Brownian motions $\mathbf{B}$ *are* $(\mathbb{F},\mathbb{P})$*-Brownian motions and the processes* $\hat{\boldsymbol{\sigma}}$ *are* $\mathbb{F}$*-predictable, the stochastic integrals* $\sum_{j=1}^{N_B}\int_0^t \hat{\sigma}_{ij}(s)dB_j(s)$ *are* $(\mathbb{F},\mathbb{P})$*-Itô integrals. Thus, albeit that each* M_i *is an* $(\mathbb{H},\mathbb{Q})$*-martingale, we don't know whether* M_i *is an* $(\mathbb{F},\mathbb{P})$*-martingale. So we cannot conclude that the bounded variation process*

$$\int_0^t \left(\hat{\xi}_i(s) \ - \ \sum_{j=1}^{N_B}\sigma_{0,j}(s)\hat{\sigma}_{ij}(s) \ - \ \sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right) ds$$

in (3.55) is zero almost surely for all $0 \le t \le T$. *In Theorem 9 we show that by subtracting processes of bounded variation from the* $(\mathbb{F},\mathbb{P})$*-Brownian motions* $\mathbf{B}$*, we are able to convert the processes* $\mathbf{B}$ *to* $(\mathbb{H},\mathbb{P})$*-Brownian motions.*

We now prove the following theorem in which we show amongst other things that the $(\mathbb{F},\mathbb{P})$-Brownian motions $\mathbf{B}$ are $(\mathbb{H},\mathbb{P})$-semimartingales.

Theorem 8 *Let* $\boldsymbol{\pi} \in \mathcal{P}_{B1}$ *be an optimal portfolio for problem* **(P1)**. *Then we have the following. (i) Let the processes* $\mathbf{M}$ *and random variable* F *be given by (3.55) and (3.57) respectively. Then for all* $0 \le t \le T, i \in \mathcal{N}_S$ *the processes*

$$\hat{M}_i(t) \ := \ M_i(t) - \int_0^t \frac{d[M_i,\bar{F}](s)}{\bar{F}(s)} \tag{3.65}$$

are $(\mathbb{H},\mathbb{P})$*-martingales, where for all* $0 \le t \le T$

$$\bar{F}(t) := \mathbb{E}[F(T)|\mathcal{H}_t]. \tag{3.66}$$

(ii) For each $i \in \mathcal{N}_S$ *the process*

$$S_i^B(t) := \sum_{j=1}^{N_B}\int_0^t \hat{\sigma}_{ij}(s)dB_j(s), \quad 0 \le t \le T$$

is an $(\mathbb{H},\mathbb{P})$*-semimartingale.*
(iii) The Brownian motions $\mathbf{B}$ *defined in (3.15) are* $(\mathbb{H},\mathbb{P})$*-semimartingales.*

(iv) Forward stochastic integrals of $\mathbb{H}$-predictable integrable stochastic processes with respect to $\mathbf{B}$ are in fact Itô integrals. In other words for all $0 \le t_1 \le t_2 \le T, j \in \mathcal{N}_B$ and $\mathbb{H}$-predictable integrable stochastic process $(X(t,\omega), 0 \le t \le T, \omega \in \Omega)$, we have almost surely that

$$\int_{t_1}^{t_2} X(t) d^- B_j(t) = \int_{t_1}^{t_2} X(t) dB_j(t). \tag{3.67}$$

Proof:
(i) Suppose the processes $\hat{\mathbf{M}} := (\hat{M}_1, \ldots, \hat{M}_{N_S})$ are $(\mathbb{H}, \mathbb{P})$-local martingales and that for each $t \in [0,T]$ the expected value of the quadratic variation of each $\hat{M}_i$ is finite. Then from Theorem 4 we have that the processes $\hat{\mathbf{M}}$ of (3.65) are $(\mathbb{H}, \mathbb{P})$-martingales and we have proved part (i).

Now for all admissible portfolios $\mathcal{P}_{B1}$ the terminal wealth value $0 < W(T)$ almost surely, thus we have that $0 < F(T)$ almost surely, where $F(T)$ is defined in (3.57). Thus we have from (3.57) that

$$d\mathbb{P} = F(T)^{-1} d\mathbb{Q} =: G(T) d\mathbb{Q}. \tag{3.68}$$

By Bayes' Theorem let $\bar{F}$ be the $(\mathbb{H}, \mathbb{Q})$-martingale defined by

$$\bar{F}(t) = \mathbb{E}_{\mathbb{Q}}[G(T)|\mathcal{H}_t] = (\mathbb{E}_{\mathbb{P}}[F(T)|\mathcal{H}_t])^{-1}\mathbb{E}_{\mathbb{P}}[F(T)G(T)|\mathcal{H}_t] = (\mathbb{E}_{\mathbb{P}}[F(T)|\mathcal{H}_t])^{-1}.$$

Now suppose $\boldsymbol{\pi} \in \mathcal{P}_{B1}$ is an optimal portfolio for problem **(P1)**. By Theorem 7 we know that the processes $\mathbf{M}$ defined in (3.55) are $(\mathbb{H}, \mathbb{Q})$-martingales. Thus from (3.68) and Corollary 4, we have that the processes $\hat{\mathbf{M}}$ are $(\mathbb{H}, \mathbb{P})$-local martingales. Secondly, from Theorem 5, for each $t \in [0,T]$ the expected value of the quadratic variation of the processes $\hat{\mathbf{M}}$ is finite.
(ii) Suppose $\boldsymbol{\pi} \in \mathcal{P}_{B1}$ is an optimal portfolio for problem **(P1)**. Then from (3.55) and (3.65) we have that

$$\begin{aligned} \sum_{j=1}^{N_B} \int_0^t \hat{\sigma}_{ij}(s) dB_j(s) &= \hat{M}_i(t) - \int_0^t \left(\hat{\xi}_i(s) - \sum_{j=1}^{N_B} \sigma_{0,j}(s) \hat{\sigma}_{ij}(s) \right. \\ &\quad \left. - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s) \pi_j(s) \right) ds + \int_0^t \bar{F}^{-1}(s) d[M_i, \bar{F}](s). \end{aligned}$$

Thus for each $i \in \mathcal{N}_S$ the expression $S_i^B(t) := \sum_{j=1}^{N_B} \int_0^t \hat{\sigma}_{ij}(s) dB_j(s)$ is an $(\mathbb{H}, \mathbb{P})$-semimartingale because it is the sum of an $(\mathbb{H}, \mathbb{P})$-martingale $\hat{M}_i$ and a process of bounded variation.
(iii) Let $\mathbf{S}^B := (S_1^B, \ldots, S_{N_S}^B)$. Then from (ii) above the expressions

$$\mathbf{S}^B(t) = \int_0^t \hat{\boldsymbol{\sigma}}(s) d\mathbf{B}(s)$$

are $(\mathbb{H}, \mathbb{P})$-semimartingales. Thus the expressions

$$\bar{\mathbf{S}}^B(t) := \int_0^t \hat{\boldsymbol{\sigma}}^T(s) d\mathbf{S}^B(s) = \int_0^t \hat{\boldsymbol{\sigma}}^T(s) \hat{\boldsymbol{\sigma}}(s) d\mathbf{B}(s) = \int_0^t \bar{\boldsymbol{\sigma}}(s) d\mathbf{B}(s)$$

are $(\mathbb{H},\mathbb{P})$-semimartingales. Thus the expressions

$$\int_0^t \bar{\boldsymbol{\sigma}}^{-1}(s)d\bar{\mathbf{S}}^B(s) = \int_0^t \bar{\boldsymbol{\sigma}}^{-1}(s)\bar{\boldsymbol{\sigma}}(s)d\mathbf{B}(s) = \mathbf{B}(t)$$

are $(\mathbb{H},\mathbb{P})$-semimartingales.
(iv) From (iii) above, since each $B_j, j \in \mathcal{N}_B$ is an $(\mathbb{H},\mathbb{P})$-semimartingale, integration of an $\mathbb{H}$-predictable integrable stochastic process X with respect to any B_j is Itô integration. ■

Note that the stochastic integrals (3.67) are not necessarily $(\mathbb{H},\mathbb{P})$-martingales since the processes $\mathbf{B}$ are not necessarily $(\mathbb{H},\mathbb{P})$-Brownian motions. We now prove the following theorem in which we derive an *optimality equation* for an optimal portfolio $\boldsymbol{\pi}$ for problem **(P1)**.

Theorem 9 *Assume $\boldsymbol{\pi}$ is an optimal portfolio for problem* **(P1)**. *Then (i) each $B_j, j \in \mathcal{N}_B$ is an $(\mathbb{H},\mathbb{P})$-semimartingale, in other words for each j there exists an $\mathbb{H}$-predictable bounded variation process H_j such that*

$$\breve{B}_j := B_j - H_j \text{ is an } (\mathbb{H},\mathbb{P})\text{-Brownian motion.} \tag{3.69}$$

(ii) Let $\hat{\boldsymbol{\xi}} := (\hat{\xi}_1, \ldots, \hat{\xi}_{N_S})$. Then if the processes $[\mathbf{M}, \bar{F}]$ and $\mathbf{H} := (H_1, \ldots, H_{N_B})$ are absolutely continuous, then we have the following explicit formula for an optimal portfolio $\boldsymbol{\pi}$ for problem **(P1)** *viz for all $0 \le t \le T$*

$$\boldsymbol{\pi}(t) \ = \ \bar{\boldsymbol{\sigma}}^{-1}(t)\left(\hat{\boldsymbol{\xi}}(t) - \boldsymbol{\sigma}^0(t) + \hat{\boldsymbol{\sigma}}(t)\frac{d}{dt}\mathbf{H}(t) - \bar{F}^{-1}(t)\frac{d}{dt}[\mathbf{M}, \bar{F}](t)\right) \tag{3.70}$$

almost surely, where for fixed $k \in \mathcal{N}_S \cup \{0\}$ the vector $\boldsymbol{\sigma}^k \equiv [\sigma_i^k] := [\sum_{j=1}^{N_B} \sigma_{kj}\hat{\sigma}_{ij}]$.

Proof:
(i) From Theorem 8(iii) for each $j \in \mathcal{N}_B$ the process B_j is an $(\mathbb{H},\mathbb{P})$-semimartingale. Thus each B_j has a Doob-Meyer decomposition ([117], Theorem III.3)

$$B_j = \breve{B}_j + H_j, \tag{3.71}$$

where each $\breve{B}_j$ is a continuous $(\mathbb{H},\mathbb{P})$-local martingale and each H_j is a process of bounded variation. From (3.71)

$$t = [B_j, B_j]_t = [\breve{B}_j, \breve{B}_j]_t.$$

From the Lévy characterisation of Brownian motion ([117], Theorem II.39), since $\breve{B}_j$ is a continuous $(\mathbb{H},\mathbb{P})$-local martingale with a quadratic variation of t, it is an $(\mathbb{H},\mathbb{P})$-Brownian motion.
(ii) From (3.65) and (3.69) we have for all $0 \le t \le T, i \in \mathcal{N}_S$ that almost surely

$$\hat{M}_i(t) + \sum_{j=1}^{N_B}\int_0^t \hat{\sigma}_{ij}(s)d\breve{B}_j(s) \quad = \quad \int_0^t \Bigg(\hat{\xi}_i(s) - \sum_{j=1}^{N_B}\sigma_{0,j}(s)\hat{\sigma}_{ij}(s) - \sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)$$

$$+\sum_{j=1}^{N_B}\hat{\sigma}_{ij}(s)\frac{d}{ds}H_j(s) - \int_0^t \bar{F}^{-1}(s)\frac{d}{ds}[M_i,\bar{F}](s)\Bigg) ds \tag{3.72}$$

since by assumption the functions $[\mathbf{M},\bar{F}]$ and $\mathbf{H}$ are absolutely continuous. Now the processes $\hat{\mathbf{M}}$ are $(\mathbb{H},\mathbb{P})$-martingales, the processes $\breve{\mathbf{B}} := (\breve{B}_1,\ldots,\breve{B}_{N_B})$ are $(\mathbb{H},\mathbb{P})$-Brownian motions and the processes $\hat{\boldsymbol{\sigma}}$ are $\mathbb{H}$-predictable, thus the left hand side of (3.72) is an $(\mathbb{H},\mathbb{P})$-local martingale which is also continuous. But the right hand side of (3.72) is a bounded variation process, thus from ([117], Corollary II.1) the process on the left hand side of (3.72) must be constant almost surely equalling in particular its initial value which is zero. Then differentiating (3.72) with respect to t and applying the fundamental theorem of calculus, (3.70) is obtained. ■

In the next section a constrained portfolio selection problem is solved assuming the insider has logarithmic utility and invests in a financial market driven by diffusions. It is also assumed that penalty functions and explicit weight constraints are present, however a money market security is not available for investment.

3.7.2 Logarithmic utility, weight constraints, penalty functions and no investment in a money market security

In this section problem **(P2)** is solved assuming the insider has logarithmic utility, invests in a financial market driven by diffusions and assuming that explicit portfolio weight constraints and penalty functions are present in the portfolio selection problem. It is assumed however that the insider cannot invest in a money market security (since this will result in the covariance matrices being singular). (In Section 3.7.3 this restriction is relaxed.) Consequently the wealth process has the form (3.27). The main result in this section is (3.82) an optimality equation which an optimal portfolio $\boldsymbol{\pi}$ for problem **(P2)** must satisfy. We now define the set of admissible portfolios $\mathcal{P}_{B2}$ for problem **(P2)**.

Definition 13 (Admissible portfolios) *A set of control processes $\boldsymbol{\pi}$ where $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ for all $0 \le t \le T$, is said to be* **admissible** *(or an* **admissible portfolio***) for problem* **(P2)** *if the following hold:*

(i) $\boldsymbol{\pi}$ satisfies Definition 10.

(ii) Let $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ be two portfolios which satisfy (i) above. Recall the form (3.27) of the insider wealth process. For all $0 \le t \le T, i \in \mathcal{N}_S$ let

$$M_i(t) \;:=\; \int_0^t \left(\xi_i(s) - \sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right) ds + \sum_{j=1}^{N_B}\int_0^t \sigma_{ij}(s)dB_j(s)$$

$$+\int_0^t \left(-\sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) - \lambda_0(s) + \sum_{j=1}^{M} \lambda_j(s) h_{ij}(s) \right) ds, \tag{3.73}$$

where the processes $\lambda_j, j \in \mathcal{N}_M \cup \{0\}$ are defined in (3.50) and from (3.17) we have that $\bar{\sigma}_{ij}(t) := \sum_{k=1}^{N_B} \sigma_{jk}(t)\sigma_{ik}(t)$. Let $\mathbf{y} := (y_1, \ldots, y_{N_S}) \in \mathbb{R}^{N_S}$. Then there must exist a $\delta > 0$ such that for each $i \in \mathcal{N}_S$ we have that $y_i \in (-\delta, \delta)$ and the family

$$\{|M_i(T, \boldsymbol{\pi} + diag(\mathbf{y})\boldsymbol{\theta})|\}_{\mathbf{0} \leq \mathbf{y} \leq \boldsymbol{\delta}}$$

is uniformly integrable, where $M_i(T) \equiv M_i(T, \boldsymbol{\pi})$ and $\boldsymbol{\delta}$ is an $N_S \times 1$ matrix with all elements equal to δ. (This is required to ensure that the partial derivatives of the Lagrangian in (3.75) below exist.)

We denote by $\mathcal{P}_{B2}$ the set of all admissible portfolios for problem **(P2)**. ♦

In what follows, Theorems 11-13 are generalisations of ([62], Theorem 5.3-5.5) respectively. For the rest of this thesis we require the following: Let $\mathbb{L}^{\dagger} : L^2([0,T]) \to L^2([0,T])$ denote the adjoint operator of $\mathbb{L}$ in the Hilbert space $L^2([0,T])$. Then for all sufficiently differentiable functions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in the domain of $\mathbb{L}$ and $\mathbb{L}^{\dagger}$ we have that

$$\mathbb{E}\left[\int_0^T \boldsymbol{\alpha}(s)\mathbb{L}(\boldsymbol{\beta}(s))ds\right] = \mathbb{E}\left[\int_0^T \mathbb{L}^{\dagger}(\boldsymbol{\alpha}(s))\boldsymbol{\beta}(s)ds\right].$$

We now prove the following theorem in which we show that the processes $\mathbf{M} := (M_1, \ldots, M_{N_S})$ defined in (3.73) are $(\mathbb{H}, \mathbb{P})$-martingales.

Theorem 10 *Assume $\boldsymbol{\pi} \in \mathcal{P}_{B2}$ is an optimal portfolio for problem* **(P2)**. *Then each M_i defined in (3.73) is an $(\mathbb{H}, \mathbb{P})$-martingale.*

Proof: Recall the definition of the Lagrangian $V_2(\boldsymbol{\pi})$ in (3.51). Let $\boldsymbol{\theta} \in \mathcal{P}_{B2}$ be another admissible portfolio of the following form. Fix $t \in [0,T)$ and $h > 0$ such that $t + h \leq T$ and

$$\theta_i(s) = \theta_i^0(t)\chi_{[t,t+h]}(s) \quad \text{for all } 0 \leq s \leq T, i \in \mathcal{N}_S, \tag{3.74}$$

where each random variable $\theta_i^0(t)$ is bounded and $\mathcal{H}_t$-measurable. Then by assumption of the optimality of $\boldsymbol{\pi}$, for $\mathbf{y} \in (-\boldsymbol{\delta}, \boldsymbol{\delta})$ the function

$$\begin{aligned} f(\mathbf{y}) &= V_2(\boldsymbol{\pi} + diag(\mathbf{y})\boldsymbol{\theta}) \\ &= J_2(\boldsymbol{\pi} + diag(\mathbf{y})\boldsymbol{\theta}) + \mathbb{E}\left[\int_0^T \lambda_0(s)\left(\Upsilon(s) - \sum_{i=1}^{N_S}(\pi_i(s) + y_i\theta_i(s))\right)ds \right. \\ &\quad \left. + \sum_{j=1}^{M}\int_0^T \lambda_j(s)\left[\sum_{i=1}^{N_S} h_{ij}(s)(\pi_i(s) + y_i\theta_i(s)) - \bar{h}_j(s)\right]ds\right] \end{aligned} \tag{3.75}$$

is maximal for $\mathbf{y} = \mathbf{0}$, hence for each $i \in \mathcal{N}_S$

$$\begin{aligned}
0 &= \left[\frac{\partial}{\partial y_i} f(\mathbf{y})\right]_{\mathbf{y}=\mathbf{0}} \\
&= \mathbb{E}\Bigg[U'(T,W(T))W(T)\frac{\partial}{\partial y_i}\Bigg(\int_0^T \Bigg(\sum_{i=1}^{N_S} \xi_i(s)(\pi_i(s)+y_i\theta_i(s)) \\
&\quad -\frac{1}{2}\sum_{j=1}^{N_B}\left(\sum_{i=1}^{N_S}\sigma_{ij}(s)(\pi_i(s)+y_i\theta_i(s))\right)^2\Bigg) ds \\
&\quad +\sum_{i=1}^{N_S}\sum_{j=1}^{N_B}\int_0^T \sigma_{ij}(s)(\pi_i(s)+y_i\theta_i(s))d^-B_j(s)\Bigg) \\
&\quad +\frac{\partial}{\partial y_i}\Bigg(-\frac{1}{2}\int_0^T \|\mathbb{L}(\boldsymbol{\pi}(s)+diag(\mathbf{y})\boldsymbol{\theta}(s))\|^2 ds \\
&\quad +\int_0^T \lambda_0(s)\left(\Upsilon(s)-\sum_{i=1}^{N_S}(\pi_i(s)+y_i\theta_i(s))\right) ds \\
&\quad +\sum_{j=1}^{M}\int_0^T \lambda_j(s)\left[\sum_{i=1}^{N_S} h_{ij}(s)(\pi_i(s)+y_i\theta_i(s))-\bar{h}_j(s)\right] ds\Bigg)\Bigg]_{\mathbf{y}=\mathbf{0}}.
\end{aligned} \tag{3.76}$$

Since the expected returns, volatilities and portfolios $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ are bounded, by the Lebesgue dominated convergence theorem the interchanging of classical differentiation and Lebesgue integration in (3.76) is justified. Since U is assumed to be logarithmic, (3.76) reduces to

$$\begin{aligned}
0 &= \mathbb{E}\Bigg[\int_0^T \left(\xi_i(s)-\sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right)\theta_i(s)ds+\sum_{j=1}^{N_B}\int_0^T \sigma_{ij}(s)\theta_i(s)d^-B_j(s) \\
&\quad +\int_0^T\left(-\sum_{j=1}^{N_S}\mathbb{L}^{\dagger}_{ji}(\mathbb{L}_{ji}(\pi_i(s)))-\lambda_0(s)+\sum_{j=1}^{M}\lambda_j(s)h_{ij}(s)\right)\theta_i(s)ds\Bigg].
\end{aligned} \tag{3.77}$$

Substituting the particular form (3.74) of $\boldsymbol{\theta}$ into (3.77) it reduces to

$$\begin{aligned}
0 &= \mathbb{E}\Bigg[\theta_i^0(t)\Bigg(\int_t^{t+h}\left(\xi_i(s)-\sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right)ds+\sum_{j=1}^{N_B}\int_t^{t+h}\sigma_{ij}(s)dB_j(s) \\
&\quad +\int_t^{t+h}\left(-\sum_{j=1}^{N_S}\mathbb{L}^{\dagger}_{ji}(\mathbb{L}_{ji}(\pi_i(s)))-\lambda_0(s)+\sum_{j=1}^{M}\lambda_j(s)h_{ij}(s)\right)ds\Bigg)\Bigg].
\end{aligned} \tag{3.78}$$

Since (3.78) is valid for all bounded $\mathcal{H}_t$-measurable random variables $\theta_i^0(t)$, we conclude that

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_t^{t+h}\left(\xi_i(s)-\sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right)ds+\sum_{j=1}^{N_B}\int_t^{t+h}\sigma_{ij}(s)dB_j(s)\right. \\ &\left.\quad+\int_t^{t+h}\left(-\sum_{j=1}^{N_S}\mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s)))-\lambda_0(s)+\sum_{j=1}^{M}\lambda_j(s)h_{ij}(s)\right)ds\,\middle|\,\mathcal{H}_t\right]. \end{aligned} \tag{3.79}$$

With M_i defined in (3.73), from (3.79) we have that

$$\mathbb{E}[M_i(t+h)-M_i(t)|\mathcal{H}_t] \quad = \quad 0. \tag{3.80}$$

Since $t \in [0,T]$ and $h>0$ are arbitrary in (3.80) and $M_i(t)$ is $\mathcal{H}_t$-measurable we have that each M_i is an $(\mathbb{H},\mathbb{P})$-martingale. ■

Remark 3 *We now explain why we have not been able to solve the constrained portfolio selection problem* **(P2)** *if the insider has a general utility function - the reasoning is the same for problem* **(P3)**. *In Theorem 10 above, if we want to further manipulate equation (3.76) assuming a general utility function, then the difficulty we encounter (using the methodology in Theorem 7) is that the random variable* $U'(T,W(T))W(T)$ *is not a coefficient of any of the constraint terms and penalty function terms in (3.76). We cannot then factorise out the random variable* $U'(T,W(T))W(T)$ *and show that the resulting expression is an* $(\mathbb{H},\mathbb{Q})$*-martingale (as in Theorem 7). To continue on from equation (3.76) we have to eliminate the random variable* $U'(T,W(T))W(T)$*. The random variable* $U'(T,W(T))W(T)$ *will be deterministic only if* $U(t,x)=h_1(t)\ln(h_2(t)x)$ *where* h_1,h_2 *are deterministic functions such that* $U(t,x)$ *is defined for all* $0\le t\le T, x\in\mathbb{R}^+$*. (This property of the logarithmic utility function is also exploited in ([72], pp151, pp156) where optimal portfolios are derived by maximising the growth rate of wealth. From [72] the consequence of using a logarithmic objective functional in a finite time horizon optimisation problem is that optimal control variables are independent of the terminal time* T*.) Now formally we could continue on from equation (3.76) and solve for an optimal portfolio* $\boldsymbol{\pi}$*, but what we shall find is that this optimal portfolio is in terms of the random variable* $U'(T,W(T))W(T)$*. Hence if managing this portfolio in practice at time* $t<T$*, then an insider must know the terminal value of his portfolio. Even for an insider, this is quite challenging to know before the terminal time* T*.*

We now prove the following theorem in which we show amongst other things that the $(\mathbb{F},\mathbb{P})$-Brownian motions $\mathbf{B}$ are $(\mathbb{H},\mathbb{P})$-semimartingales.

Theorem 11 *Suppose* $\boldsymbol{\pi}\in\mathcal{P}_{B2}$ *is an optimal portfolio for problem* **(P2)**. *Then we have the following. (i) For each* $i\in\mathcal{N}_S$ *the process*

$$S_i^B(t):=\sum_{j=1}^{N_B}\int_0^t\sigma_{ij}(s)dB_j(s), \quad 0\le t\le T$$

is an $(\mathbb{H},\mathbb{P})$*-semimartingale.*
(ii) The Brownian motions $\mathbf{B}$ *defined in (3.15) are* $(\mathbb{H},\mathbb{P})$*-semimartingales.*
(iii) Forward stochastic integrals of $\mathbb{H}$*-predictable integrable processes with respect to* $\mathbf{B}$ *are Itô integrals.*

Proof:
(i) Suppose $\boldsymbol{\pi} \in \mathcal{P}_{B2}$ is an optimal portfolio for problem **(P2)**. Then from (3.73) we have that for each $i \in \mathcal{N}_S$

$$\begin{aligned}\sum_{j=1}^{N_B}\int_0^t \sigma_{ij}(s)dB_j(s) &= M_i(t) - \int_0^t \left(\xi_i(s) - \sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right) ds \\ &\quad - \int_0^t \left(-\sum_{j=1}^{N_S}\mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) - \lambda_0(s) + \sum_{j=1}^{M}\lambda_j(s)h_{ij}(s)\right) ds.\end{aligned}$$

Thus for each $i \in \mathcal{N}_S$ the expression $S_i^B(t) = \sum_{j=1}^{N_B}\int_0^t \sigma_{ij}(s)dB_j(s)$ is an $(\mathbb{H},\mathbb{P})$-semimartingale because it is the sum of an $(\mathbb{H},\mathbb{P})$-martingale M_i and a process of bounded variation.
(ii) and (iii) See the proofs of Theorem 8(iii) and (iv). ∎

We now prove the following theorem in which we derive an optimality equation for an optimal portfolio $\boldsymbol{\pi}$ for problem (**P2**).

Theorem 12 *Suppose* $\boldsymbol{\pi}$ *is an optimal portfolio for Problem* **(P2)**. *Then (i) for each* $j \in \mathcal{N}_B$ *the Brownian motion* B_j *is an* $(\mathbb{H},\mathbb{P})$*-semimartingale, in other words there exists an* $\mathbb{H}$*-predictable bounded variation process* H_j *such that*

$$\breve{B}_j := B_j - H_j \text{ is an } (\mathbb{H},\mathbb{P})\text{-Brownian motion.} \tag{3.81}$$

(ii) Let $\boldsymbol{\xi} := (\xi_1,\ldots,\xi_{N_S})$ *and let the volatility matrix be defined as* $\boldsymbol{\sigma} \equiv [\sigma_{ij}]$. *Then if the processes* $\mathbf{H} := (H_1,\ldots,H_{N_B})$ *are absolutely continuous, then for all* $0 \le t \le T$ *the portfolio* $\boldsymbol{\pi}(t)$ *must satisfy almost surely*

$$\bar{\boldsymbol{\sigma}}(t)\boldsymbol{\pi}(t) + \mathbb{L}_{\boldsymbol{\pi}}(t) = \boldsymbol{\xi}(t) + \boldsymbol{\sigma}(t)\frac{d}{dt}\mathbf{H}(t) + \boldsymbol{\lambda}^h(t), \tag{3.82}$$

where for all $0 \le t \le T, i \in \mathcal{N}_S$

$$\mathbb{L}_{\boldsymbol{\pi},i}(t) := \sum_{j=1}^{N_S}\mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(t)))$$

is the ith element of $\mathbb{L}_{\boldsymbol{\pi}}(t)$ *and* $\boldsymbol{\lambda}^h(t)$ *is a vector with each of its* N_S *elements equal to*

$$\lambda^h(t) := -\lambda_0(t) + \sum_{j=1}^{M}\lambda_j(t)h_{ij}(t).$$

Proof:
(i) See the proof of Theorem 9(i).
(ii) From (3.73) and (3.81) we have that

$$M_i(t) - \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) d\breve{B}_j(s) = \int_0^t \left(\xi_i(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) + \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) \frac{d}{ds} H_j(s) + \int_0^t \left(- \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) - \lambda_0(s) + \sum_{j=1}^{M} \lambda_j(s) h_{ij}(s) \right) \right) ds \tag{3.83}$$

since the functions $\mathbf{H}$ are assumed to be absolutely continuous. Now the processes $\mathbf{M}$ are $(\mathbb{H}, \mathbb{P})$-martingales, the processes $\breve{\mathbf{B}} := (\breve{B}_1, \ldots, \breve{B}_{N_B})$ are $(\mathbb{H}, \mathbb{P})$-Brownian motions and the processes $\boldsymbol{\sigma}$ are $\mathbb{H}$-predictable, thus the left hand side of (3.83) is an $(\mathbb{H}, \mathbb{P})$-local martingale which is also continuous. Since the right hand side of (3.83) is a process of bounded variation, the process on the left hand side of (3.83) must be constant almost surely equalling in particular its initial value which is zero. Then differentiating (3.83) with respect to t and applying the fundamental theorem of calculus, (3.82) is obtained. ■

To emphasize the the close relationship between the Brownian motions $\mathbf{B}$ being $(\mathbb{H}, \mathbb{P})$-semimartingales and the optimality of a portfolio $\boldsymbol{\pi} \in \mathcal{P}_{B2}$ for problem **(P2)**, we prove the following theorem.

Theorem 13 *Suppose (3.81) holds, that the processes* $\mathbf{H}$ *are absolutely continuous and that* $\boldsymbol{\pi} \in \mathcal{P}_{B2}$ *is a process which solves (3.82). Then* $\boldsymbol{\pi}$ *is optimal for Problem* **(P2)**.

Proof: For simplicity, let $W(0) = 1$. Substituting (3.81) into the Lagrangian (3.51) we get that

$$\begin{aligned} V_2(\boldsymbol{\pi}) = \mathbb{E} \Bigg[& \int_0^T \left(\sum_{i=1}^{N_S} \xi_i(s)\pi_i(s) - \frac{1}{2} \sum_{j=1}^{N_B} \left(\sum_{i=1}^{N_S} \sigma_{ij}(s)\pi_i(s) \right)^2 \right) ds \\ & + \sum_{i=1}^{N_S} \sum_{j=1}^{N_B} \int_0^T \sigma_{ij}(s)\pi_i(s) \frac{d}{ds} H_j(s) ds \\ & - \frac{1}{2} \int_0^T \|\mathbb{L}(\boldsymbol{\pi}(s))\|^2 ds + \int_0^T \lambda_0(s) \left(\Upsilon(s) - \sum_{i=1}^{N_S} \pi_i(s) \right) ds \\ & + \sum_{j=1}^{M} \int_0^T \lambda_j(s) \left[\sum_{i=1}^{N_S} h_{ij}(s)\pi_i(s) - \bar{h}_j(s) \right] ds \Bigg], \end{aligned} \tag{3.84}$$

since for all $i \in \mathcal{N}_S$ the process $\int_0^t \sigma_{ij}(s)\pi_i(s) d\breve{B}_j(s)$ is an Itô diffusion integral.

Let $\delta > 0$ and for all $i \in \mathcal{N}_S$ let $y_i \in (-\delta, \delta)$. Let $\boldsymbol{\delta}$ be a vector with all its N_S elements equal to δ. Since $\mathbb{L}$ is a linear operator (3.84) is concave in each $\pi_i(t), i \in \mathcal{N}_S$. So if there exists a portfolio $\boldsymbol{\pi} \in \mathcal{P}_{B2}$ such that

$$0 = \frac{\partial}{\partial y_i}[V_2(\boldsymbol{\pi} + diag(\mathbf{y})\boldsymbol{\theta})]_{\mathbf{y}=\mathbf{0}} \qquad \text{for all } i \in \mathcal{N}_S, \boldsymbol{\theta} \in \mathcal{P}_{B2}, \mathbf{y} \in (-\boldsymbol{\delta}, \boldsymbol{\delta}),$$

then $\boldsymbol{\pi}$ is optimal for problem **(P2)**. In the same way (3.77) was derived, we get that

$$\begin{aligned} \frac{\partial}{\partial y_i}[V_2(\boldsymbol{\pi} + diag(\mathbf{y})\boldsymbol{\theta})]_{\mathbf{y}=\mathbf{0}} &= \mathbb{E}\left[\int_0^T \left(\xi_i(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) + \sum_{j=1}^{N_B} \sigma_{ij}(s)\frac{d}{ds}H_j(s) \right.\right. \\ &\left.\left. - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) - \lambda_0(s) + \sum_{j=1}^{M} \lambda_j(s)h_{ij}(s)\right)\theta_i(s)ds\right]. \end{aligned} \tag{3.85}$$

But the expression in (3.85) is zero since $\boldsymbol{\pi}$ satisfies (3.82). This proves the theorem. ■

Now we want to eliminate the multipliers $\lambda_j, j \in \mathcal{N}_M \cup \{0\}$ from (3.82), so that an optimal portfolio $\boldsymbol{\pi} \in \mathcal{P}_{B2}$ is dependent only on observable stochastic processes. To do this we have to consider specific forms of the penalty functions $\mathbb{L}$. This is done in Chapter 4, Section 4.1.1. We now consider the insider constrained portfolio selection problem assuming a money market security is available for investment.

3.7.3 Logarithmic utility, weight constraints, penalty functions and investment in a money market security

In this section problem **(P3)** is solved assuming the insider has logarithmic utility and invests in a financial market driven by diffusions. It is also assumed that explicit portfolio weight constraints and penalty functions are present in the portfolio selection problem and that a money market security is available for investment. As in (3.70), for fixed $k \in \mathcal{N}_S \cup \{0\}$ let the vector $\boldsymbol{\sigma}^k \equiv [\sigma_i^k] := [\sum_{j=1}^{N_B} \sigma_{kj}\hat{\sigma}_{ij}]$. Then the wealth process has the form (3.29) with $\xi_0(t) = r(t)$ and $\boldsymbol{\sigma}^0(t) = \mathbf{0}$ for all $0 \le t \le T$, in other words

$$\begin{aligned} W(T) &= W(t)\exp\left(\int_t^T \left(r(s) + \sum_{i=1}^{N_S}(\xi_i(s) - r(s))\pi_i(s) - \frac{1}{2}\sum_{j=1}^{N_B}\left(\sum_{i=1}^{N_S}\sigma_{ij}(s)\pi_i(s)\right)^2\right)ds \right. \\ &\left. + \sum_{j=1}^{N_B}\sum_{i=1}^{N_S}\int_t^T \sigma_{ij}(s)\pi_i(s)d^-B_j(s)\right). \end{aligned}$$

The main result in this section is (3.88) an optimality equation which an optimal portfolio $\boldsymbol{\pi}$ for problem **(P3)** must satisfy. We now define the set of admissible portfolios $\mathcal{P}_{B3}$ for problem **(P3)**.

Definition 14 (Admissible portfolios) *A set of control processes* $\boldsymbol{\pi}$ *where* $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ *for all* $0 \leq t \leq T$, *is said to be* **admissible** *(or an* **admissible portfolio***) for problem* **(P3)** *if the following hold:*

(i) $\boldsymbol{\pi}$ *satisfies Definition 11.*

(ii) Definition 13(ii) with $\lambda_0 \equiv 0$ *and for all* $0 \leq t \leq T, i \in \mathcal{N}_S$

$$M_i(t) \;:=\; \int_0^t \left(\xi_i(s) - r(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) \right) ds + \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) dB_j(s) + \int_0^t \left(-\sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) + \sum_{j=1}^{M} \lambda_j(s) h_{ij}(s) \right) ds. \tag{3.86}$$

We denote by $\mathcal{P}_{B3}$ *the set of all admissible portfolios for problem* **(P3)***.* ♦

In what follows, Theorems 15-17 are generalisations of ([62], Theorem 5.3-5.5) respectively. We now prove the following theorem in which we show that the processes $\mathbf{M} := (M_1, \ldots, M_{N_S})$ defined in (3.86) are $(\mathbb{H}, \mathbb{P})$-martingales.

Theorem 14 *Suppose* $\boldsymbol{\pi} \in \mathcal{P}_{B3}$ *is an optimal portfolio for problem* **(P3)***. Then each* M_i *defined in (3.86) is an* $(\mathbb{H}, \mathbb{P})$*-martingale.*

Proof: See the proof of Theorem 10 with $\lambda_0 \equiv 0$ and $\boldsymbol{\xi}$ replaced with $\boldsymbol{\xi} - \mathbf{r}$, where $\mathbf{r}$ is a vector with all its N_S elements equal to the money market interest rate r. ■

We now prove the following theorem in which we show amongst other things that the $(\mathbb{F}, \mathbb{P})$-Brownian motions $\mathbf{B}$ are $(\mathbb{H}, \mathbb{P})$-semimartingales.

Theorem 15 *Let* $\boldsymbol{\pi} \in \mathcal{P}_{B3}$ *be an optimal portfolio for problem* **(P3)***. Then we have the following. (i) For each* $i \in \mathcal{N}_S$ *the process*

$$S_i^B(t) := \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) dB_j(s), \quad 0 \leq t \leq T$$

is an $(\mathbb{H}, \mathbb{P})$*-semimartingale.*
(ii) The Brownian motions $\mathbf{B}$ *defined in (3.15) are* $(\mathbb{H}, \mathbb{P})$*-semimartingales.*
(iii) Forward stochastic integrals of $\mathbb{H}$*-predictable integrable stochastic processes with respect to* $\mathbf{B}$ *are in fact Itô integrals.*

Proof: See the proof of Theorem 11 with $\lambda_0 \equiv 0$ and $\boldsymbol{\xi}$ replaced with $\boldsymbol{\xi} - \mathbf{r}$. ■

We now prove the following theorem in which we derive an optimality equation for an optimal portfolio $\boldsymbol{\pi}$ for problem **(P3)**.

Theorem 16 *Suppose that there exists an optimal portfolio* $\boldsymbol{\pi}$ *for Problem* **(P3)**. *Then (i) for each* $j \in \mathcal{N}_B$ *the Brownian motion* $B_j, j \in \mathcal{N}_B$ *is an* $(\mathbb{H}, \mathbb{P})$*-semimartingale, in other words there exists an* $\mathbb{H}$*-predictable bounded variation process* H_j *such that*

$$\breve{B}_j := B_j - H_j \text{ is an } (\mathbb{H}, \mathbb{P})\text{-Brownian motion.} \tag{3.87}$$

(ii) If the processes $\mathbf{H} := (H_1, \ldots, H_{N_B})$ *are absolutely continuous, then for all* $0 \le t \le T$ *the portfolio* $\boldsymbol{\pi}(t)$ *must satisfy almost surely*

$$\bar{\boldsymbol{\sigma}}(t)\boldsymbol{\pi}(t) + \mathbb{L}_{\boldsymbol{\pi}}(t) = \boldsymbol{\xi}(t) - \mathbf{r}(t) + \boldsymbol{\sigma}(t)\frac{d}{dt}\mathbf{H}(t) + \boldsymbol{\lambda}^h(t), \tag{3.88}$$

where for all $0 \le t \le T, i \in \mathcal{N}_S$

$$\mathbb{L}_{\boldsymbol{\pi},i}(t) \quad := \quad \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(t)))$$

is the i*th element of* $\mathbb{L}_{\boldsymbol{\pi}}(t)$ *and* $\boldsymbol{\lambda}^h(t)$ *is a vector with each of its* N_S *elements equal to*

$$\lambda^h(t) \quad := \quad \sum_{j=1}^{M} \lambda_j(t) h_{ij}(t).$$

Proof: See the proof of Theorem 12 with $\lambda_0 \equiv 0$ and $\boldsymbol{\xi}$ replaced with $\boldsymbol{\xi} - \mathbf{r}$. ■

To emphasize the the close relationship between the Brownian motions $\mathbf{B}$ being $(\mathbb{H}, \mathbb{P})$-semimartingales and the optimality of a portfolio $\boldsymbol{\pi} \in \mathcal{P}_{B3}$ for problem **(P3)**, we prove the following theorem.

Theorem 17 *Suppose (3.87) holds, that the processes* $\mathbf{H}$ *are absolutely continuous and that* $\boldsymbol{\pi} \in \mathcal{P}_{B3}$ *is a process which solves (3.88). Then* $\boldsymbol{\pi}$ *is optimal for Problem* **(P3)**.

Proof: See proof of Theorem 13 with $\lambda_0 \equiv 0$ and $\boldsymbol{\xi}$ replaced with $\boldsymbol{\xi} - \mathbf{r}$. ■

Now we want to eliminate the multipliers $\lambda_j, j \in \mathcal{N}_M$ from the optimality equation (3.88) so that an optimal portfolio $\boldsymbol{\pi} \in \mathcal{P}_{B3}$ is dependent only on observable stochastic processes. To do this, we have to consider specific forms of the penalty functions $\mathbb{L}$. This is done in Chapter 4, Section 4.1.2. In the next section problems **(P1)-(P3)** are solved assuming the securities $\mathbf{S}$ exhibit jumps.

3.8 Market driven by Lévy Processes

In this section problems **(P1)-(P3)** are solved assuming the securities $\mathbf{S}$ are driven by Lévy processes of the form (3.15). The difference between this section and the previous Section 3.7 is that here the securities $\mathbf{S}$ are allowed to exhibit jumps. As mentioned in Remark 2, the reason why this case (of $\mathbf{S}$ exhibiting jumps) is dealt with separately is because here classical differentiation is not sufficient to derive insider constrained optimal portfolios. The reason for this is that it is not possible to take the variational parameters $\mathbf{y}$ outside of the forward Poisson integrals and classically differentiate with respect to $\mathbf{y}$ as is done in the continuous case in Theorems 7, 10 and 14. The variational parameters $\mathbf{y}$ cannot be taken outside of the forward Poisson integrals because they form part of the argument of the nonlinear natural logarithm found in the analytical forms of the insider wealth process (3.30) and (3.32). Also the theory for the interchanging of classical differentiation and forward stochastic integration has not yet been developed. *Malliavin calculus*, discussed in Section 3.8.1 below, is employed to circumvent this difficulty. A result (Lemma 2) of Malliavin calculus relates the expectation of a forward Poisson integral to the expectation of a Lebesgue integral of the *Malliavin derivative*. Lebesgue integration and classical differentiation can then be interchanged and constrained optimal insider portfolios can be derived.

For a general utility function U, we have only been able to solve the insider portfolio selection problem in the absence of explicit weight constraints and absence of penalty functions. With respect to the unity weight constraint (3.23), in the case of general utility, we eliminate one of the security weights with this constraint (as in Section 3.7.1). In Section 3.8.2, we solve the portfolio selection problem assuming the insider has a general utility function - this is problem **(P1)**. In Sections 3.8.3 and 3.8.4, we solve the portfolio selection problem assuming the insider has a logarithmic utility function and assuming weight constraints and penalty functions are present in the portfolio selection problem. We then solve this problem assuming a money market security is unavailable and available for investment, and these are problems **(P2)** and **(P3)** respectively. We now state the following theorem which we require in Sections 3.8.2-3.8.4.

Theorem 18 *Any local martingale $M = M(t)$ admits a unique decomposition*

$$M(t) = M^c(t) + M^d(t), \quad 0 \leq t \leq T$$

where $M(0) = M^c(0) = M^d(0) = 0$, M^c is a continuous local martingale and M^d is a purely discontinuous local martingale (in other words M^d is orthogonal to every continuous local martingale).

Proof: See ([66], Theorem I.4.18). ■

We now define *Malliavin differentiation* and show the relationship between the Malliavin derivative and forward Poisson integration.

3.8.1 Malliavin differentiation

Let $l = l(dt)$ denote the Lebesgue measure on $[0, T]$ and let $\tilde{q}$ be a compensated Poisson random measure with intensity measure $\nu = \nu(d\mathbf{z}), \mathbf{z} \in \mathbb{R}^N, N \in \mathbb{N}$.

Then following [42] and [93], for a square integrable function $f \in L^2((l \times \nu)^n), n \in \mathbb{N}$, let the *n-fold iterated integral*

$$I_n(f) \quad := \quad \int_0^T \int_{\mathbb{R}^N} \cdots \int_0^{t_2} \int_{\mathbb{R}^N} f(t_1, \mathbf{z}_1, \ldots, t_n, \mathbf{z}_n) \tilde{q}(dt_1, d\mathbf{z}_1) \ldots \tilde{q}(dt_n, d\mathbf{z}_n).$$

From [93] define the *symmetrization* of f by

$$\tilde{f}(t_1, \mathbf{z}_1, \ldots, t_n, \mathbf{z}_n) = \frac{1}{n!} \sum_{\zeta(n)} f(t_{\zeta_1}, \mathbf{z}_{\zeta_1}, \ldots, t_{\zeta_n}, \mathbf{z}_{\zeta_n}) \tag{3.89}$$

where the sum in (3.89) is taken over all permutations $\zeta(n)$ of the set $\{1, \ldots, n\}$. A function f is said to be *symmetric* if $\tilde{f} = f$. Denote by $\tilde{L}^2((l \times \nu)^n), n \in \mathbb{N}$ the space of all symmetric [93] functions f in $L^2((l \times \nu)^n)$. For each $n \in \mathbb{N}$ and $f_n \in \tilde{L}^2((l \times \nu)^n)$ let the integral

$$\bar{I}_n(f) := n! I_n(f).$$

For constants $f_0 \in \mathbb{R}$ set $\bar{I}_0(f_0) = f_0$. Then we have the following *Chaos Expansion Theorem.*

Theorem 19 (Chaos Expansion Theorem) *Every $\mathcal{F}_T$-measurable random variable $F \in L^2(\mathbb{P})$ admits the representation*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \bar{I}_n(f_n) \tag{3.90}$$

for a unique sequence of symmetric functions $f_n \in \tilde{L}^2((l \times \nu)^n)$.

Proof: See ([93], Theorem 4). ∎

Using the expansion in (3.90) we have the following definition of Malliavin differentiation taken from [41].

Definition 15 *Let $\mathbb{D}_{1,2}$ denote the set of all $\mathcal{F}_T$-measurable random variables $F \in L^2(\mathbb{P})$ admitting the chaos expansion (3.90) with norm*

$$\|F\|^2_{\mathbb{D}_{1,2}} \quad := \quad \sum_{n=1}^{\infty} n \cdot n! \|f_n\|^2_{\tilde{L}^2((l \times \nu)^n)} < \infty.$$

Then the Malliavin derivative *$\mathcal{D}_{t,\mathbf{z}}$ is a linear operator defined on $\mathbb{D}_{1,2}$ with values in $L^2(\mathbb{P} \times l \times \nu)$ and it is defined as*

$$\mathcal{D}_{t,\mathbf{z}}[F] \quad = \quad \sum_{n=1}^{\infty} n \bar{I}_{n-1}(f_n(\cdot, t, \mathbf{z})),$$

where $f_n(\cdot, t, \mathbf{z}) := f_n(t_1, \mathbf{z}_1, \ldots, t_{n-1}, \mathbf{z}_{n-1}; t, \mathbf{z})$. ♦

We prove some properties of Malliavin differentiation.

Lemma 1 *Let $X, Y \in \mathbb{D}_{1,2}$ and let l denote the Lebesgue measure.. Then we have the following:*

(i) For almost all $\omega \in \Omega$ we have that $l \times \nu$-almost everywhere

$$\mathcal{D}_{t,\mathbf{z}}[X \cdot Y] = X \cdot \mathcal{D}_{t,\mathbf{z}}[Y] + Y \cdot \mathcal{D}_{t,\mathbf{z}}[X] + \mathcal{D}_{t,\mathbf{z}}[X] \cdot \mathcal{D}_{t,\mathbf{z}}[Y]. \quad (3.91)$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then for almost all $\omega \in \Omega$ we have that $l \times \nu$-almost everywhere

$$\mathcal{D}_{t,\mathbf{z}}[f(X)] = f(X + \mathcal{D}_{t,\mathbf{z}}[X]) - f(X). \quad (3.92)$$

(iii) For $0 \leq s \leq t^+ \leq T$ let the operator $\mathcal{D}_{t^+,\mathbf{z}}$ be defined as

$$\mathcal{D}_{t^+,\mathbf{z}}[X] = \lim_{s \to t^+} \mathcal{D}_{s,\mathbf{z}}[X].$$

Suppose that $\mathcal{D}_{t^+,\mathbf{z}}[X \cdot Y], \mathcal{D}_{t^+,\mathbf{z}}[X], \mathcal{D}_{t^+,\mathbf{z}}[Y]$ exist in $L^2(\mathbb{P} \times l \times \nu)$. Then the operator $\mathcal{D}_{t^+,\mathbf{z}}$ also satisfies a product rule of the form

$$\mathcal{D}_{t^+,\mathbf{z}}[X \cdot Y] = X \cdot \mathcal{D}_{t^+,\mathbf{z}}[Y] + Y \cdot \mathcal{D}_{t^+,\mathbf{z}}[X] + \mathcal{D}_{t^+,\mathbf{z}}[X] \cdot \mathcal{D}_{t^+,\mathbf{z}}[Y]. \quad (3.93)$$

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $\mathcal{D}_{t^+,\mathbf{z}}[f(X)]$ and $\mathcal{D}_{t^+,\mathbf{z}}[X]$ exist in $L^2(\mathbb{P} \times l \times \nu)$. Then the operator $\mathcal{D}_{t^+,\mathbf{z}}$ also satisfies a chain rule of the form

$$\mathcal{D}_{t^+,\mathbf{z}}[f(X)] = f(X + \mathcal{D}_{t^+,\mathbf{z}}[X]) - f(X). \quad (3.94)$$

(v) Let $0 \leq s < t \leq T$. Then for almost all $\omega \in \Omega$ we have that $l \times \nu$-almost everywhere that $\mathcal{D}_{t,\mathbf{z}}[X] = 0$ if X is $\mathcal{F}_s$-measurable.

Proof:

(i) See ([41], Lemma 3.1).

(ii) See ([110], Lemma 3.6).

(iii) For all $0 \leq t \leq T$ we have that (3.91) holds. Replacing t with s in (3.91) and taking the limit as s tends to t^+, equation (3.93) is obtained since it was assumed that $\mathcal{D}_{t^+,\mathbf{z}}[X \cdot Y], \mathcal{D}_{t^+,\mathbf{z}}[X]$ and $\mathcal{D}_{t^+,\mathbf{z}}[Y]$ exist in $L^2(\mathbb{P} \times l \times \nu)$.

(iv) For all $0 \leq t \leq T$ we have that (3.92) holds. Replacing t with s in (3.92) and taking the limit as s tends to t^+ we get that

$$\mathcal{D}_{t^+,\mathbf{z}}[f(X)] = \lim_{s \to t^+} f(X + \mathcal{D}_{s,\mathbf{z}}[X]) - f(X)$$

which reduces to (3.94) since f is continuous.

(v) See [108].

■

From [42] we have a duality formula between the Malliavin derivative $\mathcal{D}_{t,\mathbf{z}}$ and forward Poisson integration. First we make the following definition taken from [41].

Definition 16 *Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space, let $\omega \in \Omega$, let l denote the Lebesgue measure and let $\tilde{q}$ be a compensated Poisson random measure with intensity measure $\nu = \nu(d\mathbf{z})$. Let $\mathcal{M}$ denote the set of all stochastic functions $\psi = \psi(t, \mathbf{z}, \omega), 0 \leq t \leq T, \mathbf{z} \in \mathbb{R}^N, \omega \in \Omega$ such that:*

(i) $\psi(t, \mathbf{z}, \omega) = \psi_1(t, \omega)\psi_2(t, \mathbf{z}, \omega)$ where $\psi_1 \in \mathbb{D}_{1,2}$ is càglàd, ψ_2 is $\mathbb{F}$-adapted and

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^N} \psi_2^2(t, \mathbf{z})\nu(d\mathbf{z})dt\right] < \infty.$$

(ii) $\mathcal{D}_{t^+,\mathbf{z}}[\psi]$ exists in $L^2(\mathbb{P} \times l \times \nu)$ for all $0 \leq t < T$.

(iii) $\psi + \mathcal{D}_{t^+,\mathbf{z}}[\psi]$ is forward integrable with respect to $\tilde{q}$.

Let $\mathbb{M}_{1,2}$ be the closure of the linear span of $\mathcal{M}$ with respect to the norm given by

$$\|\psi\|^2_{\mathbb{M}_{1,2}} := \|\psi\|^2_{L^2(\mathbb{P}\times l\times\nu)} + \|\mathcal{D}_{t^+,\mathbf{z}}[\psi]\|^2_{L^2(\mathbb{P}\times l\times\nu)}. \tag{3.95}$$

♦

We make the following remark.

Remark 4 *Note that*

$$\ln(1 + x) = \lim_{m\to\infty} \sum_{n=1}^{m} (-1)^{n-1}\frac{x^n}{n}, \quad |x| < 1. \tag{3.96}$$

Suppose the function $\psi_1(t)\psi_2(t, \mathbf{z}) \in \mathcal{M}$ and $|\psi_1(t)\psi_2(t, \mathbf{z})| < 1$ almost surely for all $0 \leq t \leq T, \mathbf{z} \in \mathbb{R}^N$. Then from (3.96) and Definition 16 if $\| \ln(1 + \psi_1(t)\psi_2(t, \mathbf{z}))\|^2_{\mathbb{M}_{1,2}} < \infty$, then the function $\ln(1 + \psi_1(t)\psi_2(t, \mathbf{z})) \in \mathbb{M}_{1,2}$. ♦

We now have the following relationship between Malliavin differentiation and forward Poisson integration.

Lemma 2 *Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space. Suppose ψ is forward integrable with respect to the compensated Poisson random measure $\tilde{q}$ and moreover that $\psi \in \mathbb{M}_{1,2}$. Then we have that*

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T \int_{\mathbb{R}^N} \psi(t, \mathbf{z})\tilde{q}(d^-t, d\mathbf{z})\right] = \mathbb{E}_{\mathbb{P}}\left[\int_0^T \int_{\mathbb{R}^N} \mathcal{D}_{t^+,\mathbf{z}}[\psi(t, \mathbf{z})]\nu(d\mathbf{z})dt\right]. \tag{3.97}$$

Proof: See ([41], Corollary 4.1). ■

In the next three sections 3.8.2-3.8.4 problems **(P1)-(P3)** are solved assuming the insider invests in a financial market driven by Lévy processes with jumps.

3.8.2 General utility

In this section problem **(P1)** is solved assuming the securities $\mathbf{S}$ exhibit jumps, assuming a general utility function for the insider and assuming that explicit portfolio weight constraints and penalty functions are *not* present in the portfolio selection problem. As mentioned in the introduction of Section 3.8, the unity weight constraint (3.23) is used to eliminate the security weight π_0 and so make the optimisation problem **(P1)** unconstrained. By eliminating π_0, the wealth process has the form (3.32) viz

$$\begin{aligned} W(T) \quad = \quad & W(t)\exp\left(\int_t^T\left(\xi_0(s)+\sum_{i=1}^{N_S}\hat{\xi}_i(s)\pi_i(s)\right.\right. \\ & -\frac{1}{2}\sum_{j=1}^{N_B}\left(\sigma_{0,j}(s)+\sum_{i=1}^{N_S}\hat{\sigma}_{ij}(s)\pi_i(s)\right)^2\Bigg)\,ds \\ & +\sum_{j=1}^{N_B}\int_t^T\left(\sigma_{0,j}(s)+\sum_{i=1}^{N_S}\hat{\sigma}_{ij}(s)\pi_i(s)\right)d^-B_j(s) \\ & +\sum_{j=1}^{N_q}\int_t^T\int_{A_j}\ln\left(1+G_j(s,\mathbf{z})\right)\tilde{q}_j(d^-s,d\mathbf{z}) \\ & \left.+\sum_{j=1}^{N_q}\int_t^T\int_{A_j}\left[\ln\left(1+G_j(s,\mathbf{z})\right)-G_j(s,\mathbf{z})\right]\nu_j^{\mathbb{F}}(d\mathbf{z})ds\right). \end{aligned} \tag{3.98}$$

The main result in this section is (3.119) an optimality equation (in quadratures) which an optimal portfolio $\boldsymbol{\pi}$ for problem **(P1)** must satisfy. This section is comprised of two subsections. In the first the set of admissible portfolios $\mathcal{P}_{L1}$ for problem **(P1)** is defined. In the second theorems are proved which allow us to derive the optimality equation (3.119).

Admissible portfolios for problem (P1)

In this section the set of admissible portfolios $\mathcal{P}_{L1}$ for problem **(P1)** is defined.

Definition 17 (Admissible portfolios) *A set of control processes* $\boldsymbol{\pi}$ *where* $\boldsymbol{\pi}(t)\in\mathbb{R}^{N_S}$ *for all* $0\le t\le T$*, is said to be* **admissible** *(or an* **admissible portfolio***) for problem* **(P1)** *if the following hold:*

(i) $\boldsymbol{\pi}$ *satisfies Definition 9.*

(ii) $\mathbb{E}[U'(T,W(T))W(T)]<\infty$ *where* W *has the form in (3.98). (This is required to ensure that the Radon-Nikodym derivative in (3.100) below is finite so that Bayes' Theorem can be applied.)*

(iii) *Let* $\boldsymbol{\pi}$ *and* $\boldsymbol{\theta}$ *be two portfolios which satisfy (i)-(ii) above. Recall the form (3.98) of the insider wealth process. For all* $0 \le t \le T, i \in \mathcal{N}_S$ *let*

$$\begin{aligned} M_i(t) &:= \int_0^t \left(\hat{\xi}_i(s) - \sum_{j=1}^{N_B} \sigma_{0,j}(s)\hat{\sigma}_{ij}(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) \right) ds \\ &\quad + \sum_{j=1}^{N_B} \int_0^t \hat{\sigma}_{ij}(s) dB_j(s) + \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})} \tilde{q}_j(d^- s, d\mathbf{z}) \\ &\quad + \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \left(\frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})} - \hat{g}_{ij}(s,\mathbf{z}) \right) \nu_j^{\mathbb{F}}(d\mathbf{z}) ds, \end{aligned} \tag{3.99}$$

where from (3.20) and (3.22) for all $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, i,j \in \mathcal{N}_S, k \in \mathcal{N}_B, l \in \mathcal{N}_q$

$$\begin{aligned} \hat{\xi}_i(t) &:= \xi_i(t) - \xi_0(t), & \hat{\sigma}_{ik}(t) &:= \sigma_{ik}(t) - \sigma_{0,k}(t), \\ \hat{g}_{il}(t,\mathbf{z}) &:= g_{il}(t,\mathbf{z}) - g_{0,l}(t,\mathbf{z}), & \bar{\sigma}_{ij}(t) &:= \textstyle\sum_{k=1}^{N_B} \hat{\sigma}_{ik}(t)\hat{\sigma}_{jk}(t) \end{aligned}$$

and $G_l(t,\mathbf{z}) := g_{0,l}(t,\mathbf{z}) + \sum_{i=1}^{N_S} \hat{g}_{il}(t,\mathbf{z})\pi_i(t)$*. Let* $\mathbf{y} := (y_1, \ldots, y_{N_S}) \in \mathbb{R}^{N_S}$*. Then there must exist a* $\delta > 0$ *such that for each* $i \in \mathcal{N}_S$ *we have that* $y_i \in (-\delta, \delta)$ *and the family*

$$\{U'(T, W(T, \boldsymbol{\pi}+diag(\mathbf{y})\boldsymbol{\theta}))W(T, \boldsymbol{\pi}+diag(\mathbf{y})\boldsymbol{\theta})|M_i(T, \boldsymbol{\pi}+diag(\mathbf{y})\boldsymbol{\theta})|\}_{\mathbf{0} \le \mathbf{y} \le \boldsymbol{\delta}}$$

is uniformly integrable, where $W(T) \equiv W(T, \boldsymbol{\pi})$ *and* $M_i(T) \equiv M_i(T, \boldsymbol{\pi})$ *and* $\boldsymbol{\delta}$ *is an* $N_S \times 1$ *matrix with all elements equal to* δ*. (This is required in (3.107) below to ensure that the partial derivatives of the objective functional (3.33) exist.)*

(iv) *Let*[6]

$$F(T) := (\mathbb{E}_{\mathbb{P}}[U'(T, W(T))W(T)])^{-1} U'(T, W(T))W(T) \tag{3.100}$$

and let the probability measure $\mathbb{Q}$ *be defined by*

$$d\mathbb{Q} = F(T) d\mathbb{P}. \tag{3.101}$$

Let $\delta > 0$ *and let* $\mathbf{y} := (y_1, \ldots, y_{N_S})$*. For each* $i \in \mathcal{N}_S$ *let* $y_i \in (-\delta, \delta)$ *and for all* $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, j \in \mathcal{N}_q$ *let*

$$\bar{G}_j(t,\mathbf{z},\mathbf{y}) := g_{0,j}(t,\mathbf{z}) + \sum_{i=1}^{N_S} \hat{g}_{ij}(t,\mathbf{z})(\pi_i(t) + y_i\theta_i(t)). \tag{3.102}$$

Let $\boldsymbol{\pi}$ *and* $\boldsymbol{\theta}$ *be two portfolios which satisfy (i)-(iii) above. Let* $\bar{\mathbf{G}} := (\bar{G}_1, \ldots, \bar{G}_{N_q})$ *and let* $\boldsymbol{\delta}$ *be a vector with all its* N_S *elements equal to* δ*. Then for all* $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, \mathbf{y} \in (-\boldsymbol{\delta}, \boldsymbol{\delta})$ *the following are required:*

[6] Recall that U is strictly increasing and $W > 0$ almost surely so $F(T)$ is finite almost surely.

(a) *For all $j \in \mathcal{N}_q$ we require that $1 > |\bar{G}_j(t, \mathbf{z}, \mathbf{y})|$ almost surely. (This is required since these expressions will be the argument of the natural logarithm in (3.106) below and Lemma 2 will be employed.)*

(b) *Suppose $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ satisfy (i)-(iv)(a) above. For all $i \in \mathcal{N}_S, j \in \mathcal{N}_q$ we require that the processes $\mathcal{D}_{t^+,\mathbf{z}}[\pi_i(t)]$ and $\mathcal{D}_{t^+,\mathbf{z}}[\theta_i(t)]$ exist in $L^2(\mathbb{Q} \times l \times \nu_j^{\mathbb{F}})$. (This is required in (c) below to ensure that the processes $\mathcal{D}_{t^+,\mathbf{z}}[\hat{g}_{ij}(t, \mathbf{z})\pi_i(t)]$ and $\mathcal{D}_{t^+,\mathbf{z}}[\hat{g}_{ij}(t, \mathbf{z})\theta_i(t)]$ exist in $L^2(\mathbb{Q} \times l \times \nu_j^{\mathbb{F}})$ - recall Lemma 1(v). Also see Remark 5(ii) below.)*

(c) *For all $i \in \mathcal{N}_S, j \in \mathcal{N}_q$ the processes $\hat{g}_{ij}(t, \mathbf{z})\pi_i(t) + \mathcal{D}_{t^+,\mathbf{z}}[\hat{g}_{ij}(t, \mathbf{z})\pi_i(t)]$ and $\hat{g}_{ij}(t, \mathbf{z})\theta_i(t) + \mathcal{D}_{t^+,\mathbf{z}}[\hat{g}_{ij}(t, \mathbf{z})\theta_i(t)]$ must be forward integrable with respect to the compensated Poisson random measure $\tilde{q}_j$. (Using this requirement we show in Remark 5(iii) below that for all $j \in \mathcal{N}_q$ the process $\bar{G}_j(t, \mathbf{z}, \mathbf{y})$ is an element of the set $\mathcal{M}$ (defined in Definition 16) above.)*

(d) *For all $j \in \mathcal{N}_q$ we must have that*

$$\| \ln(1+\bar{G}_j(t, \mathbf{z}, \mathbf{y}))\|^2_{L^2(\mathbb{Q}\times l\times\nu_j^{\mathbb{F}})} + \|\mathcal{D}_{t^+,\mathbf{z}}[\ln(1+\bar{G}_j(t, \mathbf{z}, \mathbf{y}))]\|^2_{L^2(\mathbb{Q}\times l\times\nu_j^{\mathbb{F}})} < \infty.$$

(This is required in Remark 5(iv)(e) below.)

(e) *We require that*

$$\mathbf{1} + \bar{\mathbf{G}}(t, \mathbf{z}, \mathbf{y}) + \mathcal{D}_{t^+,\mathbf{z}}[\bar{\mathbf{G}}(t, \mathbf{z}, \mathbf{y})] > \mathbf{0} \tag{3.103}$$

almost surely. Note that in particular (3.103) must hold for $\mathbf{y} = \mathbf{0}$. (Appendix F is referred to for details of the derivation of equation (3.108) (in Theorem 20) below. To ensure that this derivation is valid, (3.103) is required since the expressions in (3.103) are the arguments of the natural logarithm in equation (F.1) in Appendix F.)

All requirements mentioned in (iv) above are required to ensure that Lemma 2 can be employed in the proof of Theorem 20 below.

We denote by $\mathcal{P}_{L1}$ the set of all admissible portfolios for problem **(P1)**. ♦

Remark 5 *We make the following remarks:*

(i) *From Definition 17(iv)(a) we have that for all $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, j \in \mathcal{N}_q$*

$$1 + G_j(t, \mathbf{z}) \neq 0 \tag{3.104}$$

almost surely. This is stated since (3.104) is required for the objective functional (3.33) to be finite, but also for all other processes with $1 + G_j(t, \mathbf{z})$ as denominator to be finite.

(ii) *From Lemma 1(v), and Definition 17(iv)(b) and Lemma 1(iii) for all $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, \mathbf{y} \in (-\boldsymbol{\delta}, \boldsymbol{\delta}), j \in \mathcal{N}_q$ the function $\mathcal{D}_{t^+,\mathbf{z}}[\bar{G}_j(t, \mathbf{z}, \mathbf{y})]$ exists in $L^2(\mathbb{Q} \times l \times \nu_j^{\mathbb{F}})$. This result is required in Remark 5(iv)(d) below to ensure that for each $j \in \mathcal{N}_q$ the process $D_{t^+,\mathbf{z}}[\ln(1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y}))]$ exists in $L^2(\mathbb{Q} \times l \times \nu_j^{\mathbb{F}})$.*

(iii) *For all* $j \in \mathcal{N}_q$ *each term in* $\bar{G}_j(t, \mathbf{z}, \mathbf{y})$ *(defined in (3.102) above) is an element of* $\mathcal{M}$ *since:*

(a) *From (3.18) the jump coefficients* $g_{0,j}, j \in \mathcal{N}_q$ *are* $\mathbb{F}$*-adapted. Thus from condition (3.16), Lemma 1(v) and the fact that the processes* $g_{0,j}, j \in \mathcal{N}_q$ *are* $\mathbb{F}$*-adapted, each* $g_{0,j} \in \mathcal{M}$*.*

(b) *For each* $i \in \mathcal{N}_S, j \in \mathcal{N}_q$ *we have that* $\hat{g}_{ij}(t, \mathbf{z})\pi_i(t) \in \mathcal{M}$ *and* $\hat{g}_{ij}(t, \mathbf{z})\theta_i(t) \in \mathcal{M}$*. These two assertions are true because:*

(1) *Condition (3.16) holds.*

(2) *From Definition 17(iv)(b) the processes* $D_{t^+,\mathbf{z}}[\hat{g}_{ij}(t, \mathbf{z})\pi_i(t)]$ *and* $D_{t^+,\mathbf{z}}[\hat{g}_{ij}(t, \mathbf{z})\theta_i(t)]$ *exist in* $L^2(\mathbb{Q} \times l \times \nu_j^{\mathbb{F}})$*.*

(3) *Definition 17(iv)(c) holds.*

This result is used in (iv)(a) below.

(iv) *For all* $j \in \mathcal{N}_q$ *the function* $\ln(1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y})) \in \mathbb{M}_{1,2}$ *since:*

(a) *Definition 17(iv)(a) holds and from Remark 5(iii) each term in* $\bar{G}_j(t, \mathbf{z}, \mathbf{y})$ *is an element of* $\mathcal{M}$*.*

(b) $\bar{G}_j(t, \mathbf{z}, \mathbf{y})$ *is a linear span of elements of* $\mathcal{M}$*. (Recall that* $\mathbb{M}_{1,2}$ *is the closure of the linear span of* $\mathcal{M}$*.)*

(c) *From Remark 4 the function* $\ln(1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y}))$ *is thus a linear span of elements of* $\mathcal{M}$*.*

(d) *From Definition 17(iv)(a), Remark 5(ii) and Lemma 1(iv), each function* $D_{t^+,\mathbf{z}}[\ln(1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y}))]$ *exists in* $L^2(\mathbb{Q} \times l \times \nu_j^{\mathbb{F}})$*.*

(e) *Definition 17(iv)(d) holds.*

This result is used in (v) below.

(v) *From (iv) above for each* $j \in \mathcal{N}_q$ *the function* $\ln(1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y}))$ *satisfies Lemma 2 with* $\mathbb{P}$ *replaced with* $\mathbb{Q}$ *in (3.97). This result is used in the third equation in (3.107) below.*

♦

Next we prove theorems which allow us to derive the main result of this section viz the optimality equation (3.119).

Optimality theorems

In what follows, Theorems 20-22 are generalisations of ([42], Theorems 14-16). We now prove the following theorem in which we show that the processes $\mathbf{M} := (M_1, \ldots, M_{N_S})$ defined in (3.99) are $(\mathbb{H}, \mathbb{Q})$-martingales.

Theorem 20 *Suppose* $\boldsymbol{\pi} \in \mathcal{P}_{L1}$ *is an optimal portfolio for problem* **(P1)***. Then the processes* $\mathbf{M}$ *defined in (3.99) are* $(\mathbb{H}, \mathbb{Q})$*-martingales where* $\mathbb{Q}$ *is defined in (3.101).*

Proof: Recall (3.33) viz

$$\sup_{\boldsymbol{\pi}\in\mathcal{P}_{L1}} J_1(\boldsymbol{\pi}) \ := \sup_{\boldsymbol{\pi}\in\mathcal{P}_{L1}} \mathbb{E}[U(T,W(T))].$$

Let $\boldsymbol{\theta} \in \mathcal{P}_{L1}$ be another admissible portfolio of the form (3.74). In other words fix $t \in [0,T)$ and $h > 0$ such that $t+h \leq T$ and choose a portfolio $\boldsymbol{\theta}$ such that

$$\theta_i(s) = \theta_i^0(t)\chi_{[t,t+h]}(s) \qquad \text{for all } 0 \leq s \leq T, i \in \mathcal{N}_S \tag{3.105}$$

where each random variable $\theta_i^0(t)$ is bounded and $\mathcal{H}_t$-measurable. Let $\delta > 0$, let $\boldsymbol{\delta}$ be a vector with all its N_S elements equal to δ and let $\mathbf{y} := (y_1,\ldots,y_{N_S})$. Then by assumption of the optimality of $\boldsymbol{\pi}$, for $\mathbf{y} \in (-\boldsymbol{\delta},\boldsymbol{\delta})$ the function

$$f(\mathbf{y}) := J_1\left(\boldsymbol{\pi} + diag(\mathbf{y})\boldsymbol{\theta}\right) \tag{3.106}$$

is maximal for $\mathbf{y} = \mathbf{0}$. From Definition 17(iii) and Remark 5(v) we have that for all $i \in \mathcal{N}_S$

$$\begin{aligned}
0 &= \left[\frac{\partial}{\partial y_i} f(\mathbf{y})\right]_{\mathbf{y}=\mathbf{0}} \\
&= \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \left(\hat{\xi}_i(s) - \sum_{j=1}^{N_B}\sigma_{0,j}(s)\hat{\sigma}_{ij}(s) - \sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right)\theta_i(s)ds\right. \\
&\quad \left.+\sum_{j=1}^{N_B}\int_0^T \hat{\sigma}_{ij}(s)\theta_i(s)d^-B_j(s) + \sum_{j=1}^{N_q}\int_0^T\int_{\mathbb{R}^N}\left(\frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})} - \hat{g}_{ij}(s,\mathbf{z})\right)\theta_i(s)\nu_j^{\mathbb{F}}(d\mathbf{z})ds\right] \\
&\quad +\left[\frac{\partial}{\partial y_i}\mathbb{E}_{\mathbb{Q}}\left[\sum_{j=1}^{N_q}\int_0^T\int_{\mathbb{R}^N}\ln(1+\bar{G}_j(s,\mathbf{z},\mathbf{y}))\tilde{q}_j(d^-s,d\mathbf{z})\right]\right]_{\mathbf{y}=\mathbf{0}} \\
&= \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \left(\hat{\xi}_i(s) - \sum_{j=1}^{N_B}\sigma_{0,j}(s)\hat{\sigma}_{ij}(s) - \sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right)\theta_i(s)ds\right. \\
&\quad \left.+\sum_{j=1}^{N_B}\int_0^T \hat{\sigma}_{ij}(s)\theta_i(s)d^-B_j(s) + \sum_{j=1}^{N_q}\int_0^T\int_{\mathbb{R}^N}\left(\frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})} - \hat{g}_{ij}(s,\mathbf{z})\right)\theta_i(s)\nu_j^{\mathbb{F}}(d\mathbf{z})ds\right] \\
&\quad +\left[\frac{\partial}{\partial y_i}\mathbb{E}_{\mathbb{Q}}\left[\sum_{j=1}^{N_q}\int_0^T\int_{\mathbb{R}^N}\mathcal{D}_{s^+,\mathbf{z}}[\ln(1+\bar{G}_j(s,\mathbf{z},\mathbf{y}))]\nu_j^{\mathbb{F}}(d\mathbf{z})ds\right]\right]_{\mathbf{y}=\mathbf{0}}.
\end{aligned} \tag{3.107}$$

Since the expected returns, volatilities, jump coefficients and portfolios $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ are bounded, by the Lebesgue dominated convergence theorem the interchanging of classical differentiation and Lebesgue integration in (3.107) is justified. We concentrate on the integrand in the last expression of the third equation in (3.107). From Appendix F we have that for all $0 \le t < T, \mathbf{z} \in \mathbb{R}^N, \mathbf{y} \in (-\boldsymbol{\delta}, \boldsymbol{\delta}), i \in \mathcal{N}_S, j \in \mathcal{N}_q$ for almost all $\omega \in \Omega$

$$\left[\frac{\partial}{\partial y_i} \mathcal{D}_{t^+,\mathbf{z}}[\ln(1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y}))]\right]_{\mathbf{y}=\mathbf{0}} = \mathcal{D}_{t^+,\mathbf{z}}\left[\frac{\hat{g}_{ij}(t, \mathbf{z})\theta_i(t)}{1 + G_j(t, \mathbf{z})}\right] \tag{3.108}$$

$l \times \nu_j$ almost everywhere. Substituting (3.108) into (3.107) it reduces to

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{Q}}\Bigg[\int_0^T \left(\hat{\xi}_i(s) - \sum_{j=1}^{N_B} \sigma_{0,j}(s)\hat{\sigma}_{ij}(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s)\right)\theta_i(s)ds \\ &\quad + \sum_{j=1}^{N_B}\int_0^T \hat{\sigma}_{ij}(s)\theta_i(s)d^- B_j(s) + \sum_{j=1}^{N_q}\int_0^T\int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s, \mathbf{z})\theta_i(s)}{1 + G_j(s, \mathbf{z})}\tilde{q}_j(d^- s, d\mathbf{z}) \\ &\quad + \sum_{j=1}^{N_q}\int_0^T\int_{\mathbb{R}^N}\left(\frac{\hat{g}_{ij}(s, \mathbf{z})}{1 + G_j(s, \mathbf{z})} - \hat{g}_{ij}(s, \mathbf{z})\right)\theta_i(s)\nu_j^{\mathbb{F}}(d\mathbf{z})ds\Bigg]. \end{aligned} \tag{3.109}$$

Imposing the particular form (3.105) of $\boldsymbol{\theta}$ in (3.109) it reduces to

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{Q}}\Bigg[\theta_i^0(t)\Bigg(\int_t^{t+h}\left(\hat{\xi}_i(s) - \sum_{j=1}^{N_B} \sigma_{0,j}(s)\hat{\sigma}_{ij}(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s)\right)ds \\ &\quad + \sum_{j=1}^{N_B}\int_t^{t+h} \hat{\sigma}_{ij}(s)dB_j(s) + \sum_{j=1}^{N_q}\int_t^{t+h}\int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s, \mathbf{z})}{1 + G_j(s, \mathbf{z})}\tilde{q}_j(d^- s, d\mathbf{z}) \\ &\quad + \sum_{j=1}^{N_q}\int_t^{t+h}\int_{\mathbb{R}^N}\left(\frac{\hat{g}_{ij}(s, \mathbf{z})}{1 + G_j(s, \mathbf{z})} - \hat{g}_{ij}(s, \mathbf{z})\right)\nu_j^{\mathbb{F}}(d\mathbf{z})ds\Bigg)\Bigg]. \end{aligned} \tag{3.110}$$

Since (3.110) is valid for all bounded $\mathcal{H}_t$-measurable random variables $\theta_i^0(t)$, we conclude that

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{Q}}\Bigg[\int_t^{t+h}\left(\hat{\xi}_i(s) - \sum_{j=1}^{N_B} \sigma_{0,j}(s)\hat{\sigma}_{ij}(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s)\right)ds \\ &\quad + \sum_{j=1}^{N_B}\int_t^{t+h} \hat{\sigma}_{ij}(s)dB_j(s) + \sum_{j=1}^{N_q}\int_t^{t+h}\int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s, \mathbf{z})}{1 + G_j(s, \mathbf{z})}\tilde{q}_j(d^- s, d\mathbf{z}) \\ &\quad + \sum_{j=1}^{N_q}\int_t^{t+h}\int_{\mathbb{R}^N}\left(\frac{\hat{g}_{ij}(s, \mathbf{z})}{1 + G_j(s, \mathbf{z})} - \hat{g}_{ij}(s, \mathbf{z})\right)\nu_j^{\mathbb{F}}(d\mathbf{z})ds\,\Bigg|\,\mathcal{H}_t\Bigg]. \end{aligned} \tag{3.111}$$

For all $i \in \mathcal{N}_S$ let M_i be defined in (3.99). Then from (3.111) we have that

$$\mathbb{E}_{\mathbb{Q}}[M_i(t + h)|\mathcal{H}_t] = M_i(t) \tag{3.112}$$

since each $M_i(t)$ is $\mathcal{H}_t$-measurable. Since $t \in [0,T]$ and $h > 0$ are arbitrary in (3.112), we have that each process M_i is an $(\mathbb{H}, \mathbb{Q})$-martingale. ∎

We now prove the following theorem in which we show amongst other things that the $(\mathbb{F}, \mathbb{P})$-Brownian motions $\mathbf{B}$ defined in (3.15) are $(\mathbb{H}, \mathbb{P})$-semimartingales.

Theorem 21 *Let $\boldsymbol{\pi} \in \mathcal{P}_{L1}$ be an optimal portfolio for problem* **(P1)**. *Then we have the following. (i) Let the processes $\mathbf{M}$ and random variable F be given by (3.99) and (3.100) respectively. Suppose that at each $t \in [0,T]$ the expected value of the covariation of each forward Poisson integral in (3.99) and each Itô diffusion integral in (3.99) is finite. Suppose also that at each $t \in [0,T]$ the expected value of the quadratic variation of all forward Poisson integrals in (3.99) is finite. Then for all $0 \le t \le T, i \in \mathcal{N}_S$ the processes*

$$\hat{M}_i(t) \quad := \quad M_i(t) - \int_0^t \frac{d[M_i, \bar{F}](s)}{\bar{F}(s)} \tag{3.113}$$

are $(\mathbb{H}, \mathbb{P})$-martingales, where for all $0 \le t \le T$

$$\bar{F}(t) := \mathbb{E}[F(T)|\mathcal{H}_t]. \tag{3.114}$$

(ii) For each $i \in \mathcal{N}_S$ the process

$$S_i^B(t) := \sum_{j=1}^{N_B} \int_0^t \hat{\sigma}_{ij}(s) dB_j(s), \quad 0 \le t \le T$$

is an $(\mathbb{H}, \mathbb{P})$-semimartingale.
(iii) The Brownian motions $\mathbf{B}$ defined in (3.15) are $(\mathbb{H}, \mathbb{P})$-semimartingales.
(iv) Forward stochastic integrals with respect to $\mathbf{B}$ are Itô integrals.
(v) For each $i \in \mathcal{N}_S$ the process

$$S_i^q(t) := \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s, \mathbf{z})}{1 + G_j(s, \mathbf{z})} \tilde{q}_j(d^- s, d\mathbf{z}), \quad 0 \le t \le T$$

is an $(\mathbb{H}, \mathbb{P})$-semimartingale.

Proof:
(i) Suppose the processes $\hat{\mathbf{M}} := (\hat{M}_1, \ldots, \hat{M}_{N_S})$ are $(\mathbb{H}, \mathbb{P})$-local martingales and that for each $t \in [0,T]$ the expected value of the quadratic variation of each $\hat{M}_i$ is finite. Then from Theorem 4 we have that the processes $\hat{\mathbf{M}}$ of (3.113) are $(\mathbb{H}, \mathbb{P})$-martingales and we have proved part (i).

Now for all admissible portfolios $\mathcal{P}_{L1}$ the terminal wealth value $0 < W(T)$ almost surely, thus we have that $0 < F(T)$ almost surely, where $F(T)$ is defined in (3.100). Thus we have from (3.100) that

$$d\mathbb{P} = F(T)^{-1} d\mathbb{Q} =: G(T) d\mathbb{Q}. \tag{3.115}$$

By Bayes' Theorem let $\bar{F}$ be the $(\mathbb{H},\mathbb{Q})$-martingale defined by

$$\bar{F}(t)=\mathbb{E}_{\mathbb{Q}}[G(T)|\mathcal{H}_t]=(\mathbb{E}_{\mathbb{P}}[F(T)|\mathcal{H}_t])^{-1}\mathbb{E}_{\mathbb{P}}[F(T)G(T)|\mathcal{H}_t]=(\mathbb{E}_{\mathbb{P}}[F(T)|\mathcal{H}_t])^{-1}.$$

Now suppose $\boldsymbol{\pi}\in\mathcal{P}_{L1}$ is an optimal portfolio for problem **(P1)**. By Theorem 20 we know that the processes $\mathbf{M}$ defined in (3.99) are $(\mathbb{H},\mathbb{Q})$-martingales. Thus from (3.115) and Corollary 4, we have that the processes $\hat{\mathbf{M}}$ are $(\mathbb{H},\mathbb{P})$-local martingales. Secondly recall the assumptions that at each $t\in[0,T]$ the expected value of the covariation of each forward Poisson integral in (3.99) and each Itô diffusion integral in (3.99) is finite, and that at each $t\in[0,T]$ the expected value of the quadratic variation of all forward Poisson integrals in (3.99) is finite. Then from Theorem 5 for each $t\in[0,T]$ the expected value of the quadratic variation of the processes $\hat{\mathbf{M}}$ is finite.
(ii) From Theorem 18, the orthogonal decomposition of the $(\mathbb{H},\mathbb{P})$-martingales $\hat{\mathbf{M}}:=(\hat{M}_1,\dots,\hat{M}_{N_S})$ into continuous $(\mathbb{H},\mathbb{P})$-local martingales $\hat{\mathbf{M}}^c:=(\hat{M}_1^c,\dots,\hat{M}_{N_S}^c)$ and discontinuous $(\mathbb{H},\mathbb{P})$-local martingales $\hat{\mathbf{M}}^d:=(\hat{M}_1^d,\dots,\hat{M}_{N_S}^d)$, is given by

$$\hat{\mathbf{M}}(t)=\hat{\mathbf{M}}^c(t)+\hat{\mathbf{M}}^d(t),\quad 0\le t\le T. \tag{3.116}$$

In (3.116), for all $0\le t\le T, i\in\mathcal{N}_S$ we have that

$$\hat{M}_i^c(t) := \sum_{j=1}^{N_B}\int_0^t\hat{\sigma}_{ij}(s)dB_j(s)+\sum_{j=1}^{N_B}\int_0^t\hat{\sigma}_{ij}(s)\eta_{ij}^B(s)ds, \tag{3.117}$$

$$\hat{M}_i^d(t) := \sum_{j=1}^{N_q}\int_0^t\int_{\mathbb{R}^N}\frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})}\tilde{q}_j(d^-s,d\mathbf{z})+\sum_{j=1}^{N_q}\int_0^t\eta_{ij}^q(s)ds \tag{3.118}$$

and for all $i\in\mathcal{N}_S, j\in\mathcal{N}_B, k\in\mathcal{N}_q$ the $\mathbb{H}$-adapted processes η_{ij}^B and η_{ik}^q must satisfy almost surely

$$\begin{aligned}\sum_{j=1}^{N_B}\int_0^t\hat{\sigma}_{ij}(s)\eta_{ij}^B(s)ds+\sum_{j=1}^{N_q}\int_0^t\eta_{ij}^q(s)ds+\int_0^t\bar{F}^{-1}(s)d[M_i,\bar{F}](s)\\
=\int_0^t\left(\hat{\xi}_i(s)-\sum_{j=1}^{N_B}\sigma_{0,j}(s)\hat{\sigma}_{ij}(s)-\sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right)ds\\
+\sum_{j=1}^{N_q}\int_0^t\int_{\mathbb{R}^N}\left(\frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})}-\hat{g}_{ij}(s,\mathbf{z})\right)\nu_j^{\mathbb{F}}(d\mathbf{z})ds.\end{aligned}$$

Note that the processes $\hat{\mathbf{M}}^c$ are $(\mathbb{H},\mathbb{P})$-martingales by Theorem 4 since from Theorem 5 for each $t\in[0,T]$ the expected value of the quadratic variation of the processes $\hat{\mathbf{M}}^c$ is finite. (See the proof of Theorem 8(i).) Thus for each $i\in\mathcal{N}_S$ the process

$$S_i^B(t):=\sum_{j=1}^{N_B}\int_0^t\hat{\sigma}_{ij}(s)dB_j(s)=\hat{M}_i^c(t)-\sum_{j=1}^{N_B}\int_0^t\hat{\sigma}_{ij}(s)\eta_{ij}^B(s)ds,\quad 0\le t\le T$$

is an $(\mathbb{H},\mathbb{P})$-semimartingale because it is the sum of an $(\mathbb{H},\mathbb{P})$-martingale $\hat{M}_i^c$ and a process of bounded variation.
(iii) and (iv) See the proof of Theorem 8(iii)-(iv).
(v) From (3.116), for each $i \in \mathcal{N}_S$ the process

$$\begin{aligned} S_i^q(t) &:= \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})} \tilde{q}_j(d^- s, d\mathbf{z}) \\ &= \hat{M}_i(t) - \hat{M}_i^c(t) - \sum_{j=1}^{N_q} \int_0^t \eta_{ij}^q(s) ds, \qquad 0 \le t \le T. \end{aligned}$$

From (i) above the processes $\hat{\mathbf{M}}$ are $(\mathbb{H},\mathbb{P})$-martingales. From (ii) above the processes $\hat{\mathbf{M}}^c$ are $(\mathbb{H},\mathbb{P})$-martingales. Thus each process $S_i^q, i \in \mathcal{N}_S$ is an $(\mathbb{H},\mathbb{P})$-semimartingale because it is the sum of an $(\mathbb{H},\mathbb{P})$-martingale $\hat{M}_i - \hat{M}_i^c$ and a process of bounded variation. ■

We now prove the following theorem in which we derive an optimality equation for an optimal portfolio $\boldsymbol{\pi}$ for problem **(P1)**. Recall from (3.13) that the $\mathbb{F}$ compensators of the Poisson random measures $\mathbf{q}$ are $\boldsymbol{\nu}^{\mathbb{F}}$. In Theorem 22 below we consider the $\mathbb{H}$ compensators of $\mathbf{q}$.

Theorem 22 *Let $\boldsymbol{\pi} \in \mathcal{P}_{L1}$ be an optimal portfolio for problem* **(P1)***. For all $j \in \mathcal{N}_q$ let $\nu_j^{\mathbb{H}} = \nu_j^{\mathbb{H}}(dt, d\mathbf{z})$ be the $\mathbb{H}$ compensator of the Poisson random measure q_j and let $\boldsymbol{\nu}^{\mathbb{H}} := (\nu_1^{\mathbb{H}}, \ldots, \nu_{N_q}^{\mathbb{H}})$. Then for all $0 \le t \le T, i \in \mathcal{N}_S$, π_i satisfies almost surely the equation*

$$\begin{aligned} &\int_0^t \left(\hat{\xi}_i(s) - \sum_{j=1}^{N_B} \sigma_{0,j}(s)\hat{\sigma}_{ij}(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) \right) ds \\ &+ \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \left(\frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})} - \hat{g}_{ij}(s,\mathbf{z}) \right) \nu_j^{\mathbb{F}}(d\mathbf{z}) ds - \int_0^t \bar{F}^{-1}(s) d[M_i, \bar{F}](s) \\ &= \sum_{j=1}^{N_B} \int_0^t \hat{\sigma}_{ij}(s)\eta_{ij}^B(s) ds + \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})} \left(\nu_j^{\mathbb{F}} - \nu_j^{\mathbb{H}}\right)(ds, d\mathbf{z}). \end{aligned} \tag{3.119}$$

Proof: Since $\mathbb{F} \subseteq \mathbb{H}$, each Poisson random measure q_j has a unique predictable compensator with respect to $\mathbb{H}$, viz $\nu_j^{\mathbb{H}} = \nu_j^{\mathbb{H}}(dt, d\mathbf{z})$. ([66], Theorem II.1.8) In other words

(a) the processes $\int_0^t \int_{\mathbb{R}^N} X(s,\mathbf{z})(\mathbf{q} - \boldsymbol{\nu}^{\mathbb{H}})(ds, d\mathbf{z})$ are compensated Poisson integrals for any $\mathbb{H}$-predictable integrable process X,

(b) moreover the processes $\int_0^t \int_{\mathbb{R}^N} X(s,\mathbf{z})(\mathbf{q} - \boldsymbol{\nu}^{\mathbb{H}})(ds, d\mathbf{z})$ are $(\mathbb{H},\mathbb{P})$-local martingales for any $\mathbb{H}$-predictable integrable process X and

(c) in particular, for all $i \in \mathcal{N}_S, j \in \mathcal{N}_q$ the processes

$$\int_0^t \int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})}(q_j - \nu_j^{\mathbb{H}})(ds, d\mathbf{z}) = \int_0^t \int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})}(q_j - \nu_j^{\mathbb{H}})(d^- s, d\mathbf{z}) \tag{3.120}$$

are $(\mathbb{H}, \mathbb{P})$-local martingales.

Thus from (3.117), (3.118) and (3.120) we can rewrite (3.116) in index form as

$$\begin{aligned}
\hat{M}_i(t) &= \sum_{j=1}^{N_B} \int_0^t \hat{\sigma}_{ij}(s) dB_j(s) + \sum_{j=1}^{N_B} \int_0^t \hat{\sigma}_{ij}(s) \eta_{ij}^B(s) ds \\
&+ \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})}(q_j - \nu_j^{\mathbb{H}})(ds, d\mathbf{z}) \\
&+ \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})}(\nu_j^{\mathbb{H}} - \nu_j^{\mathbb{F}})(ds, d\mathbf{z}) + \sum_{j=1}^{N_q} \int_0^t \eta_{ij}^q(s) ds,
\end{aligned} \tag{3.121}$$

where it is important to note that there is no longer forward Poisson integration in (3.121). Equating (3.113) and (3.121) equation (3.119) is obtained. ■

Note that since $\boldsymbol{\nu}^{\mathbb{H}} = \boldsymbol{\nu}^{\mathbb{H}}(ds, d\mathbf{z})$ (and the quadratic variation processes $[\mathbf{M}, \bar{F}]$ are not necessarily absolutely continuous), we can't (as in Theorem 9) continue from (3.119) and show that the integrand must be zero almost surely. In Section 4.2.1, to derive an algebraic equation (and not an equation in quadratures which is (3.119)) for a constrained optimal portfolio $\boldsymbol{\pi}$, we consider specific types of Lévy process and utility function and show in these cases that in fact $\boldsymbol{\nu}^{\mathbb{H}}(ds, d\mathbf{z}) = \boldsymbol{\nu}^{\mathbb{H}}(d\mathbf{z}) ds$. We have the following corollary of Theorem 22.

Corollary 5 *Suppose the insider is in fact honest. Suppose also that the quadratic variation processes $[\mathbf{M}, \bar{F}]$ are absolutely continuous. Let l denote the Lebesgue measure. Then an optimal portfolio $\boldsymbol{\pi}$ for problem* **(P1)** *must satisfy $\mathbb{P} \times l$ almost everywhere the equation*

$$\begin{aligned}
0 &= \hat{\xi}_i(t) - \sum_{j=1}^{N_B} \sigma_{0,j}(t) \hat{\sigma}_{ij}(t) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t) \pi_j(t) - \sum_{j=1}^{N_B} \hat{\sigma}_{ij}(t) \eta_{ij}^B(t) \\
&+ \sum_{j=1}^{N_q} \int_{\mathbb{R}^N} \left(\frac{\hat{g}_{ij}(t,\mathbf{z})}{1+G_j(t,\mathbf{z})} - \hat{g}_{ij}(t,\mathbf{z}) \right) \nu_j^{\mathbb{F}}(d\mathbf{z}) - \frac{1}{\bar{F}(t)} \frac{d}{dt}[M_i, \bar{F}](t).
\end{aligned} \tag{3.122}$$

Proof: Setting $\boldsymbol{\nu}^{\mathbb{F}} \equiv \boldsymbol{\nu}^{\mathbb{H}}$ in (3.119) we get that

$$0 = \int_0^t \Bigg(\hat{\xi}_i(s) - \sum_{j=1}^{N_B} \sigma_{0,j}(s) \hat{\sigma}_{ij}(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s) \pi_j(s)$$

$$
+\sum_{j=1}^{N_q}\int_{\mathbb{R}^N}\left(\frac{\hat{g}_{ij}(s,\mathbf{z})}{1+G_j(s,\mathbf{z})}-\hat{g}_{ij}(s,\mathbf{z})\right)\nu_j^{\mathbb{F}}(d\mathbf{z})
$$

$$
-\bar{F}^{-1}(s)\frac{d}{ds}[M_i,\bar{F}](s)-\sum_{j=1}^{N_B}\hat{\sigma}_{ij}(s)\eta_{ij}^{B}(s)\Bigg)\,ds. \tag{3.123}
$$

Then using the fundamental theorem of calculus in (3.123) equation (3.122) is obtained. ■

Note that due to the presence of jumps in the securities **S**, the only way the processes $[\mathbf{M}, F]$ will be absolutely continuous is if no jumps are exhibited by **S** over the whole interval $[0, T]$. In the next section a constrained portfolio selection problem is solved assuming the insider has logarithmic utility and invests in a financial market driven by Lévy processes with jumps. It is also assumed that penalty functions and explicit weight constraints are present, however a money market security is not available for investment.

3.8.3 Logarithmic utility, weight constraints, penalty functions and no investment in a money market security

In this section problem **(P2)** is solved assuming the insider has a logarithmic utility function, invests in a financial market driven by Lévy processes with jumps and assuming that portfolio weight constraints and penalty functions are present in the portfolio selection problem. It is also assumed that the insider cannot invest in a money market security. In this case the wealth process has the form (3.30) viz

$$
\begin{aligned}
W(T) \quad = \quad & W(t)\exp\left(\int_t^T\left(\sum_{i=1}^{N_S}\xi_i(s)\pi_i(s)-\frac{1}{2}\sum_{j=1}^{N_B}\left(\sum_{i=1}^{N_S}\sigma_{ij}(s)\pi_i(s)\right)^2\right)ds\right. \\
& +\sum_{j=1}^{N_B}\sum_{i=1}^{N_S}\int_t^T\sigma_{ij}(s)\pi_i(s)d^-B_j(s) \\
& +\sum_{j=1}^{N_q}\int_t^T\int_{A_j}\ln\left(1+\sum_{i=1}^{N_S}g_{ij}(s,\mathbf{z})\pi_i(s)\right)\tilde{q}_j(d^-s,d\mathbf{z}) \\
& \left.+\sum_{j=1}^{N_q}\int_t^T\int_{A_j}\left(\ln\left(1+\sum_{i=1}^{N_S}g_{ij}(s,\mathbf{z})\pi_i(s)\right)-\sum_{i=1}^{N_S}g_{ij}(s,\mathbf{z})\pi_i(s)\right)\nu_j^{\mathbb{F}}(d\mathbf{z})ds\right).
\end{aligned} \tag{3.124}
$$

The main result in this section is (3.136) which is an optimality equation for a constrained optimal portfolio $\boldsymbol{\pi}$ for problem **(P2)**. We now define the set of admissible portfolios $\mathcal{P}_{L2}$ for problem **(P2)**.

Definition 18 (Admissible portfolios) *A set of control processes* $\boldsymbol{\pi}$ *where* $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ *for all* $0 \le t \le T$, *is said to be* **admissible** *(or an* **admissible portfolio***) for problem* **(P2)** *if the following hold:*

(i) $\boldsymbol{\pi}$ *satisfies Definition 10.*

(ii) Let $\boldsymbol{\pi}$ *and* $\boldsymbol{\theta}$ *be two portfolios which satisfy (i) above. Recall the form (3.124) of the insider wealth process. From Remark 5(i), for all* $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, i \in \mathcal{N}_S, j \in \mathcal{N}_q$ *let*

$$\vartheta_{ij}(t,\mathbf{z}) := \frac{g_{ij}(t,\mathbf{z})}{1 + \sum_{k=1}^{N_S} g_{kj}(t,\mathbf{z})\pi_k(t)}. \tag{3.125}$$

For all $0 \le t \le T, i \in \mathcal{N}_S$ *let*

$$\begin{aligned} M_i(t) \quad := \quad & \int_0^t \left(\xi_i(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) \right) ds \\ & + \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) dB_j(s) + \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s,\mathbf{z}) \tilde{q}_j(d^- s, d\mathbf{z}) \\ & - \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \left(\vartheta_{ij}(s,\mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(s,\mathbf{z})\pi_k(s) \right) \nu_j^{\mathbb{F}}(d\mathbf{z}) ds \\ & + \int_0^t \left(- \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) - \lambda_0(s) + \sum_{j=1}^{M} \lambda_j(s) h_{ij}(s) \right) ds, \end{aligned} \tag{3.126}$$

where the processes $\lambda_j, j \in \mathcal{N}_M \cup \{0\}$ *are defined in (3.50) and from (3.17) we have that* $\bar{\sigma}_{ij}(t) := \sum_{k=1}^{N_B} \sigma_{jk}(t)\sigma_{ik}(t)$. *Let* $\mathbf{y} := (y_1, \ldots, y_{N_S}) \in \mathbb{R}^{N_S}$. *Then there must exist a* $\delta > 0$ *such that for each* $i \in \mathcal{N}_S$ *we have that* $y_i \in (-\delta, \delta)$ *and the family*

$$\{|M_i(T, \boldsymbol{\pi} + diag(\mathbf{y})\boldsymbol{\theta})|\}_{\mathbf{0} \le \mathbf{y} \le \boldsymbol{\delta}}$$

is uniformly integrable, where $M_i(T) \equiv M_i(T, \boldsymbol{\pi})$ *and* $\boldsymbol{\delta}$ *is an* $N_S \times 1$ *matrix with all elements equal to* δ. *(This is required to ensure that the partial derivatives of the Lagrangian in (3.128) below exist.)*

(iii) For all $i \in \mathcal{N}_S$ *let the vector* $\mathbf{g}^i$ *denote the ith row of the jump coefficients matrix* $\mathbf{g}$. *Then* $\boldsymbol{\pi}$ *must satisfy Definition 17(iv) with* $U(t,x) = \ln x$ *and* $\boldsymbol{\sigma}^0 \equiv \mathbf{0} \equiv \mathbf{g}^0$.

We denote by $\mathcal{P}_{L2}$ *the set of all admissible portfolios for problem* **(P2)**. ♦

In what follows, Theorems 23-25 are generalisations of ([42], Theorems 14-16). We now prove the following theorem in which we show that the processes $\mathbf{M} := (M_1, \ldots, M_{N_S})$ defined in (3.126) are $(\mathbb{H}, \mathbb{P})$-martingales.

Theorem 23 *Suppose $\boldsymbol{\pi} \in \mathcal{P}_{L2}$ is an optimal portfolio for problem* **(P2)**. *Then each M_i defined in (3.126) is an $(\mathbb{H}, \mathbb{P})$-martingale.*

Proof: Recall the definition of the Lagrangian $V_2(\boldsymbol{\pi})$ in (3.51). Let $\boldsymbol{\theta} \in \mathcal{P}_{L2}$ be another admissible portfolio of the following form. Fix $t \in [0, T)$ and $h > 0$ such that $t + h \leq T$ and

$$\theta_i(s) = \theta_i^0(t)\chi_{[t,t+h]}(s) \quad \text{for all } 0 \leq s \leq T, i \in \mathcal{N}_S, \tag{3.127}$$

where each random variable $\theta_i^0(t)$ is bounded and $\mathcal{H}_t$-measurable. Then by assumption of the optimality of $\boldsymbol{\pi}$, for $\mathbf{y} \in (-\boldsymbol{\delta}, \boldsymbol{\delta})$ the function

$$\begin{aligned} f(\mathbf{y}) &= V_2(\boldsymbol{\pi} + diag(\mathbf{y})\boldsymbol{\theta}) \\ &= J_2\left(\boldsymbol{\pi} + diag(\mathbf{y})\boldsymbol{\theta}\right) + \mathbb{E}\left[\int_0^T \lambda_0(s)\left(\Upsilon(s) - \sum_{i=1}^{N_S}(\pi_i(s) + y_i\theta_i(s))\right)ds\right. \\ &\quad \left. + \sum_{j=1}^{M}\int_0^T \lambda_j(s)\left[\sum_{i=1}^{N_S} h_{ij}(s)(\pi_i(s) + y_i\theta_i(s)) - \bar{h}_j(s)\right]ds\right] \end{aligned} \tag{3.128}$$

is maximal for $\mathbf{y} = \mathbf{0}$. As in the derivation of (3.109), if we use the result (3.108) proved in Appendix F, then we have for each $i \in \mathcal{N}_S$ that

$$\begin{aligned} 0 &= \left[\frac{\partial}{\partial y_i} f(\mathbf{y})\right]_{\mathbf{y}=\mathbf{0}} \\ &= \mathbb{E}\left[\int_0^T \left(\xi_i(s) - \sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\right)\theta_i(s)ds\right. \\ &\quad + \sum_{j=1}^{N_B}\int_0^T \sigma_{ij}(s)\theta_i(s)d^-B_j(s) + \sum_{j=1}^{N_q}\int_0^T\int_{\mathbb{R}^N}\vartheta_{ij}(s,\mathbf{z})\theta_i(s)\tilde{q}_j(d^-s, d\mathbf{z}) \\ &\quad - \sum_{j=1}^{N_q}\int_0^T\int_{\mathbb{R}^N}\left(\vartheta_{ij}(s,\mathbf{z})\sum_{k=1}^{N_S} g_{kj}(s,\mathbf{z})\pi_k(s)\right)\theta_i(s)\nu_j^{\mathbb{F}}(d\mathbf{z})ds \\ &\quad \left. + \int_0^T\left(-\sum_{j=1}^{N_S}\mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) - \lambda_0(s) + \sum_{j=1}^{M}\lambda_j(s)h_{ij}(s)\right)\theta_i(s)ds\right]. \end{aligned} \tag{3.129}$$

Imposing the particular form (3.127) of $\boldsymbol{\theta}$ in (3.129) it reduces to

$$
\begin{aligned}
0 &= \mathbb{E}\Bigg[\theta_i^0(t)\Bigg(\int_t^{t+h}\Bigg(\xi_i(s)-\sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\Bigg)ds \\
&\quad +\sum_{j=1}^{N_B}\int_t^{t+h}\sigma_{ij}(s)dB_j(s)+\sum_{j=1}^{N_q}\int_t^{t+h}\int_{\mathbb{R}^N}\vartheta_{ij}(s,\mathbf{z})\tilde{q}_j(d^-s,d\mathbf{z}) \\
&\quad -\sum_{j=1}^{N_q}\int_t^{t+h}\int_{\mathbb{R}^N}\Bigg(\vartheta_{ij}(s,\mathbf{z})\sum_{k=1}^{N_S}g_{kj}(s,\mathbf{z})\pi_k(s)\Bigg)\nu_j^{\mathbb{F}}(d\mathbf{z})ds \\
&\quad +\int_t^{t+h}\Bigg(-\sum_{j=1}^{N_S}\mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s)))-\lambda_0(s)+\sum_{j=1}^{M}\lambda_j(s)h_{ij}(s)\Bigg)ds\Bigg)\Bigg].
\end{aligned} \tag{3.130}
$$

Since (3.130) is valid for all bounded $\mathcal{H}_t$-measurable functions $\theta_i^0(t)$, we conclude that

$$
\begin{aligned}
0 &= \mathbb{E}\Bigg[\int_t^{t+h}\Bigg(\xi_i(s)-\sum_{j=1}^{N_S}\bar{\sigma}_{ij}(s)\pi_j(s)\Bigg)ds \\
&\quad +\sum_{j=1}^{N_B}\int_t^{t+h}\sigma_{ij}(s)dB_j(s)+\sum_{j=1}^{N_q}\int_t^{t+h}\int_{\mathbb{R}^N}\vartheta_{ij}(s,\mathbf{z})\tilde{q}_j(d^-s,d\mathbf{z}) \\
&\quad -\sum_{j=1}^{N_q}\int_t^{t+h}\int_{\mathbb{R}^N}\Bigg(\vartheta_{ij}(s,\mathbf{z})\sum_{k=1}^{N_S}g_{kj}(s,\mathbf{z})\pi_k(s)\Bigg)\nu_j^{\mathbb{F}}(d\mathbf{z})ds \\
&\quad +\int_t^{t+h}\Bigg(-\sum_{j=1}^{N_S}\mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s)))-\lambda_0(s)+\sum_{j=1}^{M}\lambda_j(s)h_{ij}(s)\Bigg)ds\Bigg|\mathcal{H}_t\Bigg].
\end{aligned} \tag{3.131}
$$

With M_i defined in (3.126), from (3.131) we have that

$$
\mathbb{E}[M_i(t+h)-M_i(t)|\mathcal{H}_t] = 0. \tag{3.132}
$$

Since $t\in[0,T]$ and $h>0$ are arbitrary in (3.132) and $M_i(t)$ is $\mathcal{H}_t$-measurable we have that each M_i is an $(\mathbb{H},\mathbb{P})$-martingale. ■

We now prove the following theorem in which we show amongst other things that the $(\mathbb{F},\mathbb{P})$-Brownian motions $\mathbf{B}$ defined in (3.15) are $(\mathbb{H},\mathbb{P})$-semimartingales.

Theorem 24 *Let $\boldsymbol{\pi} \in \mathcal{P}_{L2}$ be an optimal portfolio for problem* **(P2)**. *Then we have the following. (i) For each $i \in \mathcal{N}_S$ the process*

$$S_i^B(t) := \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) dB_j(s), \quad 0 \leq t \leq T$$

is an $(\mathbb{H}, \mathbb{P})$-semimartingale.
(ii) The Brownian motions $\mathbf{B}$ *defined in (3.15) are $(\mathbb{H}, \mathbb{P})$-semimartingales.*
(iii) Forward stochastic integrals with respect to $\mathbf{B}$ *are Itô integrals.*
(iv) For each $i \in \mathcal{N}_S$ the process

$$S_i^q(t) := \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z}) \tilde{q}_j(d^- s, d\mathbf{z}), \quad 0 \leq t \leq T$$

is an $(\mathbb{H}, \mathbb{P})$-semimartingale.

Proof:
(i) From Theorem 18, the orthogonal decomposition of the $(\mathbb{H}, \mathbb{P})$-martingales $\mathbf{M}$ into continuous $(\mathbb{H}, \mathbb{P})$-local martingales $\mathbf{M}^c := (M_1^c, \ldots, M_{N_S}^c)$ and discontinuous $(\mathbb{H}, \mathbb{P})$-local martingales $\mathbf{M}^d := (M_1^d, \ldots, M_{N_S}^d)$, is given by

$$\mathbf{M}(t) = \mathbf{M}^c(t) + \mathbf{M}^d(t), \quad 0 \leq t \leq T. \tag{3.133}$$

In (3.133), for all $0 \leq t \leq T, i \in \mathcal{N}_S$, we have that

$$M_i^c(t) = \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) dB_j(s) + \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) \eta_{ij}^B(s) ds \quad \text{and} \tag{3.134}$$

$$M_i^d(t) = \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z}) \tilde{q}_j(d^- s, d\mathbf{z}) + \sum_{j=1}^{N_q} \int_0^t \eta_{ij}^q(s) ds, \tag{3.135}$$

where for all $0 \leq t \leq T, i \in \mathcal{N}_S, j \in \mathcal{N}_B, k \in \mathcal{N}_q$ the $\mathbb{H}$-adapted processes η_{ij}^B and η_{ik}^q must satisfy almost surely

$$\begin{aligned}
\sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) \eta_{ij}^B(s) ds + \sum_{j=1}^{N_q} \int_0^t \eta_{ij}^q(s) ds - \int_0^t \left(\xi_i(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s) \pi_j(s) \right) ds \\
= -\sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \left(\vartheta_{ij}(s, \mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(s, \mathbf{z}) \pi_k(s) \right) \nu_j^{\mathbb{F}}(d\mathbf{z}) ds \\
+ \int_0^t \left(-\sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) - \lambda_0(s) + \sum_{j=1}^{M} \lambda_j(s) h_{ij}(s) \right) ds.
\end{aligned}$$

Now by Theorem 4 the $(\mathbb{H},\mathbb{P})$-local martingales $\mathbf{M}^c$ are in fact $(\mathbb{H},\mathbb{P})$-martingales since from Theorem 5 for each $t \in [0,T]$ the expected value of the quadratic variation of the processes $\mathbf{M}^c$ is finite. Thus for each $i \in \mathcal{N}_S$ the process

$$S_i^B(t) := \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) dB_j(s) = M_i^c(t) - \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s)\eta_{ij}^B(s) ds, \quad 0 \le t \le T$$

is an $(\mathbb{H},\mathbb{P})$-semimartingale because it is the sum of an $(\mathbb{H},\mathbb{P})$-martingale M_i^c and a process of bounded variation.
(ii) and (iii) See the proof of Theorem 21(iii)-(iv).
(iv) From (3.133), for each $i \in \mathcal{N}_S$ the process

$$\sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s,\mathbf{z}) \tilde{q}_j(d^-s, d\mathbf{z}) = M_i(t) - M_i^c(t) - \sum_{j=1}^{N_q} \int_0^t \eta_{ij}^q(s) ds.$$

Thus the process $\sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s,\mathbf{z}) \tilde{q}_j(d^-s, d\mathbf{z})$ is an $(\mathbb{H},\mathbb{P})$-semimartingale because it is the sum of an $(\mathbb{H},\mathbb{P})$-martingale $M_i - M_i^c$ and a process of bounded variation. ∎

We now prove the following theorem in which we derive an optimality equation for an optimal portfolio $\boldsymbol{\pi}$ for problem **(P2)**. Recall from (3.13) that the $\mathbb{F}$ compensators of the Poisson random measures $\mathbf{q}$ are $\boldsymbol{\nu}^{\mathbb{F}}$. In Theorem 25 below we consider the $\mathbb{H}$ compensators of $\mathbf{q}$.

Theorem 25 *Let $\boldsymbol{\pi} \in \mathcal{P}_{L2}$ be an optimal portfolio for problem* **(P2)**. *For all $j \in \mathcal{N}_q$ let $\nu_j^{\mathbb{H}} = \nu_j^{\mathbb{H}}(dt, d\mathbf{z})$ be the $\mathbb{H}$ compensator of the Poisson random measure q_j and let $\boldsymbol{\nu}^{\mathbb{H}} := (\nu_1^{\mathbb{H}}, \ldots, \nu_{N_q}^{\mathbb{H}})$. Then for all $0 \le t \le T, i \in \mathcal{N}_S$, π_i satisfies almost surely the equation*

$$\begin{aligned} \int_0^t \left(\xi_i(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) \right) ds - \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \left(\vartheta_{ij}(s,\mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(s,\mathbf{z})\pi_k(s) \right) \nu_j^{\mathbb{F}}(d\mathbf{z}) ds \\ = \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s)\eta_{ij}^B(s) ds + \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s,\mathbf{z})(\nu_j^{\mathbb{F}} - \nu_j^{\mathbb{H}})(ds, d\mathbf{z}) \\ - \int_0^t \left(-\sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) - \lambda_0(s) + \sum_{j=1}^{M} \lambda_j(s) h_{ij}(s) \right) ds. \end{aligned} \tag{3.136}$$

Proof: With $\boldsymbol{\nu}^{\mathbb{H}}$ defined in the proof of Theorem 22(i), from (3.134)-(3.135) we can rewrite (3.133) in index form as

$$\begin{aligned} M_i(t) &= \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) dB_j(s) + \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) \eta_{ij}^B(s) ds \\ &+ \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z})(q_j - \nu_j^{\mathbb{H}})(ds, d\mathbf{z}) \\ &+ \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z})(\nu_j^{\mathbb{H}} - \nu_j^{\mathbb{F}})(ds, d\mathbf{z}) + \sum_{j=1}^{N_q} \int_0^t \eta_{ij}^q(s) ds, \end{aligned} \tag{3.137}$$

where it is important to note that there is no longer forward Poisson integration in (3.137) (due to the presence of the $\mathbb{H}$ compensators $\boldsymbol{\nu}^{\mathbb{H}}$). Equating (3.126) and (3.137) equation (3.136) is obtained. ■

We have the following corollary of Theorem 25.

Corollary 6 *Suppose the insider is in fact honest. Then an optimal portfolio* $\boldsymbol{\pi}$ *for problem* **(P2)** *must satisfy almost surely the equation*

$$\begin{aligned} 0 &= \xi_i(t) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t)\pi_j(t) - \sum_{j=1}^{N_B} \sigma_{ij}(t)\eta_{ij}^B(t) - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(t))) \\ &+ \lambda_0(t) + \sum_{j=1}^{M} \lambda_j(t) h_{ij}(t) - \sum_{j=1}^{N_q} \int_{\mathbb{R}^N} \vartheta_{ij}(t, \mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(t, \mathbf{z}) \pi_k(t) \nu_j^{\mathbb{F}}(d\mathbf{z}). \end{aligned}$$

Proof: Setting $\boldsymbol{\nu}^{\mathbb{F}} = \boldsymbol{\nu}^{\mathbb{H}}$ in (3.136) we get that

$$\begin{aligned} 0 &= \int_0^t \Bigg(\xi_i(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) - \sum_{j=1}^{N_B} \sigma_{ij}(s)\eta_{ij}^B(s) - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) \\ &+ \lambda_0(s) + \sum_{j=1}^{M} \lambda_j(s) h_{ij}(s) - \sum_{j=1}^{N_q} \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(s, \mathbf{z}) \pi_k(s) \nu_j^{\mathbb{F}}(d\mathbf{z}) \Bigg) ds. \end{aligned} \tag{3.138}$$

Then use the fundamental theorem of calculus as in the proof of Corollary 5. ■

Now we want to eliminate the multipliers $\lambda_j, j \in \mathcal{N}_M \cup \{0\}$ from (3.136) so that an optimal portfolio $\boldsymbol{\pi}$ for problem **(P2)** is dependent only on observable stochastic processes. To do this, we have to consider specific forms of the penalty functions $\mathbb{L}$. This is done in Chapter 4, Section 4.2.2. In the next section a constrained portfolio selection problem is solved assuming the insider has logarithmic utility and invests in a financial market driven by Lévy processes with jumps. It is also assumed that penalty functions and explicit weight constraints are present in the portfolio selection problem and that a money market security is available for investment.

3.8.4 Logarithmic utility, weight constraints, penalty functions and investment in a money market security

In this section problem **(P3)** is solved assuming the insider has logarithmic utility and invests in a financial market driven by Lévy processes with jumps. It is also assumed that explicit portfolio weight constraints and penalty functions are present in the portfolio selection problem and that a money market security is available for investment. In this case the wealth process has the form (3.32) with $\xi_0 \equiv r, \boldsymbol{\sigma}^0 \equiv \mathbf{0}$ and $\mathbf{g}^0 \equiv \mathbf{0}$, in other words

$$
\begin{aligned}
W(T) \quad = \quad & W(t) \exp\left(\int_t^T \left(r(s) + \sum_{i=1}^{N_S} (\xi_i(s) - r(s))\pi_i(s) - \frac{1}{2} \sum_{j=1}^{N_B} \left(\sum_{i=1}^{N_S} \sigma_{ij}(s)\pi_i(s) \right)^2 \right) ds \right. \\
& + \sum_{j=1}^{N_B} \sum_{i=1}^{N_S} \int_t^T \sigma_{ij}(s)\pi_i(s) d^- B_j(s) + \sum_{j=1}^{N_q} \int_t^T \int_{A_j} \ln\left(1 + g_{ij}(s,\mathbf{z})\pi_i(s)\right) \tilde{q}_j(d^- s, d\mathbf{z}) \\
& \left. + \sum_{j=1}^{N_q} \int_t^T \int_{A_j} \left[\ln\left(1 + g_{ij}(s,\mathbf{z})\pi_i(s)\right) - g_{ij}(s,\mathbf{z})\pi_i(s) \right] \nu_j^{\mathbb{F}}(d\mathbf{z}) ds \right).
\end{aligned}
$$

The main result in this section is (3.140) which is an optimality equation for a constrained optimal portfolio $\boldsymbol{\pi}$ for problem **(P3)**. We now define the set of admissible portfolios $\mathcal{P}_{L3}$ for problem **(P3)**.

Definition 19 (Admissible portfolios) *A set of control processes* $\boldsymbol{\pi}$ *where* $\boldsymbol{\pi}(t) \in \mathbb{R}^{N_S}$ *for all* $0 \le t \le T$*, is said to be* **admissible** *(or an* **admissible portfolio***) for problem* **(P3)** *if the following hold:*

(i) $\boldsymbol{\pi}$ *satisfies Definition 11.*

(ii) Definition 18(ii) with $\lambda_0 \equiv 0$ *and for all* $0 \le t \le T, i \in \mathcal{N}_S$

$$
\begin{aligned}
M_i(t) \quad := \quad & \int_0^t \left(\xi_i(s) - r(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) \right) ds \\
& + \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) dB_j(s) + \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s,\mathbf{z}) \tilde{q}_j(d^- s, d\mathbf{z}) \\
& - \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \left(\vartheta_{ij}(s,\mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(s,\mathbf{z})\pi_k(s) \right) \nu_j^{\mathbb{F}}(d\mathbf{z}) ds \\
& + \int_0^t \left(- \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) + \sum_{j=1}^{M} \lambda_j(s) h_{ij}(s) \right) ds.
\end{aligned}
\tag{3.139}
$$

(iii) Definition 18(iii).

We denote by $\mathcal{P}_{L3}$ the set of all admissible portfolios for problem **(P3)***.* ♦

In what follows, Theorems 26-28 are generalisations of ([42], Theorems 14-16). We now prove the following theorem in which we show that the processes $\mathbf{M} := (M_1, \ldots, M_{N_S})$ defined in (3.139) are $(\mathbb{H}, \mathbb{P})$-martingales.

Theorem 26 *Assume $\boldsymbol{\pi} \in \mathcal{P}_{L3}$ is an optimal portfolio for problem* **(P3)***. Then each M_i defined in (3.139) is an $(\mathbb{H}, \mathbb{P})$-martingale.*

Proof: See the proof of Theorem 23 with $\lambda_0 \equiv 0$ and $\boldsymbol{\xi}$ replaced with $\boldsymbol{\xi} - \mathbf{r}$. ■

We now prove the following theorem in which we show amongst other things that the $(\mathbb{F}, \mathbb{P})$-Brownian motions $\mathbf{B}$ defined in (3.15) are $(\mathbb{H}, \mathbb{P})$-semimartingales.

Theorem 27 *Let $\boldsymbol{\pi} \in \mathcal{P}_{L3}$ be an optimal portfolio for problem* **(P3)***. Then we have the following. (i) For each $i \in \mathcal{N}_S$ the process*

$$S_i^B(t) := \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) dB_j(s), \quad 0 \leq t \leq T$$

is an $(\mathbb{H}, \mathbb{P})$-semimartingale.
(ii) The Brownian motions $\mathbf{B}$ defined in (3.15) are $(\mathbb{H}, \mathbb{P})$-semimartingales.
(iii) Forward stochastic integrals with respect to $\mathbf{B}$ are Itô integrals.
(iv) For each $i \in \mathcal{N}_S$ the process

$$S_i^q(t) := \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z}) \tilde{q}_j(d^- s, d\mathbf{z}), \quad 0 \leq t \leq T$$

is an $(\mathbb{H}, \mathbb{P})$-semimartingale.

Proof: See the proof of Theorem 24 with $\lambda_0 \equiv 0$ and $\boldsymbol{\xi}$ replaced with $\boldsymbol{\xi} - \mathbf{r}$. ■

We now prove the following theorem in which we derive an optimality equation for an optimal portfolio $\boldsymbol{\pi}$ for problem **(P3)**.

Theorem 28 *Let $\boldsymbol{\pi} \in \mathcal{P}_{L3}$ be an optimal portfolio for problem* **(P3)***. For each $i \in \mathcal{N}_S$, π_i satisfies almost surely the equation*

$$\begin{aligned} 0 \quad = \quad & \int_0^t \Bigg(\xi_i(s) - r(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) - \sum_{j=1}^{N_B} \sigma_{ij}(s)\eta_{ij}^B(s) - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) \\ & + \sum_{j=1}^{M} \lambda_j(s) h_{ij}(s) - \sum_{j=1}^{N_q} \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(s, \mathbf{z})\pi_k(s)\nu_j^{\mathbb{F}}(d\mathbf{z}) \Bigg) ds \\ & - \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z})(\nu_j^{\mathbb{F}} - \nu_j^{\mathbb{H}})(ds, d\mathbf{z}). \end{aligned} \tag{3.140}$$

Proof: See the proof of Theorem 25 with $\lambda_0 \equiv 0$ and $\boldsymbol{\xi}$ replaced with $\boldsymbol{\xi} - \mathbf{r}$. ∎

We have the following corollary of Theorem 28.

Corollary 7 *Suppose the insider is in fact honest. Then an optimal portfolio* $\boldsymbol{\pi}$ *for problem* **(P3)** *must satisfy almost surely the equation*

$$\begin{aligned} 0 &= \xi_i(t) - r(t) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t)\pi_j(t) - \sum_{j=1}^{N_B} \sigma_{ij}(t)\eta_{ij}^B(t) - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(t))) \\ &\quad + \sum_{j=1}^{M} \lambda_j(t) h_{ij}(t) - \sum_{j=1}^{N_q} \int_{\mathbb{R}^N} \vartheta_{ij}(t,\mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(t,\mathbf{z})\pi_k(t)\nu_j^{\mathbb{F}}(d\mathbf{z}). \end{aligned}$$

Proof: See the proof of Corollary 6. ∎

Now we want to eliminate the multipliers $\lambda_j, j \in \mathcal{N}_M$ from (3.140) so that an optimal portfolio $\boldsymbol{\pi}$ for problem **(P3)** is dependent only on observable stochastic processes. To do this, we have to consider specific forms of the penalty functions $\mathbb{L}$. This is done in Chapter 4, Section 4.2.3.

Chapter 4

Constrained portfolio selection with non-Markov processes and Insiders (II)

In this chapter the theoretical results from Chapter 3 are used to find constrained optimal portfolios $\boldsymbol{\pi}$ which are dependent only on observable stochastic processes and not on the multipliers $\lambda_j, j \in \mathcal{N}_M \cup \{0\}$ defined in (3.50). Thus in this chapter we are mainly interested in Sections 3.7.2, 3.7.3, 3.8.3 and 3.8.4 where explicit portfolio weight constraints are present and the insider is assumed to have a logarithmic utility function. (Recall that in the general utility cases, viz Sections 3.7.1 and 3.8.2, the portfolio selection problem **(P1)** is solved in the *absence* of penalty functions and *absence* of portfolio weight constraints. See Remark 3 in Section 3.7.2 for an explanation of why.) In this chapter we also consider specific forms of the penalty functions $\mathbb{L}$ and analytical and numerical examples are provided. Before we give an outline of the rest of this chapter we state the following.

Recall the form (3.44)-(3.45) of the integral constraints in problem **(P2)** viz

$$\mathbb{E}\left[\int_0^T k_0^1(t)\left|\Upsilon(t)-\sum_{i=1}^{N_S}\pi_i(t)\right|dt\right] = 0 \tag{4.1}$$

$$\mathbb{E}\left[\int_0^T k_j^1(t)\left|\sum_{i=1}^{N_S}h_{ij}(t)\pi_i(t)-\bar{h}_j(t)\right|dt\right] = 0, \quad j=1,\dots,M\in\mathbb{N}. \tag{4.2}$$

The most important specifications in this chapter are that specific types of integral portfolio weight constraints (4.1)-(4.2) are considered, viz that of the form (3.48) for example

$$\mathbb{E}\left[\int_0^T k_3^1(t)\left|\pi_6(t)-b_6(t)\right|dt\right] = 0 \tag{4.3}$$

where the function $b_6 = b_6(t, \omega)$ could be an *a priori* defined upper bound of π_6 and the function $k_3^1 = k_3^1(t, \omega)$ could have the value 1 only over the time intervals $[0, 1]$ and $[3, 5]$ (with $T = 10$ say) and zero otherwise. When required, these are used to set active the algebraic inequality constraints

$$\mathbf{a}(t) \ \leq \ \boldsymbol{\pi}(t) \ \leq \ \mathbf{b}(t) \quad \text{for all } 0 \leq t \leq T. \tag{4.4}$$

In other words some portfolio weights $\pi_i, i \in \mathcal{N}_S$ are constrained to particular values at some time over the time interval $[0, T]$. This allows us to ensure that an insider constrained optimal portfolio satisfies the inequality constraints (4.4). To set active any one or group of the constraints (4.4) the use of $2N_S$ different integral constraints of the form (4.3) is required (since sometimes the upper and sometimes lower bound constraint of each portfolio weight π_i must be set active). Thus in (4.2) we have that $M = 2N_S$ and the functions h_{ij} and $\bar{h}_j, i, j \in \mathcal{N}_M$ must have the form

$$h_{ij}(t) \quad \equiv \quad \begin{cases} 1 & \text{if } i = j \text{ and } j \leq N_S \\ -1 & \text{if } i = j \text{ and } j > N_S \\ 0 & \text{otherwise} \end{cases}$$

and

$$\bar{h}_j(t) \quad \equiv \quad \begin{cases} b_j(t) & \text{if } j \leq N_S \\ -a_{j-N_S}(t) & \text{if } j > N_S \end{cases}$$

for all $0 \leq t \leq T$. Thus the integral constraints (4.1)-(4.2) reduce to

$$\mathbb{E}\left[\int_0^T k_0^1(t) \left| \Upsilon(t) - \sum_{i=1}^{N_S} \pi_i(t) \right| dt\right] \quad = \quad 0 \tag{4.5}$$

$$\mathbb{E}\left[\int_0^T k_i^1(t) \left|\pi_i(t) - b_i(t)\right| dt\right] \quad = \quad 0 \quad \text{for } i \in \mathcal{N}_S \tag{4.6}$$

$$\mathbb{E}\left[\int_0^T \bar{k}_j^1(t) \left|-\pi_j(t) + a_j(t)\right| dt\right] \quad = \quad 0 \quad \text{for } j \in \mathcal{N}_S. \tag{4.7}$$

Multiplying (4.5)-(4.7) by the Lagrange multipliers $k_0^2, k_i^2, k_j^2, i, j \in \mathcal{N}_S$ respectively defined in (3.49), equations (4.5)-(4.7) reduce to

$$\mathbb{E}\left[\int_0^T \lambda(t) \left(\Upsilon(t) - \sum_{i=1}^{N_S} \pi_i(t) \right) dt\right] \quad = \quad 0 \tag{4.8}$$

$$\mathbb{E}\left[\int_0^T \mu_i(t) \left[\pi_i(t) - b_i(t)\right] dt\right] \quad = \quad 0 \quad \text{for } i \in \mathcal{N}_S \tag{4.9}$$

$$\mathbb{E}\left[\int_0^T \bar{\mu}_j(t) \left[-\pi_j(t) + a_j(t)\right] dt\right] \quad = \quad 0 \quad \text{for } j \in \mathcal{N}_S, \tag{4.10}$$

where from (3.50) for all $0 \leq t \leq T, i, j \in \mathcal{N}_S$

$$
\begin{array}{rclcl}
\lambda(t) & := & \lambda_0(t) & = & \left\{ \begin{array}{ll} k_0^2 k_0^1(t) & \text{if } \Upsilon(t) - \sum_{i=1}^{N_S} \pi_i(t) \geq 0 \\ -k_0^2 k_0^1(t) & \text{otherwise,} \end{array} \right. \\
\mu_i(t) & := & \lambda_i(t) & = & \left\{ \begin{array}{ll} k_i^2 k_i^1(t) & \text{if } \pi_i(t) - b_i(t) \geq 0 \\ -k_i^2 k_i^1(t) & \text{otherwise,} \end{array} \right. \\
 & & \text{and} & & \\
\bar{\mu}_j(t) & := & \lambda_{N_S+j}(t) & = & \left\{ \begin{array}{ll} k_{N_S+j}^2 k_{N_S+j}^1(t) & \text{if } -\pi_j(t) + a_j(t) \geq 0 \\ -k_{N_S+j}^2 k_{N_S+j}^1(t) & \text{otherwise.} \end{array} \right.
\end{array}
\tag{4.11}
$$

Since the integrands in (4.8)-(4.10) are nonnegative almost surely these imply that for all $0 \leq t \leq T$

$$
\lambda(t) \left(\Upsilon(t) - \sum_{i=1}^{N_S} \pi_i(t) \right) = 0 \tag{4.12}
$$

$$
\mu_i(t) \left[\pi_i(t) - b_i(t) \right] = 0 \qquad \text{for all } i \in \mathcal{N}_S \tag{4.13}
$$

$$
\bar{\mu}_j(t) \left[-\pi_j(t) + a_j(t) \right] = 0 \qquad \text{for all } j \in \mathcal{N}_S. \tag{4.14}
$$

Note that the equations (4.12)-(4.14) will not always be active since for some $t \in [0, T]$ the multipliers λ and $\boldsymbol{\mu} := (\mu_1, \ldots, \mu_{N_S})$ and $\bar{\boldsymbol{\mu}} := (\bar{\mu}_1, \ldots, \bar{\mu}_{N_S})$ will be zero. Using equations (4.12)-(4.14) and the optimality equations derived in Chapter 3, we show below that we have a system of equations involving a constrained optimal portfolio $\boldsymbol{\pi}$ and the multipliers $\lambda, \boldsymbol{\mu}, \bar{\boldsymbol{\mu}}$. The rest of this chapter is organised as follows. It is split into four Sections 4.1-4.4.

- In Section 4.1 it is assumed that the securities **S** are driven by diffusions and in this case the optimality equations (3.82) and (3.88) derived in Sections 3.7.2 and 3.7.3 respectively are made independent of the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$. In Section 4.1.1 it is assumed that the securities **S** are driven by diffusions and that a money money market security is not available for investment. Specific forms of the penalty functions $\mathbb{L}$ are considered and consequently the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ are eliminated from the optimality equation (3.82). In Section 4.1.2 it is assumed that the securities **S** are driven by diffusions and that a money money market security is available for investment. Specific forms of the penalty functions $\mathbb{L}$ are considered and consequently the multipliers $\boldsymbol{\mu}, \bar{\boldsymbol{\mu}}$ are eliminated from the optimality equation (3.88).

- In Section 4.2 it is assumed that the securities **S** are driven by Lévy processes with jumps and in this case the optimality equations (3.136) and (3.140) derived in Sections 3.8.3 and 3.8.4 respectively are made independent of the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$. In Section 4.2.2 it is assumed that the securities **S** are driven by Lévy processes with jumps and that a money market security is not available for investment. Specific forms of the penalty functions $\mathbb{L}$ are considered and consequently the multipliers

$\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ are eliminated from the optimality equation (3.136). In Section 4.2.3 it is assumed that the securities $\mathbf{S}$ are driven by Lévy processes with jumps and that a money market security is available for investment. Specific forms of the penalty functions $\mathbb{L}$ are considered and consequently the multipliers $\boldsymbol{\mu}, \bar{\boldsymbol{\mu}}$ are eliminated from the optimality equation (3.140).

- In Section 4.3 a procedure for calculating constrained optimal portfolios is provided.
- In Section 4.4 several examples are provided.

4.1 Market driven by Diffusions

This section is an extension of Section 3.7 where it is assumed that the securities $\mathbf{S}$ are driven by diffusions. Particular forms of the penalty functions $\mathbb{L}$ are considered and the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ are eliminated from the optimality equations (3.82) and (3.88). In Section 4.1.1 it is assumed that a money market security is not available for investment, whereas in Section 4.1.2 it is assumed that it is.

4.1.1 Logarithmic utility, weight constraints, penalty functions and no investment in a money market security

This section is an extension of Section 3.7.2. Here specific forms of the penalty functions $\mathbb{L}$ are considered so that the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ can be eliminated from the optimality equation (3.82) which from (4.11) reduces to

$$\bar{\boldsymbol{\sigma}}(t)\boldsymbol{\pi}(t) + \mathbb{L}_{\boldsymbol{\pi}}(t) = \boldsymbol{\xi}(t) + \boldsymbol{\sigma}(t)\frac{d}{dt}\mathbf{H}(t) - \boldsymbol{\lambda}(t) + \boldsymbol{\mu}(t) - \bar{\boldsymbol{\mu}}(t). \tag{4.15}$$

First non-differential penalty functions $\mathbb{L}$ are considered and the multipliers are eliminated from (4.15). Second differential penalty functions are considered and the multipliers are eliminated from (4.15).

Specific forms of $\mathbb{L}$ - (I) Non-differential

The most important result in this subsection is equation (4.26) which is the optimality equation (4.15) with the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ eliminated. The operator $\mathbb{L}$ is assumed to be diagonal with

$$\mathbb{L}_{ij}(\pi_i(t)) = \begin{cases} \kappa_i(t)\pi_i(t) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$

for all $0 \leq t \leq T, i, j \in \mathcal{N}_S$, where each function $\kappa_i, i \in \mathcal{N}_S$ is deterministic and for notational simplicity we define $\kappa_i = \kappa_{ii}$. Recall that form (4.16) implies that the insider is being penalised for large investments in security S_i and penalisation is different for each security. Substituting (4.16) into (4.15) we get that

$$\boldsymbol{\pi}(t) = \boldsymbol{\sigma}^{\kappa}(t)\left(\boldsymbol{\xi}(t) + \boldsymbol{\sigma}(t)\frac{d}{dt}\mathbf{H}(t) - \boldsymbol{\lambda}(t) + \boldsymbol{\mu}(t) - \bar{\boldsymbol{\mu}}(t)\right), \tag{4.17}$$

where for all $0 \le t \le T$ the matrix $\boldsymbol{\sigma}^{\kappa}(t) := [\bar{\boldsymbol{\sigma}}(t) + (diag(\boldsymbol{\kappa}(t)))^2]^{-1}$ assuming it is invertible, $\boldsymbol{\kappa} := (\kappa_1, \ldots, \kappa_{N_S})$ is now a vector and $\boldsymbol{\lambda}(t)$ is a vector with all its N_S elements equal to $\lambda(t)$. Substituting (4.17) into the unity weight constraint (4.12) we get that

$$\lambda(t) = \sum_{p=1}^{N_S}\sum_{m=1}^{N_S}\frac{\sigma^{\kappa}_{pm}(t)}{\Gamma(t)}\left(\xi_m(t) + \sum_{n=1}^{N_B}\sigma_{mn}(t)\frac{d}{dt}H_n(t) + \mu_m(t) - \bar{\mu}_m(t)\right) - \frac{\Upsilon(t)}{\Gamma(t)}, \tag{4.18}$$

where for all $0 \le t \le T$ the function $\Gamma(t) := \sum_{i=1}^{N_S}\sum_{k=1}^{N_S}\sigma^{\kappa}_{ik}(t)$. Substituting (4.18) into (4.17) we get that for all $0 \le t \le T, i \in \mathcal{N}_S$

$$\begin{aligned}\pi_i(t) \quad = \quad & \Upsilon(t)(\Gamma(t))^{-1}\sum_{k=1}^{N_S}\sigma^{\kappa}_{ik}(t) + \sum_{k=1}^{N_S}\sigma^{\kappa}_{ik}(t)\left(\xi_k(t) - (\Gamma(t))^{-1}\sum_{l=1}^{N_S}\sum_{m=1}^{N_S}\sigma^{\kappa}_{lm}(t)\xi_m(t)\right.\\ & + \sum_{j=1}^{N_B}\sigma_{kj}(t)\frac{d}{dt}H_j(t) - (\Gamma(t))^{-1}\sum_{l=1}^{N_S}\sum_{m=1}^{N_S}\sigma^{\kappa}_{lm}(t)\sum_{n=1}^{N_B}\sigma_{mn}(t)\frac{d}{dt}H_n(t)\\ & \left. + \mu_k(t) - \bar{\mu}_k(t) - (\Gamma(t))^{-1}\sum_{l=1}^{N_S}\sum_{m=1}^{N_S}\sigma^{\kappa}_{lm}(t)(\mu_m(t) - \bar{\mu}_m(t))\right).\end{aligned} \tag{4.19}$$

As in Section 2.3.3, for any portfolio weight $\pi_i, i \in \mathcal{N}_S$, regardless of whether the upper bound or lower bound constraint (4.4) or neither is active, the expression $\mu_i(t) - \bar{\mu}_i(t)$ will always reduce to exactly one of the following, viz for all $0 \le t \le T$

$$\mu^*_i(t) \; := \; \begin{cases} \mu_i(t) & \text{if } \pi_i(t) \le b_i(t) \text{ is active (and } -\pi_i(t) \le -a_i(t) \text{ inactive)},\\ -\bar{\mu}_i(t) & \text{if } -\pi_i(t) \le -a_i(t) \text{ is active (and } \pi_i(t) \le b_i(t) \text{ inactive)},\\ 0 & \text{otherwise.}\end{cases} \tag{4.20}$$

So from (4.20) we can rewrite (4.19) as

$$\pi_i(t) = \bar{\Gamma}_i(t) + \bar{C}_i(t) + \sum_{k\in\mathcal{C}^*(t)}\sigma^{\kappa}_{ik}(t)\left(\mu^*_k(t) - \sum_{n\in\mathcal{C}^*(t)}\bar{\Gamma}_n(t)\mu^*_n(t)\right), \tag{4.21}$$

where for all $0 \le t \le T$

$$\begin{aligned}
\bar{\Gamma}_i(t) &:= \Upsilon(t)(\Gamma(t))^{-1} \sum_{k=1}^{N_S} \sigma^{\kappa}_{ik}(t) \quad \text{and} \\
\bar{C}_i(t) &:= \sum_{k=1}^{N_S} \sigma^{\kappa}_{ik}(t) \Bigg(\xi_k(t) - (\Gamma(t))^{-1} \sum_{l=1}^{N_S} \sum_{m=1}^{N_S} \sigma^{\kappa}_{lm}(t) \xi_m(t) \\
&\quad + \sum_{j=1}^{N_B} \sigma_{kj}(t) \frac{d}{dt} H_j(t) - (\Gamma(t))^{-1} \sum_{l=1}^{N_S} \sum_{n=1}^{N_S} \sigma^{\kappa}_{ln}(t) \sum_{p=1}^{N_B} \sigma_{np}(t) \frac{d}{dt} H_p(t) \Bigg).
\end{aligned}$$

In (4.21) the set $\mathcal{C}(t)(\bar{\mathcal{C}}(t)) \subseteq \mathcal{N}_S$ is the index set of time-t active upper (lower) bound constraints and $\mathcal{C}(t) \cup \bar{\mathcal{C}}(t) =: \mathcal{C}^*(t) := \{\alpha_1, \alpha_2, \ldots, \alpha_{m(t)}\}$, where $m(t)$ is the number of active inequality constraints at time t. Each number $\alpha_j \in \mathcal{C}^*(t)$ denotes that at time t an inequality constraint of security S_{α_j} is active. So if we have an opportunity set of 5 securities and only the upper (lower) bound constraint of $\pi_1(t)$ and the lower (upper) bound constraint of $\pi_4(t)$ are active, then $\mathcal{C}^*(t) = \{1, 4\}$ in both cases. For $t \in [0, T]$ fixed, in (4.4), as $\mathbf{a}(t)$ and $\mathbf{b}(t)$ tend to $-\infty$ and $+\infty$ respectively, the number of elements in $\mathcal{C}^*(t)$ decreases since fewer control variables $\boldsymbol{\pi}(t)$ will hit the boundaries $\mathbf{a}(t)$ and $\mathbf{b}(t)$. Thus as $\mathbf{a}(t)$ and $\mathbf{b}(t)$ tend to $-\infty$ and $+\infty$, optimal solutions of the constrained optimisation problem **(P2)** tend toward optimal solutions of the unconstrained optimisation problem (3.43) (subject only to (3.44)). Since $\mathbf{a}(t) < \mathbf{b}(t)$ the upper and lower weight constraints of no security can be active at the same time.

Continuing, for all $0 \le t \le T$ we can rewrite the two inequality constraints $\pi_i(t) \le b_i(t)$ and $-\pi_i(t) \le -a_i(t)$ in a more compact form as

$$(-1)^{d_i(t)} \pi_i(t) \le c_i(t), \qquad i = 1, \ldots, N_S, \tag{4.22}$$

where for all $0 \le t \le T, i \in \mathcal{N}_S$

$$\left\{ \begin{array}{l} c_i(t) = b_i(t) \\ d_i(t) = 0 \end{array} \right\} \qquad \text{for the constraint } \pi_i(t) \le b_i(t) \quad \text{and}$$

$$\left\{ \begin{array}{l} c_i(t) = -a_i(t) \\ d_i(t) = 1 \end{array} \right\} \qquad \text{for the constraint } -\pi_i(t) \le -a_i(t).$$

Thus (4.13)-(4.14) we can be rewritten as

$$\mu^*_i(t)[(-1)^{d_i(t)} \pi_i(t) - c_i(t)] = 0, \tag{4.23}$$

where for all $0 \le t \le T, i \in \mathcal{N}_S$ the multiplier $\mu^*_i(t) = 0$ if (4.22) is not active and nonzero otherwise. Substituting (4.21) into (4.23) we find that for each

$\alpha_i \in \mathcal{C}^*(t)$

$$
\begin{aligned}
(-1)^{-d_{\alpha_i}(t)} c_{\alpha_i}(t) &= \bar{\Gamma}_{\alpha_i}(t) + \bar{C}_{\alpha_i}(t) + \mu^*_{\alpha_1}(t) \left(\sigma^\kappa_{\alpha_i,\alpha_1}(t) - \bar{\Gamma}_{\alpha_1}(t) \sum_{\alpha_k \in \mathcal{C}^*(t)} \sigma^\kappa_{\alpha_i,\alpha_k}(t) \right) \\
&\quad + \ldots + \mu^*_{\alpha_j}(t) \left(\sigma^\kappa_{\alpha_i,\alpha_j}(t) - \bar{\Gamma}_{\alpha_j}(t) \sum_{\alpha_k \in \mathcal{C}^*(t)} \sigma^\kappa_{\alpha_i,\alpha_k}(t) \right) + \ldots \\
&\quad + \mu^*_{\alpha_m}(t) \left(\sigma^\kappa_{\alpha_i,\alpha_m}(t) - \bar{\Gamma}_{\alpha_m}(t) \sum_{\alpha_k \in \mathcal{C}^*(t)} \sigma^\kappa_{\alpha_i,\alpha_k}(t) \right).
\end{aligned} \tag{4.24}
$$

With $\bar{c}_{\alpha_i}(t) := (-1)^{-d_{\alpha_i}(t)} c_{\alpha_i}(t)$ for all $0 \leq t \leq T, \alpha_i \in \mathcal{C}^*(t)$ we can rewrite (4.24) in matrix form as

$$
\boldsymbol{\mu}^*(t) = \boldsymbol{\Psi}^{-1}(t) \left(\bar{\mathbf{c}}(t) - \bar{\boldsymbol{\Gamma}}(t) - \bar{\mathbf{C}}(t) \right), \tag{4.25}
$$

where all vectors in (4.25) are of length $m(t)$, $\bar{\mathbf{C}} := (\bar{C}_{\alpha_1}, \ldots, \bar{C}_{\alpha_{m(t)}})$, $\boldsymbol{\Psi}(t) \equiv [\Psi_{ij}(t)]$ is an $m(t) \times m(t)$ matrix with $\Psi_{ij}(t) := \sigma^\kappa_{\alpha_i,\alpha_j}(t) - \bar{\Gamma}_{\alpha_j}(t) \sum_{\alpha_k \in \mathcal{C}^*(t)} \sigma^\kappa_{\alpha_i,\alpha_k}(t)$ and the invertibility of $\boldsymbol{\Psi}(t)$ can be verified before projecting the model from the current time to the next. Let the matrix $\boldsymbol{\Psi}^{-1}(t) \equiv [\varsigma_{jk}(t)]$. Then substituting (4.25) into (4.21) we find that

$$
\begin{aligned}
\pi_i(t) &= \bar{\Gamma}_i(t) + \bar{C}_i(t) + \sum_{k \in \mathcal{C}^*(t)} \sigma^\kappa_{ik}(t) \left(\sum_{l \in \mathcal{C}^*(t)} \varsigma_{kl}(t) \left(\bar{c}_l(t) - \bar{\Gamma}_l(t) - \bar{C}_l(t) \right) \right. \\
&\quad \left. - \sum_{n \in \mathcal{C}^*(t)} \bar{\Gamma}_n(t) \sum_{l \in \mathcal{C}^*(t)} \varsigma_{nl}(t) \left(\bar{c}_l(t) - \bar{\Gamma}_l(t) - \bar{C}_l(t) \right) \right).
\end{aligned} \tag{4.26}
$$

This shows that if the penalty functions $\mathbb{L}$ are of the form (4.16), then an optimal portfolio $\boldsymbol{\pi}(t)$ for problem **(P2)** is the unconstrained portfolio $\bar{\Gamma}_i(t) + \bar{C}_i(t), i \in \mathcal{N}_S$ plus the terms present if $\boldsymbol{\pi}(t)$ are constrained. Let

$$
\bar{N} := 1 + \sum_{k=2}^{M} \left[\binom{2M}{k} - M \binom{2M}{k-2} \right] \tag{4.27}
$$

where M is the number of integral constraints (4.2) and $\binom{k}{0} := 1, k \in \mathbb{N}$. Then with $M = 2N_S$ in (4.27), to find a constrained optimal portfolio for problem **(P2)**, we must consider at most $\bar{N}$ combinations of active inequality constraints $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$. For each combination of active inequality constraints we need to calculate the objective functional value (3.43). A constrained optimal portfolio $\boldsymbol{\pi}(t)$ is that which satisfies the inequality constraints (4.4) and has the largest objective functional value (3.43). A worked example is provided in Section 4.4. The formula (4.27) was determined as follows:

- the unity weight constraint is always active,
- the upper and lower bound constraint on some security weight cannot be active at the same time,
- if a money market security is (not) available for investment, then at most N_S ($N_S - 1$) inequality constraints can be active at the same time.

From (4.27), in the case of the inequality constraints (4.4), if $N_S = 2, 3, 4, \ldots$, then at each time one has to calculate at least one and at most $5, 19, 65, \ldots$ objective functional values to find a time-t constrained optimal portfolio. We now consider differential penalty functions $\mathbb{L}$ and eliminate the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ from (4.15).

Specific forms of $\mathbb{L}$ - (II) Differential

The main result in this subsection is equation (4.36) which is the discrete form of the optimality equation (4.15) with the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ eliminated. The operator $\mathbb{L}$ is assumed to be differential and diagonal with

$$\mathbb{L}_{ij}(\pi_i(t)) = \begin{cases} \bar{\kappa}_i(t)\frac{d}{dt}\pi_i(t) + \kappa_i(t)\pi_i(t) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{4.28}$$

where the functions $\boldsymbol{\kappa}(t) := (\kappa_1(t), \ldots, \kappa_{N_S}(t))$ and $\bar{\boldsymbol{\kappa}}(t) := (\bar{\kappa}_1(t), \ldots, \bar{\kappa}_{N_S}(t))$, $\bar{\kappa}_i(t) \neq 0$ for all $0 \leq t \leq T, i \in \mathcal{N}_S$ are deterministic and for notational simplicity we define $\kappa_i = \kappa_{ii}$ and $\bar{\kappa}_i = \bar{\kappa}_{ii}$. Form (4.28) implies that the insider is being penalised for large investments as well as large investment fluctuations in security S_i. Also penalisation is different for each S_i. From [143] for each $i \in \mathcal{N}_S$ we have that the adjoint operator $\mathbb{L}^{\dagger}_{ii} = -\bar{\kappa}_i \frac{d}{dt} + (-\bar{\kappa}'_i + \kappa_i)$. We now constrain the admissible portfolios $\mathcal{P}_{B2}$ (Definition 13) to include only those portfolios $\boldsymbol{\pi}$ which are also at least twice continuously differentiable and satisfy

$$\boldsymbol{\pi}(0) = \boldsymbol{\pi}_{BM} \text{ and } \boldsymbol{\pi}'(0) = 0, \tag{4.29}$$

where $\boldsymbol{\pi}_{BM}$ are some predefined weights such that $\mathbf{a}(0) \leq \boldsymbol{\pi}_{BM} \leq \mathbf{b}(0)$. (We use the subscript BM to refer to some benchmark portfolio.) Substituting (4.28) into (4.15) we get that for all $0 \leq t \leq T$

$$-\mathbf{D}(t) = -\bar{\boldsymbol{\sigma}}(t)\boldsymbol{\pi}(t) + \mathbf{U}(t)\boldsymbol{\pi}''(t) + \mathbf{V}(t)\boldsymbol{\pi}'(t) + \mathbf{Y}(t)\boldsymbol{\pi}(t), \tag{4.30}$$

where

$$\begin{aligned} \mathbf{D}(t) &:= \boldsymbol{\xi}(t) + \boldsymbol{\sigma}(t)\frac{d}{dt}\mathbf{H}(t) - \boldsymbol{\lambda}(t) + \boldsymbol{\mu}(t) - \bar{\boldsymbol{\mu}}(t), \\ \mathbf{U}(t) &:= (diag(\bar{\boldsymbol{\kappa}}(t)))^2, \\ \mathbf{V}(t) &:= 2 \times diag(\bar{\boldsymbol{\kappa}}(t))diag(\bar{\boldsymbol{\kappa}}'(t)) \quad \text{and} \\ \mathbf{Y}(t) &:= diag(\bar{\boldsymbol{\kappa}}(t))diag(\boldsymbol{\kappa}'(t)) + diag(\bar{\boldsymbol{\kappa}}'(t))diag(\boldsymbol{\kappa}(t)) - (diag(\boldsymbol{\kappa}(t)))^2. \end{aligned} \tag{4.31}$$

We convert (4.30) to a system of first order ordinary differential equations by defining

$$\bar{\boldsymbol{\pi}}(t) = \boldsymbol{\pi}'(t) = \frac{d}{dt}\boldsymbol{\pi}(t), \quad 0 \le t \le T. \tag{4.32}$$

Since $\bar{\kappa}_i(t) \neq 0$ for all $0 \le t \le T, i \in \mathcal{N}_S$ the matrix $\mathbf{U}(t)$ is invertible for all $0 \le t \le T$ so (4.30) can be written as

$$\bar{\boldsymbol{\pi}}'(t) = -\mathbf{U}^{-1}(t)\mathbf{D}(t) + \mathbf{U}^{-1}(t)(\bar{\boldsymbol{\sigma}}(t) - \mathbf{Y}(t))\boldsymbol{\pi}(t) - \mathbf{U}^{-1}(t)\mathbf{V}(t)\bar{\boldsymbol{\pi}}(t). \tag{4.33}$$

Combining (the first equation in) (4.32) and (4.33) we get that

$$\hat{\boldsymbol{\pi}}'(t) = \mathbf{f}(t, \hat{\boldsymbol{\pi}}(t)), \tag{4.34}$$

where for all $0 \le t \le T$

$$\begin{aligned}
\hat{\boldsymbol{\pi}}(t) &:= \begin{pmatrix} \bar{\boldsymbol{\pi}}(t) \\ \boldsymbol{\pi}(t) \end{pmatrix}, \\
\mathbf{K}_1(t) &:= -\mathbf{U}^{-1}(t)\mathbf{V}(t), \\
\mathbf{K}_2(t) &:= \mathbf{U}^{-1}(t)(\bar{\boldsymbol{\sigma}}(t) - \mathbf{Y}(t)) \quad \text{and} \\
\mathbf{f}(t, \hat{\boldsymbol{\pi}}(t)) &:= \begin{pmatrix} & \mathbf{K}_1(t) & & \mathbf{K}_2(t) & & \\ 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \hat{\boldsymbol{\pi}}(t) - \begin{pmatrix} \mathbf{U}^{-1}(t)\mathbf{D}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\end{aligned} \tag{4.35}$$

We solve (4.34) discretely via a first order scheme. (See Section 4.3 and Appendix G for an explanation of why we do this.) Partition the interval $[0, T]$ into n intervals of equal length Δt with $0 = t_0 < t_1 < \ldots < t_n = T$. For each $j \in \{0, 1, \ldots, n\}$ let

$$\hat{\boldsymbol{\pi}}_j := \hat{\boldsymbol{\pi}}(t_j), \quad \lambda_j := \lambda(t_j), \quad \boldsymbol{\mu}_j^* := \boldsymbol{\mu}^*(t_j), \quad \boldsymbol{\kappa}_j := \boldsymbol{\kappa}(t_j), \quad \bar{\boldsymbol{\kappa}}_j := \bar{\boldsymbol{\kappa}}(t_j), \quad \mathbf{f}_j := \mathbf{f}(t_j, \hat{\boldsymbol{\pi}}(t_j)).$$

The values $\hat{\boldsymbol{\pi}}_0$ are determined by the initial conditions (4.29). From (4.29), $\boldsymbol{\mu}_0^* = \mathbf{0}$ since by construction the initial portfolio $\boldsymbol{\pi}_{BM}$ satisfies the constraints $\mathbf{a}(0) \le \boldsymbol{\pi}_{BM} \le \mathbf{b}(0)$. From (4.28) and (4.29)

$$\mathbb{L}(0, \boldsymbol{\pi}_0) = diag(\bar{\boldsymbol{\kappa}}_0)\boldsymbol{\pi}_0' + diag(\boldsymbol{\kappa}_0)\boldsymbol{\pi}_0 = diag(\boldsymbol{\kappa}_0)\boldsymbol{\pi}_{BM},$$

thus

$$\mathbb{L}^\dagger(0, \mathbb{L}(0, \boldsymbol{\pi}_0)) = (diag(\boldsymbol{\kappa}_0))^2\boldsymbol{\pi}_{BM}.$$

Thus with $t = t_0$ in (4.30) the value λ_0 can be calculated.

Assume the values $\hat{\boldsymbol{\pi}}_j$ and $\mathbf{f}_j$ have been calculated. We then calculate the values $\hat{\boldsymbol{\pi}}_{j+1}$ via the first order scheme

$$\hat{\boldsymbol{\pi}}_{j+1} = \hat{\boldsymbol{\pi}}_j + \Delta t \mathbf{f}_j. \tag{4.36}$$

To calculate the values $\hat{\boldsymbol{\pi}}_{j+1}$, we have $3N_S+1$ equations (4.12), (4.23) and (4.36), to solve for the $3N_S+1$ variables $\hat{\boldsymbol{\pi}}_{j+1}, \lambda_{j+1}$ and $\boldsymbol{\mu}^*_{j+1}$. So at each time t_j we can calculate constrained portfolios depending on which subset of inequality constraints $\mathbf{a}(t_j) \leq \boldsymbol{\pi}(t_j) \leq \mathbf{b}(t_j)$ has been set active. To find a constrained optimal portfolio, we must consider at most $\bar{N}$ combinations of active inequality constraints $\mathbf{a}(t_j) \leq \boldsymbol{\pi}(t_j) \leq \mathbf{b}(t_j)$ (where $\bar{N}$ is defined in (4.27)). For each combination of active inequality constraints we need to calculate the objective functional value (3.43). A constrained optimal portfolio $\boldsymbol{\pi}(t_j)$ is that which satisfies the inequality constraints $\mathbf{a}(t_j) \leq \boldsymbol{\pi}(t_j) \leq \mathbf{b}(t_j)$ and has the largest objective functional value (3.43). A worked example is provided in Section 4.4.7.

4.1.2 Logarithmic utility, weight constraints, penalty functions and investment in a money market security

This section is an extension of Section 3.7.3. Here specific forms of the penalty functions $\mathbb{L}$ are considered so that the multipliers $\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ can be eliminated from the optimality equation (3.88) which from (4.11) reduces to

$$\bar{\boldsymbol{\sigma}}(t)\boldsymbol{\pi}(t) + \mathbb{L}_{\boldsymbol{\pi}}(t) = \boldsymbol{\xi}(t) - \mathbf{r}(t) + \boldsymbol{\sigma}(t)\frac{d}{dt}\mathbf{H}(t) + \boldsymbol{\mu}(t) - \bar{\boldsymbol{\mu}}(t). \tag{4.37}$$

First non-differential penalty functions are considered and the multipliers $\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ are eliminated from (4.37). Second differential penalty functions are considered and the multipliers are eliminated from (4.37).

Specific forms of $\mathbb{L}$ - (I) Non-differential

The most important result in this subsection is (4.41) which is the optimality equation (4.37) with the multipliers $\boldsymbol{\mu}, \bar{\boldsymbol{\mu}}$ eliminated. The operator $\mathbb{L}$ is assumed to be of the form (4.16) in other words

$$\mathbb{L}_{ij}(\pi_i(t)) = \begin{cases} \kappa_i(t)\pi_i(t) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{4.38}$$

for all $0 \leq t \leq T, i, j \in \mathcal{N}_S$ where each function $\kappa_i, i \in \mathcal{N}_S$ is deterministic and for notational simplicity we define $\kappa_i = \kappa_{ii}$. Substituting (4.38) into (4.37) we have from (4.20) that

$$\pi_i(t) \quad = \quad \hat{\Gamma}_i(t) + \sum_{j \in \mathcal{C}^*(t)}^{N_S} \sigma^{\kappa}_{ij}(t)\mu^*_j(t), \tag{4.39}$$

where for all $0 \leq t \leq T, i \in \mathcal{N}_S$

$$\begin{aligned} \boldsymbol{\sigma}^{\kappa}(t) &:= [\bar{\boldsymbol{\sigma}}(t) + (diag(\boldsymbol{\kappa}(t)))^2]^{-1} \quad \text{(assuming it is invertible) and} \\ \hat{\Gamma}_i(t) &:= \sum_{j=1}^{N_S} \sigma^{\kappa}_{ij}(t)\left(\xi_j(t) - r(t) + \sum_{k=1}^{N_B} \sigma_{jk}(t)\frac{d}{dt}H_k(t)\right). \end{aligned}$$

In (4.39) the set $\mathcal{C}(t)(\bar{\mathcal{C}}(t)) \subseteq \mathcal{N}_S$ is the index set of time-t active upper (lower) bound constraints and $\mathcal{C}(t) \cup \bar{\mathcal{C}}(t) =: \mathcal{C}^*(t) := \{\alpha_1, \alpha_2, \ldots, \alpha_{m(t)}\}$, where $m(t)$ is the number of active inequality constraints. Each number $\alpha_j \in \mathcal{C}^*(t)$ denotes that an inequality constraint of security S_{α_j} is active. Substituting (4.39) into (4.23) we find that for all $0 \le t \le T$

$$\boldsymbol{\mu}^*(t) \quad = \quad \boldsymbol{\Psi}^{-1}(t)\left(\bar{\mathbf{c}}(t) - \hat{\boldsymbol{\Gamma}}(t)\right), \tag{4.40}$$

where each $\alpha_i \in \mathcal{C}^*(t)$, $\bar{c}_{\alpha_i}(t) := (-1)^{-d_{\alpha_i}(t)} c_{\alpha_i}(t)$, all vectors in (4.40) are of length $m(t)$, $\boldsymbol{\Psi}(t)$ is an $m(t) \times m(t)$ matrix with $\Psi_{ij}(t) := \sigma^{\kappa}_{\alpha_i,\alpha_j}(t)$, $\boldsymbol{\Psi}^{-1}(t) \equiv [\varsigma_{jk}(t)]$ and the invertibility of $\boldsymbol{\Psi}(t)$ can be verified before projecting the model from the current time to the next. Substituting (4.40) into (4.39) we find that for all $0 \le t \le T, i \in \mathcal{N}_S$

$$\pi_i(t) \quad = \quad \hat{\Gamma}_i(t) + \sum_{j \in \mathcal{C}^*(t)} \sigma^{\kappa}_{ij}(t) \sum_{k \in \mathcal{C}^*(t)} \varsigma_{jk}(t)\left(\bar{c}_k(t) - \hat{\Gamma}_k(t)\right), \tag{4.41}$$

which shows that if the penalty functions $\mathbb{L}$ are of the form (4.38), then an optimal portfolio $\boldsymbol{\pi}(t)$ for problem **(P3)** is the unconstrained portfolio $\hat{\Gamma}_i(t), i \in \mathcal{N}_S$ plus the terms present if $\boldsymbol{\pi}(t)$ are constrained. To find a constrained optimal portfolio of **(P3)** we must consider at most $\bar{N}$ combinations of active inequality constraints $\mathbf{a}(t) \le \boldsymbol{\pi}(t) \le \mathbf{b}(t)$. For each combination of active inequality constraints we need to calculate the objective functional value (3.52). A constrained optimal portfolio is that which satisfies the constraints $\mathbf{a}(t) \le \boldsymbol{\pi}(t) \le \mathbf{b}(t)$ and has the largest objective functional value (3.52). Note that (4.26) and (4.41) are not special cases of each other. This emphasizes why we consider separately the cases of a money market security being unavailable and available for investment (by the insider) when finding constrained optimal portfolios.

Specific forms of $\mathbb{L}$ - (II) Differential

The most important result in this subsection is equation (4.56) which is the discrete form of the optimality equation (4.37) with the multipliers $\boldsymbol{\mu}, \bar{\boldsymbol{\mu}}$ eliminated. The operator $\mathbb{L}$ is assumed to be diagonal with

$$\mathbb{L}_{ij}(\pi_i(t)) = \begin{cases} \bar{\kappa}_i(t)\frac{d}{dt}\pi_i(t) + \kappa_i(t)\pi_i(t) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{4.42}$$

where the functions $\boldsymbol{\kappa} := (\kappa_1, \ldots, \kappa_{N_S})$ and $\bar{\boldsymbol{\kappa}} := (\bar{\kappa}_1, \ldots, \bar{\kappa}_{N_S})$ are deterministic and for notational simplicity we define $\kappa_i = \kappa_{ii}$ and $\bar{\kappa}_i = \bar{\kappa}_{ii}$. Form (4.42) implies that the insider is being penalised for large investments as well as large investment fluctuations in security S_i. Also penalisation is different for each security. The important difference between (4.28) and (4.42) is that some functions $\bar{\boldsymbol{\kappa}}$ are now allowed to be zero at some time $t \in [0, T]$. This possibility was disallowed in (4.28) because the invertibility of the matrices $\mathbf{U}(t), 0 \le t \le T$ in (4.33) was required. (If any $\bar{\kappa}_i(t) = 0$ for some $t \in [0, T], i \in \mathcal{N}_S$, then $\mathbf{U}(t)$ will not be invertible.) To continue however we assume that at each time $t \in [0, T]$

at least one function $\bar{\kappa}_i, i \in \mathcal{N}_S$ has a nonzero value. Substituting (4.42) into (4.37) we get from (4.20) that for all $0 \leq t \leq T, i \in \mathcal{N}_S$

$$\begin{aligned} 0 &= D_i(t) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t)\pi_j(t) + \bar{\kappa}_i^2(t)\pi_i''(t) + 2\bar{\kappa}_i(t)\bar{\kappa}_i'(t)\pi_i'(t) \\ &\quad +(\bar{\kappa}_i(t)\kappa_i'(t) + \bar{\kappa}_i'(t)\kappa_i(t) - \kappa_i^2(t))\pi_i(t), \end{aligned} \tag{4.43}$$

where

$$D_i(t) := \xi_i(t) - r(t) + \sum_{j=1}^{N_B} \sigma_{ij}(t)\frac{d}{dt}H_j(t) + \mu_i^*(t). \tag{4.44}$$

For all $0 \leq t \leq T$ let the set $L(t) := \{L_1, \ldots, L_{d(t)}\} \subseteq \mathcal{N}_S$, where $d(t) \in \mathcal{N}_S$ is its cardinality. Let $L(t)$ denote for which securities S_i the corresponding penalty function $\mathbb{L}_{ii}(t)$ has a nonzero coefficient value $\bar{\kappa}_i(t)$. For all $0 \leq t \leq T$ let $\boldsymbol{\pi}_L(t) := (\pi_{L_1}(t), \ldots, \pi_{L_{d(t)}}(t))$. Then from (4.43) we have that

$$-\mathbf{D}_L(t) = -\bar{\boldsymbol{\sigma}}_L(t)\boldsymbol{\pi}_L(t) + \mathbf{U}_L(t)\boldsymbol{\pi}_L''(t) + \mathbf{V}_L(t)\boldsymbol{\pi}_L'(t) + \mathbf{Y}_L(t)\boldsymbol{\pi}_L(t), \tag{4.45}$$

where $\mathbf{r}_L(t)$ is a vector of cardinality $d(t)$ with all elements equal to $r(t)$ and

$$\begin{aligned} \boldsymbol{\xi}_L(t) &:= (\xi_{L_1}(t), \ldots, \xi_{L_{d(t)}}(t)), \\ \boldsymbol{\sigma}_L(t) &:= [\sigma_{jk}(t)], j \in L(t), k \in \mathcal{N}_B, \\ \boldsymbol{\mu}_L^*(t) &:= (\mu_{L_1}^*(t), \ldots, \mu_{L_{d(t)}}^*(t)), \\ \bar{\boldsymbol{\sigma}}_L(t) &:= [\bar{\sigma}_{L_j,L_n}(t)], j, n \in \{1, \ldots, d(t)\}, \\ \mathbf{D}_L(t) &:= \boldsymbol{\xi}_L(t) - \mathbf{r}_L(t) + \boldsymbol{\sigma}_L(t)\frac{d}{dt}\mathbf{H}(t) + \boldsymbol{\mu}_L^*(t), \\ \boldsymbol{\kappa}_L(t) &:= (\kappa_{L_1}(t), \ldots, \kappa_{L_{d(t)}}(t)), \\ \bar{\boldsymbol{\kappa}}_L(t) &:= (\bar{\kappa}_{L_1}(t), \ldots, \bar{\kappa}_{L_{d(t)}}(t)), \\ \mathbf{U}_L(t) &:= (diag(\bar{\boldsymbol{\kappa}}_L(t)))^2, \\ \mathbf{V}_L(t) &:= 2 \times diag(\bar{\boldsymbol{\kappa}}_L(t))diag(\bar{\boldsymbol{\kappa}}_L'(t)) \quad \text{and} \\ \mathbf{Y}_L(t) &:= diag(\bar{\boldsymbol{\kappa}}_L(t))diag(\boldsymbol{\kappa}_L'(t)) - diag(\bar{\boldsymbol{\kappa}}_L'(t))diag(\boldsymbol{\kappa}_L(t)) + (diag(\boldsymbol{\kappa}_L(t)))^2. \end{aligned} \tag{4.46}$$

We now constrain the admissible portfolios $\mathcal{P}_{B3}$ (Definition 14) to include only those portfolios $\boldsymbol{\pi}$ also with the subset $\boldsymbol{\pi}_L$ of portfolio security weights at least twice continuously differentiable and which satisfy

$$\boldsymbol{\pi}_L(0) = \boldsymbol{\pi}_{L,BM} \quad \text{and} \quad \boldsymbol{\pi}_L'(0) = \mathbf{0}. \tag{4.47}$$

In (4.47) the values $\boldsymbol{\pi}_{L,BM}$ are predefined weights such that $\mathbf{a}_L(0) \leq \boldsymbol{\pi}_{L,BM} \leq \mathbf{b}_L(0)$, where for all $0 \leq t \leq T$

$$\mathbf{a}_L(t) := (a_{L_1}(t), \ldots, a_{L_{d(t)}}(t)) \quad \text{and} \tag{4.48}$$

$$\mathbf{b}_L(t) := (b_{L_1}(t), \ldots, b_{L_{d(t)}}(t)). \tag{4.49}$$

As in (4.29) we use the subscript BM to refer to some benchmark portfolio. We convert (4.45) to a system of first order ordinary differential equations by defining

$$\bar{\boldsymbol{\pi}}_L(t) = \boldsymbol{\pi}'_L(t) = \frac{d}{dt}\boldsymbol{\pi}_L(t), \quad 0 \le t \le T. \tag{4.50}$$

Then since $\bar{\boldsymbol{\kappa}}_L(t) \neq \mathbf{0}, 0 \le t \le T$, the $d(t) \times d(t)$ matrix $\mathbf{U}_L(t)$ is nonsingular and so (4.45) can be written as

$$\bar{\boldsymbol{\pi}}'_L(t) = -\mathbf{U}_L^{-1}(t)\mathbf{D}_L(t) + \mathbf{U}_L^{-1}(t)(\bar{\boldsymbol{\sigma}}_L(t) - \mathbf{Y}_L(t))\boldsymbol{\pi}_L(t) - \mathbf{U}_L^{-1}(t)\mathbf{V}_L(t)\bar{\boldsymbol{\pi}}_L(t). \tag{4.51}$$

Combining (the first equation in) (4.50) and (4.51) we get that for all $0 \le t \le T$

$$\hat{\boldsymbol{\pi}}'_L(t) = \mathbf{f}_L(t, \hat{\boldsymbol{\pi}}_L(t)), \tag{4.52}$$

where

$$\begin{aligned}
\hat{\boldsymbol{\pi}}_L(t) &:= \begin{pmatrix} \bar{\boldsymbol{\pi}}_L(t) \\ \boldsymbol{\pi}_L(t) \end{pmatrix}, \\
\mathbf{K}_{L,1}(t) &:= -\mathbf{U}_L^{-1}(t)\mathbf{V}_L(t), \\
\mathbf{K}_{L,2}(t) &:= \mathbf{U}_L^{-1}(t)(\bar{\boldsymbol{\sigma}}_L(t) - \mathbf{Y}_L(t)) \quad \text{and} \\
\mathbf{f}_L(t, \hat{\boldsymbol{\pi}}_L(t)) &:= \begin{pmatrix} & \mathbf{K}_{L,1}(t) & & & \mathbf{K}_{L,2}(t) & \\ 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \hat{\boldsymbol{\pi}}_L(t) - \begin{pmatrix} \mathbf{U}_L^{-1}(t)\mathbf{D}_L(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\end{aligned} \tag{4.53}$$

As with (4.34) we solve (4.52) discretely via a first order scheme. So partition the interval $[0, T]$ into n intervals of equal length Δt with $0 = t_0 < t_1 < \ldots < t_n = T$. Let $\hat{\pi}_{L,i}(t_j), j \in \{0, 1, \ldots, n\}$ denote the value of the ith element of the vector $\hat{\boldsymbol{\pi}}_L(t_j)$. Firstly, the values $\pi_i(t_0), i \in L(t_0)$ are obtained from the first set of initial conditions in (4.47). We solve for $\pi_i(t_0), i \in \mathcal{N}_S \backslash L(t_0)$. Recall that, from the initial conditions (4.47), $\boldsymbol{\mu}_L^*(t_0) = \mathbf{0}$ since by construction the initial portfolio $\boldsymbol{\pi}_{L,BM}$ satisfies the constraints $\mathbf{a}_L(0) \le \boldsymbol{\pi}_{L,BM} \le \mathbf{b}_L(0)$. From the second (differential) set of initial conditions in (4.47), with $t = t_0$ in (4.43), it becomes for all $i \in \mathcal{N}_S \backslash L(t_0)$

$$0 = \xi_i(t_0) - r(t_0) + \sum_{j=1}^{N_B} \sigma_{ij}(t_0) \frac{d}{dt} H_j(t_0) + \mu_i^*(t_0) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t_0)\pi_j(t_0) - \kappa_i^2(t_0)\pi_i(t_0). \tag{4.54}$$

Thus in (4.23) and (4.54) we have $2(N_S - d(t_0))$ equations for $2(N_S - d(t_0))$ unknowns $\mu_i^*(t_0), \pi_i(t_0), i \in \mathcal{N}_S \backslash L(t_0)$.

Assume the values $\boldsymbol{\pi}(t_j)$ have been computed. Then we calculate the values $\pi_i(t_{j+1}), i \in \mathcal{N}_S \backslash L(t_{j+1})$ via the equation

$$\begin{aligned} 0 &= \xi_i(t_{j+1}) - r(t_{j+1}) + \sum_{j=1}^{N_B} \sigma_{ij}(t_{j+1}) \frac{d}{dt} H_j(t_{j+1}) + \mu_i^*(t_{j+1}) \\ &\quad - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t_{j+1}) \pi_j(t_{j+1}) - \kappa_i^2(t_{j+1}) \pi_i(t_{j+1}) \end{aligned} \tag{4.55}$$

and we calculate the values $\hat{\boldsymbol{\pi}}_L(t_{j+1})$ via the first order scheme

$$\hat{\boldsymbol{\pi}}_L(t_{j+1}) = \hat{\boldsymbol{\pi}}_L(t_j) + \Delta t \mathbf{f}_L(t_j, \hat{\boldsymbol{\pi}}_L(t_j)). \tag{4.56}$$

In (4.23), (4.55) and (4.56) we have $2N_S + d(t_{j+1})$ equations for $2N_S + d(t_{j+1})$ unknowns $\boldsymbol{\mu}^*(t_{j+1}), \bar{\boldsymbol{\pi}}_L(t_{j+1})$ and $\boldsymbol{\pi}(t_{j+1})$. So at each time t_j we can calculate constrained portfolios depending on which subset of inequality constraints $\mathbf{a}(t_j) \leq \boldsymbol{\pi}(t_j) \leq \mathbf{b}(t_j)$ has been set active. To find a constrained optimal portfolio, we must consider at most $\bar{N}$ combinations of active inequality constraints $\mathbf{a}(t_j) \leq \boldsymbol{\pi}(t_j) \leq \mathbf{b}(t_j)$. For each combination of active inequality constraints we need to calculate the objective functional value (3.52). A constrained optimal portfolio $\boldsymbol{\pi}(t_j)$ is that which satisfies the constraints $\mathbf{a}(t_j) \leq \boldsymbol{\pi}(t_j) \leq \mathbf{b}(t_j)$ and has the largest objective functional value (3.52).

4.2 Market driven by Lévy Processes

This section is an extension of Section 3.8 where it is assumed that the securities $\mathbf{S}$ are driven by Lévy processes with jumps. Constrained optimal portfolios are found assuming particular forms of the penalty functions $\mathbb{L}$ and assuming particular types of jumps for the securities $\mathbf{S}$. Consequently the multipliers $\lambda_j, j \in \mathcal{N}_M \cup \{0\}$ are eliminated from the optimality equations (3.136) and (3.140) which from (4.11) reduce respectively to

$$\begin{aligned} &\int_0^t \left(\xi_i(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s) \pi_j(s) \right) ds - \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \left(\vartheta_{ij}(s, \mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(s, \mathbf{z}) \pi_k(s) \right) \nu_j^{\mathbb{F}}(d\mathbf{z}) ds \\ &= \sum_{j=1}^{N_B} \int_0^t \sigma_{ij}(s) \eta_{ij}^B(s) ds + \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z}) (\nu_j^{\mathbb{F}} - \nu_j^{\mathbb{H}})(ds, d\mathbf{z}) \\ &\quad - \int_0^t \left(- \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) - \lambda(s) + \mu_i(s) - \bar{\mu}_i(s) \right) ds. \end{aligned} \tag{4.57}$$

and

$$
\begin{aligned}
0 \quad = \quad & \int_0^t \left(\xi_i(s) - r(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) - \sum_{j=1}^{N_B} \sigma_{ij}(s)\eta_{ij}^B(s) - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) \right. \\
& \left. + \mu_i(s) - \bar{\mu}_i(s) - \sum_{j=1}^{N_q} \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(s, \mathbf{z})\pi_k(s)\nu_j^{\mathbb{F}}(d\mathbf{z}) \right) ds \\
& - \sum_{j=1}^{N_q} \int_0^t \int_{\mathbb{R}^N} \vartheta_{ij}(s, \mathbf{z})(\nu_j^{\mathbb{F}} - \nu_j^{\mathbb{H}})(ds, d\mathbf{z}).
\end{aligned} \tag{4.58}
$$

In Section 4.2.1 specific types of Lévy process are considered. In Section 4.2.2 it is assumed that a money market security is not available for investment, whereas in Section 4.2.3 it is, and the multipliers $\lambda, \boldsymbol{\mu}, \bar{\boldsymbol{\mu}}$ are eliminated from the optimality equations (4.57) and (4.58) respectively in each section.

4.2.1 Specific types of Lévy process

The difficulty with finding analytical forms of even unconstrained portfolios in the Lévy financial market (3.14)-(3.15) is due to $\boldsymbol{\pi}$ appearing in the denominator in (3.125) viz

$$
\vartheta_{ij}(t, \mathbf{z}) = \frac{g_{ij}(t, \mathbf{z})}{1 + \sum_{k=1}^{N_S} g_{kj}(t, \mathbf{z})\pi_k(t)} \tag{4.59}
$$

for all $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, i \in \mathcal{N}_S, j \in \mathcal{N}_q$. Thus to eliminate the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ from the optimality equations (4.57) and (4.58) we follow [42] and consider specific types of Lévy process (and find constrained optimal insider portfolios in these cases). We consider separately the cases where a money market security is unavailable and available for investment by the insider and these are Sections 4.2.2 and 4.2.3 respectively. Constrained optimal portfolios are derived only in the case where the penalty functions $\mathbb{L}$ are not differential. The analysis is similar (to the differential cases in Sections 4.1.1 and 4.1.2) if some penalty function $\mathbb{L}_{ij}$ is differential. To continue, from [42] the following assumptions are made.

Assumption 1 *To derive analytical forms of constrained optimal portfolios it is assumed that:*

(i) The jump coefficients $\mathbf{g}$ *are independent of time in other words*

$$
\mathbf{g} = \mathbf{g}(\mathbf{z}). \tag{4.60}
$$

(ii) Define the pure jump processes $\boldsymbol{\varphi} := (\varphi_1, \dots, \varphi_{N_q})$ *as*

$$
\varphi_j(t) := \sum_{i=1}^{N_S} \int_0^t \int_{\mathbb{R}^N} g_{ij}(\mathbf{z})\tilde{q}_j(ds, d\mathbf{z}), \quad 0 \le t \le T, j \in \mathcal{N}_q. \tag{4.61}
$$

Then as in [42] we assume that the insider has at most knowledge about the value of the underlying driving processes $\mathbf{B}(T_0)$ *and* $\boldsymbol{\varphi}(T_0)$ *for some time* $T_0 > T$. *This means that the insider filtration* $\mathbb{H}$ *is such that* $\mathbb{F} \subseteq \mathbb{H} \subseteq \mathbb{H}'$, *where*

$$\mathcal{H}'_t := \sigma(\mathcal{F}_t \cup \sigma(\mathbf{B}(T_0), \boldsymbol{\varphi}(T_0))) \quad \text{for all } 0 \le t \le T. \tag{4.62}$$

♦

From [42] we state the following two propositions.

Proposition 4 *Let* $\boldsymbol{\varphi}$ *and the filtration* $\mathbb{H}'$ *be defined as in Assumption 1. Then the processes*

$$\boldsymbol{\varphi}(t) - \int_0^t \frac{\mathbb{E}[\boldsymbol{\varphi}(T_0)|\mathcal{H}_s] - \boldsymbol{\varphi}(s)}{T_0 - s} ds \quad \text{and} \quad \mathbf{B}(t) - \int_0^t \frac{\mathbb{E}[\mathbf{B}(T_0)|\mathcal{H}_s] - \mathbf{B}(s)}{T_0 - s} ds$$

are $(\mathbb{H}, \mathbb{P})$*-martingales.*

Proof: See ([42], Proposition 18). ■

We state the following proposition in which the $\mathbb{H}$ compensators $\boldsymbol{\nu}^{\mathbb{H}}$ of the Poisson random measures $\mathbf{q}$ are deduced.

Proposition 5 *Suppose Assumption 1 holds. Then the* $\mathbb{H}$*-compensating measures* $\boldsymbol{\nu}^{\mathbb{H}}$ *of the Poisson random measures* $\mathbf{q}$ *are given by*

$$\begin{aligned} \boldsymbol{\nu}^{\mathbb{H}}(ds, d\mathbf{z}) &= \boldsymbol{\nu}^{\mathbb{F}}(d\mathbf{z})ds + \mathbb{E}\left[\frac{1}{T_0 - s} \int_s^{T_0} \tilde{\mathbf{q}}(dr, d\mathbf{z}) \middle| \mathcal{H}_s \right] ds \\ &= \mathbb{E}\left[\frac{1}{T_0 - s} \int_s^{T_0} \mathbf{q}(dr, d\mathbf{z}) \middle| \mathcal{H}_s \right] ds. \end{aligned} \tag{4.63}$$

Proof: See ([42], Proposition 19). ■

In the next two sections we simplify the optimality equations (4.57) and (4.58) by making use of Assumption 1 and Propositions 4 and 5.

4.2.2 No investment in a money market security

In this section the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ are eliminated from the optimality equation (4.57) so that a constrained optimal portfolio $\boldsymbol{\pi}$ is dependent only on observable stochastic processes. It is assumed that a money market security is not available for investment. It is also assumed that the jump coefficients $\mathbf{g}$ are independent of the security prices $\mathbf{S}$, in other words for all $i \in \mathcal{N}_S, j \in \mathcal{N}_q$

$$g_{ij} = g_j, \tag{4.64}$$

where $g_j = g_j(\mathbf{z})$. From Assumption 1 and (4.64), for all $0 \le t \le T, \mathbf{z} \in \mathbb{R}^N, i \in \mathcal{N}_S, j \in \mathcal{N}_q$ equation (4.59) reduces to

$$\vartheta_{ij}(t,\mathbf{z}) \ = \ \vartheta_j(\mathbf{z}) \ = \ \frac{g_j(\mathbf{z})}{1+g_j(\mathbf{z})}. \tag{4.65}$$

Thus from Proposition 4 with

$$\eta_{ij}^B(t) = -\frac{\mathbb{E}[B_j(T_0)|\mathcal{H}_t] - B_j(t)}{T_0 - t}, \quad 0 \le t \le T, i \in \mathcal{N}_S, j \in \mathcal{N}_B, \tag{4.66}$$

we have from (4.20), Proposition 5 and (4.65) that the optimality equation (4.57) reduces to

$$\begin{aligned} 0 \ = \ & \int_0^t \Bigg(\xi_i(s) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(s)\pi_j(s) - \sum_{j=1}^{N_B} \sigma_{ij}(s)\eta_{ij}^B(s) - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(s))) - \lambda(s) \\ & + \mu_i^*(s) - \sum_{j=1}^{N_q} \int_{\mathbb{R}^N} \vartheta_j(\mathbf{z}) g_j(\mathbf{z}) \nu_j^{\mathbb{F}}(d\mathbf{z}) + \mathbb{E}\left[\sum_{j=1}^{N_q} \int_s^{T_0} \int_{\mathbb{R}^N} \frac{\vartheta_j(\mathbf{z})}{T_0 - s} \tilde{q}_j(dr, d\mathbf{z}) \Bigg| \mathcal{H}_s \right] \Bigg) ds. \end{aligned} \tag{4.67}$$

Taking the time derivative in (4.67) we have from the fundamental theorem of calculus that for all $0 \le t \le T, i \in \mathcal{N}_S$

$$0 \ = \ \xi_i(t) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t)\pi_j(t) - \sum_{j=1}^{N_B} \sigma_{ij}(t)\eta_{ij}^B(t) - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(t))) - \lambda(t) + \mu_i^*(t) + \rho(t), \tag{4.68}$$

where

$$\rho(t) := -\sum_{j=1}^{N_q} \int_{\mathbb{R}^N} \vartheta_j(\mathbf{z}) g_j(\mathbf{z}) \nu_j^{\mathbb{F}}(d\mathbf{z}) + \mathbb{E}\left[\sum_{j=1}^{N_q} \int_t^{T_0} \int_{\mathbb{R}^N} \frac{\vartheta_j(\mathbf{z})}{T_0 - t} \tilde{q}_j(dr, d\mathbf{z}) \Bigg| \mathcal{H}_t \right]. \tag{4.69}$$

Now we want to eliminate the multipliers $\lambda(t)$ and $\boldsymbol{\mu}^*(t)$ from (4.68) so that a constrained optimal portfolio $\boldsymbol{\pi}$ is dependent only on observable stochastic processes. To do this we have to consider specific forms of the penalty functions $\mathbb{L}$. We assume that the penalty functions $\mathbb{L}$ have the form (4.16), viz $\mathbb{L}$ is diagonal with

$$\mathbb{L}_{ij}(\pi_i(t)) = \begin{cases} \kappa_i(t)\pi_i(t) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{4.70}$$

for all $0 \le t \le T, i, j \in \mathcal{N}_S$ where each function $\kappa_i, i \in \mathcal{N}_S$ is deterministic and for notational simplicity we define $\kappa_i = \kappa_{ii}$. Substituting (4.70) into (4.68) it reduces to

$$\boldsymbol{\pi}(t) \quad = \quad \boldsymbol{\sigma}^{\kappa}(t) \left(\boldsymbol{\xi}(t) + \bar{\boldsymbol{\eta}}^B(t) - \boldsymbol{\lambda}(t) + \boldsymbol{\mu}^*(t) + \boldsymbol{\rho}(t) \right), \tag{4.71}$$

where for all $0 \leq t \leq T, i \in \mathcal{N}_S$ we have that $\boldsymbol{\sigma}^{\kappa}(t) := [\bar{\boldsymbol{\sigma}}(t) + (diag(\boldsymbol{\kappa}(t)))^2]^{-1}$ assuming it is invertible, $\bar{\eta}_i^B(t) := \sum_{j=1}^{N_B} \sigma_{ij}(t)\eta_{ij}^B(t)$, $\bar{\boldsymbol{\eta}}^B(t) \equiv [\bar{\eta}_i^B(t)]$ and $\boldsymbol{\rho}(t)$ is a vector with each of its N_S elements equal to $\rho(t)$ defined in (4.69). To eliminate $\lambda(t)$ we substitute (4.71) into (4.12) to get that

$$\lambda(t) = \sum_{p=1}^{N_S} \sum_{m=1}^{N_S} \frac{\sigma_{pm}^{\kappa}(t)}{\Gamma(t)} \left(\xi_m(t) + \sum_{n=1}^{N_B} \sigma_{mn}(t)\eta_{mn}^B(t) + \mu_m^*(t) + \rho(t) \right) - \frac{\Upsilon(t)}{\Gamma(t)}, \tag{4.72}$$

where $\Gamma(t) := \sum_{i=1}^{N_S} \sum_{k=1}^{N_S} \sigma_{ik}^{\kappa}(t), 0 \leq t \leq T$. Substituting (4.72) into (4.71) we get that for all $0 \leq t \leq T, i \in \mathcal{N}_S$

$$\pi_i(t) = \bar{\Gamma}_i(t) + \bar{C}_i(t) + \sum_{k \in \mathcal{C}^*(t)} \sigma_{ik}^{\kappa}(t) \left(\mu_k^*(t) - \sum_{n \in \mathcal{C}^*(t)} \bar{\Gamma}_n(t)\mu_n^*(t) \right), \tag{4.73}$$

where for all $0 \leq t \leq T, i \in \mathcal{N}_S, j \in \mathcal{N}_B$ the process η_{ij}^B has the form in (4.66),

$$\begin{aligned}
\bar{\Gamma}_i(t) &:= \Upsilon(t)(\Gamma(t))^{-1} \sum_{k=1}^{N_S} \sigma_{ik}^{\kappa}(t) \quad \text{and} \\
\bar{C}_i(t) &:= \sum_{k=1}^{N_S} \sigma_{ik}^{\kappa}(t) \Bigg(\xi_k(t) - (\Gamma(t))^{-1} \sum_{l=1}^{N_S} \sum_{m=1}^{N_S} \sigma_{lm}^{\kappa}(t)\xi_m(t) \\
&\quad + \sum_{j=1}^{N_B} \sigma_{kj}(t)\eta_{kj}^B(t) - (\Gamma(t))^{-1} \sum_{l=1}^{N_S} \sum_{m=1}^{N_S} \sigma_{lm}^{\kappa}(t) \sum_{n=1}^{N_B} \sigma_{mn}(t)\eta_{mn}^B(t) \Bigg).
\end{aligned} \tag{4.74}$$

In (4.73) the set $\mathcal{C}(t)(\bar{\mathcal{C}}(t)) \subseteq \mathcal{N}_S$ is the index set of time-t active upper (lower) bound constraints and $\mathcal{C}(t) \cup \bar{\mathcal{C}}(t) =: \mathcal{C}^*(t) := \{\alpha_1, \alpha_2, \ldots, \alpha_{m(t)}\}$, where $m(t)$ is the number of active inequality constraints at time t. Each number $\alpha_j \in \mathcal{C}^*(t)$ denotes that an inequality constraint of security S_{α_j} is active. As in Section 4.1.1 we can derive (4.23) and so substituting (4.73) into (4.23), solving for $\boldsymbol{\mu}^*(t)$ and substituting $\boldsymbol{\mu}^*(t)$ into (4.73), we get that

$$\begin{aligned}
\pi_i(t) &= \bar{\Gamma}_i(t) + \bar{C}_i(t) + \sum_{k \in \mathcal{C}^*(t)} U_{ik}^{\kappa}(t) \Bigg(\sum_{l \in \mathcal{C}^*(t)} \varsigma_{kl}(t) \left(\bar{c}_l(t) - \tilde{\Gamma}_l(t) - \bar{C}_l(t) \right) \\
&\quad - \sum_{n \in \mathcal{C}^*(t)} \tilde{\Gamma}_n(t) \sum_{l \in \mathcal{C}^*(t)} \varsigma_{nl}(t) \left(\bar{c}_l(t) - \tilde{\Gamma}_l(t) - \bar{C}_l(t) \right) \Bigg),
\end{aligned} \tag{4.75}$$

where $\alpha_i \in \mathcal{C}^*(t)$, $\bar{c}_{\alpha_i}(t) := (-1)^{-d_{\alpha_i}(t)} c_{\alpha_i}(t)$, $\boldsymbol{\Psi}(t)$ is an $m(t) \times m(t)$ matrix with $\Psi_{ij}(t) := \sigma_{\alpha_i,\alpha_j}^{\kappa}(t) - \bar{\Gamma}_{\alpha_j}(t) \sum_{\alpha_k \in \mathcal{C}^*(t)} \sigma_{\alpha_i,\alpha_k}^{\kappa}(t)$, the matrix $\boldsymbol{\Psi}^{-1}(t) \equiv [\varsigma_{jk}(t)]$ and the invertibility of $\boldsymbol{\Psi}(t)$ can be verified before projecting the model from the current time to the next. Equation (4.75) shows that if the penalty functions $\mathbb{L}$ are of the form (4.70), then a constrained optimal portfolio $\boldsymbol{\pi}(t)$ for problem **(P2)** is the unconstrained portfolio $\bar{\Gamma}_i(t)+\bar{C}_i(t), i \in \mathcal{N}_S$ plus the terms present if $\boldsymbol{\pi}(t)$ are constrained. To find a constrained optimal portfolio of **(P2)**, we must consider at most $\bar{N}$ (defined in (4.27)) combinations of active inequality constraints $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$. For each combination of active inequality constraints, we need to calculate the objective functional value (3.43). A time-t constrained optimal portfolio is that which satisfies the constraints $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$ and has the largest objective functional value (3.43).

4.2.3 Investment in a money market security

In this section we eliminate the multipliers $\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$ from the optimality equation (4.58), so that constrained optimal insider portfolios for problem **(P3)** are dependent only on observable stochastic processes. As in the derivation of (4.68), if we make Assumption 1, then (4.58) reduces to

$$0 = \xi_i(t) - r(t) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t)\pi_j(t) - \sum_{j=1}^{N_B} \sigma_{ij}(t)\eta_{ij}^B(t) - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(t))) + \mu_i^*(t) + \rho_i(t), \tag{4.76}$$

where for all $0 \leq t \leq T, i \in \mathcal{N}_S, j \in \mathcal{N}_q$ the process η_{ij}^B has the form in (4.66),

$$\begin{aligned} \rho_i(t) \quad := \quad & -\sum_{j=1}^{N_q} \int_{\mathbb{R}^N} \vartheta_{ij}(t,\mathbf{z}) \sum_{k=1}^{N_S} g_{kj}(\mathbf{z})\pi_k(t)\nu_j^{\mathbb{F}}(d\mathbf{z}) \\ & + \mathbb{E}\left[\sum_{j=1}^{N_q} \int_t^{T_0} \int_{\mathbb{R}^N} \frac{\vartheta_{ij}(t,\mathbf{z})}{T_0 - t} \tilde{q}_j(dr, d\mathbf{z}) \,\middle|\, \mathcal{H}_t \right] \end{aligned} \tag{4.77}$$

and

$$\vartheta_{ij}(t,\mathbf{z}) \quad = \quad \frac{g_{ij}(\mathbf{z})}{1 + \sum_{k=1}^{N_S} g_{kj}(\mathbf{z})\pi_k(t)}. \tag{4.78}$$

Now even if we assume (as in Section 4.2.2 equation (4.64)) that the jump coefficients $\mathbf{g}$ are independent of the security prices $\mathbf{S}$, the portfolio weight values $\boldsymbol{\pi}$ cannot be eliminated from the integrand in (4.77). (Recall that here a money market security is available for investment, thus we have for all $0 \leq t \leq T$ that $\sum_{i=1}^{N_S} \pi_i(t) = 1 - \pi_0(t)$ which is not independent of $\boldsymbol{\pi}$.) Thus as in ([42], equation (72)), if a money market security is available for investment, then for each $t \in [0, T]$, there is potentially more than one optimal portfolio $\boldsymbol{\pi}(t)$ calculable from (4.76). Unlike in [42] however, here many of these solutions will likely be eliminated by the inequality constraints $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$. Equation (4.76) must be solved in conjunction with (4.23) to derive constrained optimal portfolios. See Section 4.4 for worked examples. Next we provide a procedure for calculating constrained optimal portfolios.

4.3 Procedure for calculating constrained optimal insider portfolios

In this section a procedure for calculating constrained optimal (insider) portfolios is provided using the models in Section 3.7.2 (and Sections 3.7.3, 3.8.3 and 3.8.4). (Recall that in the case of general utility only unconstrained portfolios are derived.) For notational simplicity it is assumed that the securities $\mathbf{S}$ are driven by diffusions and that the penalty functions $\mathbb{L}$ are not differential operators, but the procedure is analogous even if $\mathbf{S}$ are driven by Lévy processes with jumps and/or some penalty function $\mathbb{L}_{ij}, i, j \in \mathcal{N}_S$ is differential. For the rest of this section, when we refer to a result in Section 3.7.2, we shall on the first occasion include in brackets the analogous results in Sections 3.7.3, 3.8.3 and 3.8.4 since the procedure is the same. For the rest of this chapter, if the time variable is (not) included, then it must be understood that the variable $\boldsymbol{\pi}(t)$ ($\boldsymbol{\pi}$) denotes the values of the security weights at time t (over the entire horizon $[0, T]$). In both cases though $\boldsymbol{\pi}(t)$ and $\boldsymbol{\pi}$ will be referred to as portfolios.

The way to solve for a time t constrained optimal portfolio is first to solve for a time-t unconstrained portfolio $\boldsymbol{\pi}(t)$. If $\boldsymbol{\pi}(t)$ satisfies the inequality constraints $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$, then the unconstrained portfolio $\boldsymbol{\pi}(t)$ is in fact also the time-t constrained optimal portfolio. Otherwise we have to discretise the problem and at each discrete time we have to consider at most $\bar{N}$ (defined in (4.27)) different combinations of active and inactive integral constraints (4.1)-(4.2) in the calculation of a constrained optimal portfolio. (See below and Appendix G to understand why the problem must be discretised.) In other words at each discrete time we have to consider at most $\bar{N}$ different combinations of where the multipliers $\lambda_j, j \in \mathcal{N}_M \cup \{0\}$ are zero and nonzero. In follow up work we hope to prove that the discrete constrained optimal portfolios in fact tend to the continuous-time constrained optimal portfolios as the time increment tends to zero. A constrained optimal portfolio $\boldsymbol{\pi}(t)$ is that which satisfies the constraints (4.4) and has the largest objective functional value (3.43) (or (3.52)). In the following paragraph we summarise how to calculate constrained optimal portfolios:

> *After determining the* really good portfolios*, let time evolve and calculate the unconstrained portfolios at each time. If the time-t unconstrained portfolio satisfies the constraints (4.4), then it is also the time-t constrained optimal portfolio, otherwise the time-t really good portfolio is the time-t constrained optimal portfolio.*

(4.79)

In the rest of this section we elaborate on the procedure in (4.79) in particular defining in (iii) below what we mean by *really good portfolios*. So unpacking (4.79), constrained optimal portfolios are determined as follows:

(i) Let $t \in [0, T]$. Then with $\mathcal{C}^*(t)$ empty in (4.26) (or (4.30), (4.41) or (4.43)), calculate the unconstrained portfolio $\boldsymbol{\pi}(t)$.

(ii) If $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$, then the unconstrained portfolio $\boldsymbol{\pi}(t)$ is in fact also the time-t constrained optimal portfolio. Proceed to step (i) above, increment time and calculate the next constrained optimal portfolio. If however at any time t the unconstrained portfolio $\mathbf{a}(t) \nleq \boldsymbol{\pi}(t)$ or $\boldsymbol{\pi}(t) \nleq \mathbf{b}(t)$, then we have to solve discretely for future constrained optimal portfolios $\boldsymbol{\pi}(s), s \in [t, T]$. In this case proceed to step (iii) below.

(iii) From (ii), this step will be reached if either $\mathbf{a}(t) \nleq \boldsymbol{\pi}(t)$ or $\boldsymbol{\pi}(t) \nleq \mathbf{b}(t)$. As mentioned in (ii), in this case, all time-$s, t \leq s \leq T$ constrained optimal portfolios must be determined discretely and we now explain why. Suppose at time t_j that the financial market parameter values from time t_j to time T are known (in other words the financial market parameters are deterministic). At the end of this section we return to this assumption. Note that even in this (deterministic) case, an insider (constrained portfolio selection) problem must still be solved since the disturbances $\mathbf{B}$ (and $\mathbf{q}$) are $\mathbb{F}$-adapted and insider portfolios $\boldsymbol{\pi}$ are $\mathbb{H}$-adapted resulting in the presence of forward stochastic integrals in the insider wealth process W.

Now a time-t constrained optimal portfolio is that which satisfies the constraints (4.4) and has the largest objective functional value (3.43). So an objective functional value must be determined for at most $\bar{N}$ constrained portfolios at time t and (that which satisfies (4.4) and) that with the largest objective functional value is the time-t constrained optimal portfolio. To calculate one objective functional value, many different possible sample paths of the disturbances $\mathbf{B}$ (and $\mathbf{q}$) must be considered. From these sample paths of $\mathbf{B}$ different terminal (time T) portfolio wealth values are obtained and the objective functional (3.43) can be evaluated by taking the expectation of the utility of terminal wealth. Now from (3.30) (and (3.32)) the insider wealth process is driven by the portfolio weights (processes) $\boldsymbol{\pi}$. So the expectation in (3.43) is actually taken over all future constrained optimal portfolios. So for each sample path of $\mathbf{B}$, to determine the resulting terminal portfolio wealth value (for that sample path of $\mathbf{B}$), only constrained optimal portfolios can be used.

Suppose we are at time-t. To determine the time-t constrained optimal portfolio we need to calculate an objective functional value (3.43) for at most $\bar{N}$ constrained portfolios at time t. But the evaluation of each objective functional value requires information about terminal wealth values and each wealth value is dependent on constrained optimal portfolios for each sample path of $\mathbf{B}$ over the entire interval $[t, T]$. Each of those future constrained optimal portfolios are determined by calculating objective functional values associated with at most $\bar{N}$ constrained portfolios at that time. But the evaluation of each of those objective functional values is dependent on constrained optimal portfolios from that time up to time T for each sample path of $\mathbf{B}$. So one must work backwards from time T to find a time-t constrained optimal portfolio. This is why constrained optimal portfolios have to be determined discretely the moment an unconstrained portfolio violates the inequality constraints (4.4).

Given this, a more detailed procedure for calculating constrained optimal portfolios is provided in (iv)-(vi) below. There references are made to what are referred to as *really good portfolios* and we now define these. Let $0 = t_0 < t_1 < \cdots < t_n = T, n \in \mathbb{N}$ be an equally spaced partition of $[0, T]$ and let the current time be t_j. Now the evaluation of the objective functional (3.43) always involves the determination of the time-t_{n-1} constrained optimal portfolio for each sample path of $\mathbf{B}$. In fact, if we are at time t_j, then for evaluation of *each* objective functional value (3.43), for each sample path of $\mathbf{B}$ constrained optimal portfolios are required from times $t_k, k \geq j$ up to time t_{n-1}. Now for each sample path of $\mathbf{B}$ the time-t_{n-1} unconstrained portfolio can be calculated from (4.26) (since it was assumed above that the financial market parameters are deterministic). If this unconstrained portfolio violates (4.4), then at most $\bar{N}$ time-t_{n-1} constrained portfolios must be evaluated and that which satisfies (4.4) and has the largest objective functional value (3.43) will be the time-t_{n-1} constrained optimal portfolio (for that sample path of $\mathbf{B}$). Now for all sample paths of $\mathbf{B}$ where the time-t_{n-1} unconstrained portfolio violates (4.4), the resultant time-t_{n-1} constrained optimal portfolio will be the same. This portfolio is calculated once and it is called the time-t_{n-1} *really good portfolio*. It is that portfolio with at least one active inequality constraint (4.4) and has the largest objective functional value (3.43). Then for each sample path of $\mathbf{B}$ the time-t_{n-1} unconstrained portfolio is calculated and if it violates (4.4), then the time-t_{n-1} really good portfolio will be the time-t_{n-1} constrained optimal portfolio (for that sample path of $\mathbf{B}$). In fact, before calculating constrained optimal portfolios from times t_j to t_{n-1}, all really good portfolios from the current time t_j to time t_{n-1} are calculated at time t_j. (This can be done since it was assumed above that the financial market parameters are deterministic.) This is analogous to Chapter 2, Section 2.5 where a grid of future constrained optimal portfolios is produced. Then as time evolved each time t constrained optimal portfolio was simply read off from the grid.

(iv) We now describe how to calculate the time-t_{n-1} really good portfolio. Recall the partition defined in (iii) above. First, amongst all time-t_{n-1} portfolios with *at least one* active (inequality) constraint (4.4), find that which satisfies (4.4) and has the largest objective functional value (3.43). We refer to this portfolio as the *time-t_{n-1} really good portfolio*. To find the time-t_{n-1} really good portfolio, we consider at most $\bar{N}$ ways of constraining the time-t_{n-1} security weights $\boldsymbol{\pi}(t_{n-1})$, in other words at most $\bar{N}$ ways of setting active the upper and lower inequality portfolio weight constraints $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$. The time-$t_{n-1}$ really good portfolio is calculated as follows:

(a) Set active only the lower bound constraint on $\pi_1(t_{n-1})$. Then the set $\mathcal{C}^*(t_{n-1}) = \{1\}$ and in (4.26), $\bar{c}_1(t_{n-1}) = a_1(t_{n-1})$ and $\bar{c}_i(t_{n-1}) = 0$ for all $i \in \mathcal{N}_S \backslash \{1\}$. From (iii) above (since the financial market parameters are assumed to be deterministic) calculate the optimal values of $\pi_2(t_{n-1}), \ldots, \pi_{N_S}(t_{n-1})$ using (4.26). Then by simulating different values of $\mathbf{B}(t_n)$, calculate the objective functional value (3.43) associated with this constrained portfolio.

(b) Set active only the upper bound constraint on $\pi_1(t_{n-1})$. Then the set $\mathcal{C}^*(t_{n-1}) = \{1\}$ and in (4.26), $\bar{c}_1(t_{n-1}) = b_1(t_{n-1})$ and $\bar{c}_i(t_{n-1}) = 0$ for all $i \in \mathcal{N}_S \backslash \{1\}$. From (iii) above calculate the optimal values of $\pi_2(t_{n-1}), \ldots, \pi_{N_S}(t_{n-1})$ using (4.26). Then by simulating different values of $\mathbf{B}(t_n)$, calculate the objective functional value (3.43) associated with this constrained portfolio.

(c) Repeat steps (a) and (b) for all securities $\mathbf{S}$.

(d) From (a)-(c) above one of the following will be done:

   1) If there is at least one portfolio in (a)-(c) which satisfies $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$, then the time-$t_{n-1}$ really good portfolio is that which satisfies $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$ and has the largest objective functional value (3.43). Save this portfolio. Then calculate the time-t_{n-2} really good portfolio by moving to step (v) below.

   2) Proceed to step (e) to set active more time-t_{n-1} inequality constraints $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$ to determine the time-t_{n-1} really good portfolio.

(e) Set active only the lower bound constraints on $\pi_1(t_{n-1})$ and $\pi_2(t_{n-1})$. Then the set $\mathcal{C}^*(t_{n-1}) = \{1, 2\}$ and in (4.26), $\bar{c}_1(t_{n-1}) = a_1(t_{n-1})$, $\bar{c}_2(t_{n-1}) = a_2(t_{n-1})$ and $\bar{c}_i(t_{n-1}) = 0$ for all $i \in \mathcal{N}_S \backslash \{1, 2\}$. From (iii) above calculate the optimal values of $\pi_3(t_{n-1}), \ldots, \pi_{N_S}(t_{n-1})$ using (4.26). Then by simulating different values of $\mathbf{B}(t_n)$, calculate the objective functional value (3.43) associated with this constrained portfolio.

(f) Set active only the lower bound constraints on $\pi_1(t_{n-1})$ and $\pi_3(t_{n-1})$. Then the set $\mathcal{C}^*(t_{n-1}) = \{1, 3\}$ and in (4.26), $\bar{c}_1(t_{n-1}) = a_1(t_{n-1})$, $\bar{c}_3(t_{n-1}) = a_3(t_{n-1})$ and $\bar{c}_i(t_{n-1}) = 0$ for all $i \in \mathcal{N}_S \backslash \{1, 3\}$. From (iii) above calculate the optimal values of $\pi_2(t_{n-1}), \pi_4(t_{n-1}), \ldots, \pi_{N_S}(t_{n-1})$ using (4.26). Then by simulating different values of $\mathbf{B}(t_n)$, calculate the objective functional value (3.43) associated with this constrained portfolio.

(g) Repeat steps (e) and (f) for all security weight pairs with different active upper and lower constraints $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$.

(h) From (e)-(g) above one of the following will be done:

   1) If there is at least one portfolio in (e)-(g) which satisfies $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$, then the time-$t_{n-1}$ really good portfolio is that which satisfies $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$ and has the largest objective functional value (3.43). Save this portfolio. Then calculate the time-t_{n-2} really good portfolio by moving to step (v) below.

2) Proceed to step (i) to set active more time-t_{n-1} inequality constraints $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$ to determine the time-t_{n-1} really good portfolio.

(i) Find constrained portfolios $\boldsymbol{\pi}(t_{n-1})$ for different sets of active upper and lower portfolio weight inequality constraints $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$. Choose the time-$t_{n-1}$ really good portfolio as that which satisfies $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$ and has the largest objective functional value (3.43). Save this portfolio.

(v) The time-t_{n-2} really good portfolio is determined by applying step (iv) above and using knowledge of the time-t_{n-1} really good portfolio (which has just been found in (iv) above). To calculate the time-t_{n-2} objective functional values (to compare the time-t_{n-2} constrained portfolios) we need to simulate different sample paths of $\mathbf{B}$ over the interval $[t_{n-2}, t_n]$. To calculate one time-t_{n-2} objective functional value (corresponding to one time-t_{n-2} constrained portfolio), we calculate the expectation (of the utility of terminal wealth) over different sample paths of $\mathbf{B}$ over $[t_{n-2}, t_n]$.

We now explain how knowledge of the time-t_{n-1} really good portfolio (calculated in step (iv) above) is incorporated into the calculation of the time-t_{n-2} really good portfolio. Since we assumed in step (iii) that the financial market parameter values over $[t, T]$ are known at time t, for each sample path of $\mathbf{B}$ (over $[t_{n-2}, t_n]$) we can calculate unconstrained portfolios $\boldsymbol{\pi}(t_{n-1})$. If $\mathbf{a}(t_{n-1}) \leq \boldsymbol{\pi}(t_{n-1}) \leq \mathbf{b}(t_{n-1})$, then use this portfolio in the calculation of the objective functional value (3.43) at time t_{n-2}. Otherwise use the time-t_{n-1} really good portfolio in the calculation of (3.43) to compare time-t_{n-2} constrained portfolios. We shall then be determining the time-t_{n-2} really good portfolio as that which satisfies (4.4) and has the largest objective functional value calculated by taking the expectation over constrained optimal portfolios in each sample path of $\mathbf{B}$.

(vi) We repeat step (v) until we find the time-t_j really good portfolio.

Returning to the current time t_j, if we found that the time-t_j unconstrained portfolio violates the constraints $\mathbf{a}(t_j) \leq \boldsymbol{\pi}(t_j) \leq \mathbf{b}(t_j)$, then the time-$t_j$ constrained optimal portfolio will in fact be the time-t_j really good portfolio. We then project to the next (discrete) time. If we find that $\mathbf{a}(t_{j+1}) \nleq \boldsymbol{\pi}(t_{j+1})$ or $\boldsymbol{\pi}(t_{j+1}) \nleq \mathbf{b}(t_{j+1})$, then the time-$t_{j+1}$ constrained optimal portfolio will be the time-t_{j+1} really good portfolio. We evolve to the next time step until the time-t_{n-1} unconstrained portfolio $\boldsymbol{\pi}(t_{n-1})$ is calculated. If we find that $\mathbf{a}(t_{n-1}) \nleq \boldsymbol{\pi}(t_{n-1})$ or $\boldsymbol{\pi}(t_{n-1}) \nleq \mathbf{b}(t_{n-1})$, then the time-$t_{n-1}$ constrained optimal portfolio will in fact be the time-t_{n-1} really good portfolio.

If in contrast to step (iii) above the future financial market parameter values are unknown at time t_j, then we have to repeat steps (iv)-(vi) above for different sample paths of the non-deterministic financial market parameters. Then a time-t really good portfolio is that which satisfies (4.4) and has the largest objective functional value, where in particular the expectation in (3.43) is taken over all the stochastic financial market parameters. This is unfortunately computationally intensive. See Section 4.4 below for examples and where numerical methods are employed, time estimates are provided of how long it took to solve the constrained portfolio selection problems.

4.4 Examples

4.4.1 Example 1

Recall equation (3.70) viz

$$\boldsymbol{\pi}(t) = \bar{\boldsymbol{\sigma}}^{-1}(t)\left(\hat{\boldsymbol{\xi}}(t) - \boldsymbol{\sigma}^0(t) + \hat{\boldsymbol{\sigma}}(t)\frac{d}{dt}\mathbf{H}(t) - \bar{F}^{-1}(t)\frac{d}{dt}[\mathbf{M},\bar{F}](t)\right). \qquad (4.80)$$

In this example we simplify (4.80) assuming the insider has logarithmic utility. Let S_0 be a money market security with interest rate $r(t)$ and let $U(t,x) \equiv \ln x$. Then

$$\hat{\boldsymbol{\xi}}(t) = \boldsymbol{\xi}(t) - \mathbf{r}(t), \quad \boldsymbol{\sigma}^0 \equiv \mathbf{0}, \quad \hat{\boldsymbol{\sigma}}(t) = \boldsymbol{\sigma}(t), \quad \bar{F} \equiv 1, \quad [\mathbf{M},\bar{F}] \equiv \mathbf{0}$$

and from ([18], equation (3.35)), $\mathbf{H}(t) = \int_0^t \boldsymbol{\gamma}_t(s)ds$ for some functions $\boldsymbol{\gamma}_t(s) := (\gamma_{t,1}(s),\ldots,\gamma_{t,N_B}(s))$ where $\boldsymbol{\gamma}_t$ are $\mathbb{H}$-adapted (and the processes $\int_0^t \boldsymbol{\gamma}_t(s)ds$ are of bounded variation). Then (4.80) reduces to

$$\boldsymbol{\pi}(t) \quad = \quad \bar{\boldsymbol{\sigma}}^{-1}(t)\left(\boldsymbol{\xi}(t) - \mathbf{r}(t) + \boldsymbol{\sigma}(t)\boldsymbol{\gamma}_t(t)\right). \qquad (4.81)$$

In particular if

$$N_S = 1 = N_B, \quad \xi_1(t) = \xi(t), \quad \sigma_{11}(t) = \sigma(t), \quad 0 \le t \le T$$

then $\boldsymbol{\gamma}_t(t) = \gamma_t(t)$ and $\bar{\boldsymbol{\sigma}}(t) = \hat{\boldsymbol{\sigma}}(t)\hat{\boldsymbol{\sigma}}^T(t) = \sigma^2(t)$ and (4.81) reduces to

$$\pi(t) = \sigma^{-2}(t)(\xi(t) - r(t) + \sigma(t)\gamma_t(t))$$

which is exactly the result derived by (Biagini, Øksendal [18], equation (3.36)).

4.4.2 Example 2

Recall equation (4.41) viz

$$\pi_i(t) = \hat{\Gamma}_i(t) + \sum_{j\in\mathcal{C}^*(t)} \sigma_{ij}^{\kappa}(t) \sum_{k\in\mathcal{C}^*(t)} \varsigma_{jk}(t)\left(\bar{c}_k(t) - \hat{\Gamma}_k(t)\right).$$

In this example we simplify (4.41) assuming a special form of the insider's filtration $\mathbb{H}$. Suppose the sets $\mathcal{C}^*(t), 0 \le t \le T$ are empty, in other words there are no active inequality constraints at any time $0 \le t \le T$. Then for all $0 \le t \le T, i \in \mathcal{N}_S$ (4.41) reduces to

$$\pi_i(t) \quad = \quad \sum_{j=1}^{N_S} \sigma^\kappa_{ij}(t) \left(\xi_j(t) - r(t) + \sum_{k=1}^{N_B} \sigma_{jk}(t) \frac{d}{dt} H_k(t) \right), \qquad (4.82)$$

where for all $0 \le t \le T$ the matrix $\boldsymbol{\sigma}^\kappa(t) = [\bar{\boldsymbol{\sigma}}(t) + (diag(\boldsymbol{\kappa}(t)))^2]^{-1}$ (assuming it is invertible). Let $T < T_0$. Then as in ([62], (5.26)), suppose the insider's filtration $\mathbb{H}$ is such that for all $0 \le t \le T$

$$\mathcal{H}_t = \sigma(\mathcal{F}_t \cup \sigma(\mathbf{B}(T_0))). \qquad (4.83)$$

Then from ([62], (5.27)) for all $0 \le t \le T$ the bounded variation processes $\mathbf{H}$ defined in (3.87) have the form

$$\mathbf{H}(t) = \int_0^t \frac{\mathbf{B}(T_0) - \mathbf{B}(s)}{T_0 - s} ds \qquad (4.84)$$

and (4.82) reduces to

$$\pi_i(t) \quad = \quad \sum_{j=1}^{N_S} \sigma^\kappa_{ij}(t) \left(\xi_j(t) - r(t) + \sum_{k=1}^{N_B} \sigma_{jk}(t) \frac{B_k(T_0) - B_k(t)}{T_0 - t} \right). \qquad (4.85)$$

Let $N_B = 1 = N_S$ and let $\sigma_{11} = \sigma$ and $\kappa_1 = \kappa$ be functions. Then we have that $\bar{\sigma}_{11} = \sigma^2$ and $\sigma^\kappa_{11} = (\sigma^2 + \kappa^2)^{-1}$. So with $\pi_1 = \pi,\ \xi_1 = \xi,\ B_1 = B$ equation (4.85) reduces to

$$\pi(t) \quad = \quad (\sigma^2(t) + \kappa^2(t))^{-1} \left(\xi(t) - r(t) + \sigma(t) \frac{B(T_0) - B(t)}{T_0 - t} \right). \qquad (4.86)$$

Assuming κ and σ are constants and replacing κ with $\kappa\sigma$, (4.86) reduces to the result derived by (Hu, Øksendal [62], equation (5.28)). As in ([62], Theorem 5.7) we can show that in the special case (4.86) the objective functional (3.52) is finite. This finiteness of the objective functional is a direct consequence of the inclusion of the penalty functions (4.38).

4.4.3 Example 3

Recall equation (4.43) viz for all $0 \le t \le T, i \in \mathcal{N}_S$

$$\begin{aligned} 0 \quad = \quad & D_i(t) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t)\pi_j(t) + \bar{\kappa}_i^2(t)\pi_i''(t) + 2\bar{\kappa}_i(t)\bar{\kappa}_i'(t)\pi_i'(t) \\ & + (\bar{\kappa}_i(t)\kappa_i'(t) + \bar{\kappa}_i'(t)\kappa_i(t) - \kappa_i^2(t))\pi_i(t). \end{aligned}$$

In this example we find a particular solution of (4.43) assuming a particular form of the insider's filtration $\mathbb{H}$. Suppose there are no active inequality constraints at any time $t \in [0,T]$. Then for all $0 \leq t \leq T$ the set $\mathcal{C}^*(t)$ is empty and $\boldsymbol{\mu}^*(t) = \mathbf{0}$ almost surely. With $N_B = 1 = N_S$, $\pi_1 = \pi$, $\xi_1 = \xi$, $B_1 = B$ and $\sigma_{11} = \sigma$, $\kappa_1 = \kappa$, $\bar{\kappa}_1 = \bar{\kappa}$ constants, we have that (4.43) reduces to

$$-D(t) = -\bar{\sigma}\pi(t) + U(t)\pi''(t) + V(t)\pi'(t) + Y(t)\pi(t), \tag{4.87}$$

where for all $0 \leq t \leq T$

$$\begin{aligned} D(t) &= \xi(t) - r(t) + \sigma(t)\frac{d}{dt}H(t), \\ U(t) &= \bar{\kappa}^2, \\ V(t) &= 0 \quad \text{and} \\ Y(t) &= -\kappa^2. \end{aligned}$$

Using variation of parameters we can show that the unique solution of (4.87) is

$$\begin{aligned} \pi(t) = & \left(c_1(0) - v_1^{-1}\bar{\kappa}^{-2}(u_1 - u_2)^{-1} \int_0^t D(s)e^{-u_1 s}ds\right) v_1 e^{u_1 t} \\ & + \left(c_2(0) + \bar{\kappa}^{-2}v_2^{-1}(u_1 - u_2)^{-1} \int_0^t D(s)e^{-u_2 s}ds\right) v_2 e^{u_2 t}, \end{aligned} \tag{4.88}$$

where $u_{1,2} = \pm\sqrt{\bar{\kappa}^{-2}(\sigma^2 - \kappa^2)}$ are the eigenvalues of the matrix

$$\begin{pmatrix} 0 & \bar{\kappa}^{-2}(\sigma^2 - \kappa^2) \\ 1 & 0 \end{pmatrix}. \tag{4.89}$$

In (4.88) the variables

$$\begin{pmatrix} \bar{v}_1 \\ v_1 \end{pmatrix} =: \hat{\mathbf{v}}_1$$

are the eigenvectors of the matrix (4.89) and

$$c_2(t) = c_2(0) + \bar{\kappa}^{-2}v_2^{-1}(u_1 - u_2)^{-1} \int_0^t D(s)e^{-u_2 s}ds, \tag{4.90}$$

$$c_1(t) = c_1(0) - v_1^{-1}\bar{\kappa}^{-2}(u_1 - u_2)^{-1} \int_0^t D(s)e^{-u_1 s}ds, \tag{4.91}$$

where the constants $c_i(0), i = 1, 2$ are obtained from the initial conditions (4.29) in other words

$$\boldsymbol{\pi}(0) = \boldsymbol{\pi}_{BM} \text{ and } \boldsymbol{\pi}'(0) = 0.$$

If $\kappa = 0$ and $\bar{\kappa} = 1 = \sigma$, then we have that the eigenvalues and eigenvectors of (4.89) are respectively

$$u_1 = 1, \ \hat{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad u_2 = -1, \ \hat{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In this case (4.88) reduces to

$$\pi(t) = c_1(0)e^t - c_2(0)e^{-t} + \int_0^t D(s)\sinh(s-t)ds.$$

Using the initial condition $\pi(0) = 0$ we have that $c_1(0) = c_2(0)$ thus

$$\pi(t) = 2c_1(0)\sinh t + \int_0^t D(s)\sinh(s-t)ds. \tag{4.92}$$

With $N_B = 1$ if we use the filtration (4.83), then from (4.84) equation (4.92) reduces to

$$\pi(t) = 2c_1(0)\sinh t + \int_0^t \left(\xi(s) - r(s) + \frac{B(T_0) - B(s)}{T_0 - s}\right)\sinh(s-t)ds$$

which is exactly the result derived by (Hu, Øksendal [62], equation (5.35)). As shown in ([62], Example 5.8), in this case the objective functional of the insider is finite regardless of how close T_0 is to T. This finiteness of the objective functional is a direct consequence of the inclusion of the penalty functions (4.42).

4.4.4 Example 4

Recall equation (4.76) viz for all $0 \le t \le T, i \in \mathcal{N}_S$

$$0 = \xi_i(t) - r(t) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t)\pi_j(t) - \sum_{j=1}^{N_B} \sigma_{ij}(t)\eta_{ij}^B(t) - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(t))) + \mu_i^*(t) + \rho_i(t).$$

In this example we simplify (4.76) assuming particular types of jumps exhibited by the securities $\mathbf{S}$. With $N_S = 1 = N_q, \boldsymbol{\sigma} \equiv \mathbf{0} \equiv \mathbb{L} \equiv \boldsymbol{\mu}^*, \pi_1 = \pi, \xi_1 = \xi, q_1 = q, \nu_1 = \nu, \rho_1 = \rho, g_{11} = z$ in (4.76) it reduces to

$$0 = \xi(t) - r(t) - \int_{\mathbb{R}} \frac{z^2\pi(t)}{1 + z\pi(t)}\nu^{\mathbb{F}}(dz) + \mathbb{E}\left[\int_t^{T_0}\int_{\mathbb{R}} \frac{z}{(1 + z\pi(t))(T_0 - t)}\tilde{q}(dr, dz)\,\middle|\, \mathcal{H}_t\right]. \tag{4.93}$$

From Assumption 1, if we assume that the pure jump process $\varphi = \varphi_1$ associated with the Poisson random measure q is in fact a Poisson process with intensity $\bar{\rho}$, then we have that

$$\nu^{\mathbb{F}}(dz)ds = \bar{\rho}\delta_{\{1\}}(dz)ds$$

where $\delta_{\{1\}}(dz)$ is the unit point mass at $z = 1$. With $P = P(t)$ denoting a Poisson process of intensity $\bar{\rho}$, equation (4.93) reduces to

$$\pi(t) = \frac{\mathbb{E}[P(T_0)|\mathcal{H}_t] - P(t)}{(T_0 - t)(r(t) + \bar{\rho} - \xi(t))} - 1,$$

which is exactly the result derived by (Di Nunno, *et al.* [42], equation (67)).

4.4.5 Example 5

Recall equation (4.76) viz for all $0 \le t \le T, i \in \mathcal{N}_S$

$$0 = \xi_i(t) - r(t) - \sum_{j=1}^{N_S} \bar{\sigma}_{ij}(t)\pi_j(t) - \sum_{j=1}^{N_B} \sigma_{ij}(t)\eta_{ij}^B(t) - \sum_{j=1}^{N_S} \mathbb{L}_{ji}^{\dagger}(\mathbb{L}_{ji}(\pi_i(t))) + \mu_i^*(t) + \rho_i(t).$$

In this example we simplify (4.76) assuming particular types of jumps exhibit by the securities $\mathbf{S}$. From Assumption 1, if we assume that the pure jump processes $\boldsymbol{\varphi}$ associated with the Poisson random measures $\mathbf{q}$ are Poisson processes with intensities $\bar{\boldsymbol{\rho}} := (\bar{\rho}_1, \ldots, \bar{\rho}_{N_q})$, then with $N_S = 1 = N_q,\ \pi_1 = \pi,\ \xi_1 = \xi,\ \mathbb{L} \equiv \mathbf{0} \equiv \boldsymbol{\mu}^*,\ \rho_1 = \rho,\ \eta_{11}^B = \eta^B$, we have that (4.76) reduces to

$$\pi(t) = \frac{1}{2\sigma^2(t)}\left[\bar{\eta}^B(t) \pm \sqrt{(\bar{\eta}^B(t))^2 + 4\sigma^2(t)\left(\bar{\eta}^B(t) + \sigma^2(t) + \frac{\mathbb{E}[P(T_0)|\mathcal{H}_t] - P(t)}{(T_0 - t)}\right)}\,\right],$$

where $\bar{\eta}^B(t) := \xi(t) - r(t) - \sigma(t)\eta^B(t) - \bar{\rho} - \sigma^2(t)$ for all $0 \le t \le T$. This is exactly the result derived by (Di Nunno, *et al.* [42], equation (72)).

4.4.6 Example 6

Recall equation (4.26) viz for all $0 \le t \le T, i \in \mathcal{N}_S$

$$\begin{aligned}\pi_i(t) = {} & \bar{\Gamma}_i(t) + \bar{C}_i(t) + \sum_{k \in \mathcal{C}^*(t)} \sigma_{ik}^{\kappa}(t)\left(\sum_{l \in \mathcal{C}^*(t)} \varsigma_{kl}(t)\left(\bar{c}_l(t) - \bar{\Gamma}_l(t) - \bar{C}_l(t)\right)\right. \\ & \left. - \sum_{n \in \mathcal{C}^*(t)} \bar{\Gamma}_n(t) \sum_{l \in \mathcal{C}^*(t)} \varsigma_{nl}(t)\left(\bar{c}_l(t) - \bar{\Gamma}_l(t) - \bar{C}_l(t)\right)\right).\end{aligned} \tag{4.94}$$

In this example we evaluate $\boldsymbol{\pi}$ defined in (4.94) for problem **(P2)** and in particular we want to ensure that $\boldsymbol{\pi}$ satisfies the inequality constraints (4.4). We assume the securities $\mathbf{S}$ do not exhibit jumps and that

$$T = 1,\ T_0 = 2,\ N_S = 3,\ N_B = 4,\ \Upsilon \equiv 1,\ \boldsymbol{\kappa} \equiv \mathbf{1}.$$

We assume that $\mathbb{H}$ is of the form (4.83) viz

$$\mathcal{H}_t = \sigma(\mathcal{F}_t \cup \sigma(\mathbf{B}(T_0))) \tag{4.95}$$

such that the bounded variation processes $\mathbf{H}$ are of the form (4.84) viz

$$\mathbf{H}(t) = \int_0^t \frac{\mathbf{B}(T_0) - \mathbf{B}(s)}{T_0 - s} ds.$$

We also assume that the financial market parameters have the values given in Tables 4.1 and 4.2. We assume these values are known at time 0 (which implies

t	0	0.25	0.5	0.75
a_1	0.1	0.1	0.1	0.1
a_2	0.1	0.1	0.1	0.1
a_3	0.1	0.1	0.1	0.1
b_1	0.7	0.7	0.7	0.7
b_2	0.7	0.7	0.7	0.7
b_3	0.7	0.7	0.7	0.7
ξ_1	0.1405	0.1292	0.1620	0.1542
ξ_2	0.0198	0.0481	0.0521	0.0444
ξ_3	0.0855	0.1080	0.1415	0.1744

Table 4.1: Time-t values of financial market parameters for securities S_1, S_2, S_3 for Example 6.

that these financial market parameters are deterministic). (We see from Table 4.1 that we require insider constrained optimal portfolios to be bounded between 10% and 70% over the entire horizon $[0, T]$.) We also assume that the particular sample paths over $[0, T_0]$ of the Brownian motions $\mathbf{B}$ which are realised are those listed in Table 4.3. Given these inputs we found that the constrained optimal portfolios are those listed in Table 4.4 and we now describe how these were calculated.

First, from (4.94), with $\mathcal{C}^*(t)$ empty for all $0 \leq t \leq T$, the unconstrained portfolios are those given in Table 4.5. From Table 4.5 we see that the time-0 unconstrained portfolio $\boldsymbol{\pi}(0)$ does not satisfy the constraints $\mathbf{a}(0) \leq \boldsymbol{\pi}(0) \leq \mathbf{b}(0)$, where $\mathbf{a}(0)$ and $\mathbf{b}(0)$ are defined in Table 4.1. Thus from Section 4.3 we do the following to find a constrained optimal portfolio for problem **(P2)**: Let $0 = t_0 < t_1 < \cdots < t_n = T, n = 4$ be an equally spaced partition of $[0, T]$. We find all really good portfolios from time $t_{n-1} = 0.75$ up to time $t_0 = 0$.

The time-t_3 really good portfolio was determined as follows: At time t_3, with $M = 6$ in (4.27), we need to consider at most $\bar{N} = 19$ constrained portfolios and their corresponding objective functional values to find the time-t_3 really good portfolio. These 19 time-t_3 constrained portfolios are listed in Table 4.6. (From Section 4.3 there should be 19 portfolios between which we should choose. In Table 4.6 there are 18 portfolios; the nineteenth portfolio is $\boldsymbol{\pi}(t_3)$, the unconstrained portfolio in Table 4.5.) Only the portfolios which satisfy the constraints $\mathbf{a}(t_3) \leq \boldsymbol{\pi}(t_3) \leq \mathbf{b}(t_3)$, and their corresponding objective functional values, are listed in Table 4.7. Using the form (4.16) of the penalty functions $\mathbb{L}$ and the partition defined above, from (3.43), the objective functional $\bar{J}_{2,t_3}$ has

t	$\boldsymbol{\sigma}$
0	$\begin{pmatrix} 0.1920 & 0.1440 & -0.0246 & 0.0282 \\ -0.2142 & -0.4873 & 0.0232 & -0.5793 \\ -0.1288 & -0.4847 & -0.4536 & -0.2740 \end{pmatrix}$
0.25	$\begin{pmatrix} -0.0483 & 0.0575 & -0.4053 & -0.0715 \\ 0.1621 & -0.2180 & -0.1205 & -0.0769 \\ 0.0867 & -0.2906 & 0.0725 & -0.0025 \end{pmatrix}$
0.5	$\begin{pmatrix} 0.0219 & -0.4226 & -0.4129 & -0.2403 \\ -0.4801 & -0.0289 & -0.0763 & -0.0794 \\ -0.5090 & -0.3261 & -0.3908 & -0.4542 \end{pmatrix}$
0.75	$\begin{pmatrix} -0.2726 & 0.1443 & -0.2679 & -0.4211 \\ -0.2711 & -0.5645 & -0.5977 & -0.1366 \\ -0.0571 & 0.1386 & -0.3306 & 0.0230 \end{pmatrix}$

Table 4.2: Time-t volatility matrices of securities S_1, S_2, S_3 for Example 6.

t	0	0.25	0.5	0.75	1	2
B_1	0	0.4275	0.3404	1.6325	2.7859	3.3035
B_2	0	0.3674	0.6679	1.7251	1.2242	0.1495
B_3	0	0.1044	0.1811	0.0457	0.8158	-1.3385
B_4	0	-0.0818	0.0109	-0.2847	0.0431	-0.5280

Table 4.3: Time-t values of Brownian motions B_1, B_2, B_3, B_4 for Example 6.

t	0	0.25	0.5	0.75
$\boldsymbol{\pi}$	$\begin{pmatrix} 0.67 \\ 0.10 \\ 0.23 \end{pmatrix}$	$\begin{pmatrix} 0.21 \\ 0.16 \\ 0.63 \end{pmatrix}$	$\begin{pmatrix} 0.20 \\ 0.10 \\ 0.70 \end{pmatrix}$	$\begin{pmatrix} 0.44 \\ 0.10 \\ 0.46 \end{pmatrix}$

Table 4.4: Time-t constrained optimal portfolios for securities S_1, S_2, S_3 for Example 6.

t	0	0.25	0.5	0.75
$\boldsymbol{\pi}$	$\begin{pmatrix} 0.73 \\ 0.00 \\ 0.27 \end{pmatrix}$	$\begin{pmatrix} 0.21 \\ 0.16 \\ 0.63 \end{pmatrix}$	$\begin{pmatrix} 0.92 \\ -0.21 \\ 0.29 \end{pmatrix}$	$\begin{pmatrix} -0.06 \\ 0.96 \\ 0.10 \end{pmatrix}$

Table 4.5: Time-t unconstrained portfolios for Example 6.

$\begin{pmatrix} -0.50 \\ 0.80 \\ 0.70 \end{pmatrix}$ $\begin{pmatrix} 0.70 \\ 0.70 \\ -0.40 \end{pmatrix}$ $\begin{pmatrix} 0.20 \\ 0.10 \\ 0.70 \end{pmatrix}$ $\begin{pmatrix} 0.10 \\ 0.70 \\ 0.20 \end{pmatrix}$	$\begin{pmatrix} 0.09 \\ 0.70 \\ 0.21 \end{pmatrix}$ $\begin{pmatrix} -0.06 \\ -0.96 \\ 0.10 \end{pmatrix}$ $\begin{pmatrix} 0.70 \\ 0.10 \\ 0.20 \end{pmatrix}$ $\begin{pmatrix} 0.10 \\ 0.80 \\ 0.10 \end{pmatrix}$	$\begin{pmatrix} -0.40 \\ 0.70 \\ 0.70 \end{pmatrix}$ $\begin{pmatrix} 0.20 \\ 0.70 \\ 0.10 \end{pmatrix}$ $\begin{pmatrix} 0.80 \\ 0.10 \\ 0.10 \end{pmatrix}$ $\begin{pmatrix} 0.10 \\ 0.10 \\ 0.80 \end{pmatrix}$	$\begin{pmatrix} 0.70 \\ 0.71 \\ -0.41 \end{pmatrix}$ $\begin{pmatrix} 0.70 \\ 0.20 \\ 0.10 \end{pmatrix}$ $\begin{pmatrix} 0.10 \\ 0.91 \\ -0.01 \end{pmatrix}$	$\begin{pmatrix} 0.70 \\ -0.40 \\ 0.70 \end{pmatrix}$ $\begin{pmatrix} 0.44 \\ 0.10 \\ 0.46 \end{pmatrix}$ $\begin{pmatrix} 0.10 \\ 0.20 \\ 0.70 \end{pmatrix}$

Table 4.6: All time-t_3 constrained portfolios for Example 6.

$\bar{J}_{2,t_3}$	13.818	13.831	13.841	13.837
$\boldsymbol{\pi}(t_3)$	$\begin{pmatrix} 0.20 \\ 0.70 \\ 0.10 \end{pmatrix}$	$\begin{pmatrix} 0.70 \\ 0.20 \\ 0.10 \end{pmatrix}$	$\begin{pmatrix} 0.44 \\ 0.10 \\ 0.46 \end{pmatrix}$	$\begin{pmatrix} 0.20 \\ 0.10 \\ 0.70 \end{pmatrix}$
$\bar{J}_{2,t_3}$	13.836	13.834	13.810	
$\boldsymbol{\pi}(t_3)$	$\begin{pmatrix} 0.70 \\ 0.10 \\ 0.20 \end{pmatrix}$	$\begin{pmatrix} 0.10 \\ 0.20 \\ 0.70 \end{pmatrix}$	$\begin{pmatrix} 0.10 \\ 0.70 \\ 0.20 \end{pmatrix}$	

Table 4.7: Valid time-t_3 constrained portfolios and corresponding objective functional values for Example 6.

the form

$$\begin{aligned}
\bar{J}_{2,t_3} &= \sup_{\boldsymbol{\pi}\in\mathcal{P}_{B2}} \mathbb{E}\left[U(T,W(T)) - \frac{1}{2}\int_0^T \|\mathbb{L}(\boldsymbol{\pi}(s))\|^2 ds\right] \\
&= \sup_{\boldsymbol{\pi}\in\mathcal{P}_{B2}} \mathbb{E}\left[U(t_4,W(t_4)) - \frac{1}{2}\sum_{i=1}^{3}(\mathbb{L}_{ii}(\pi_i(t_3)))^2\Delta t - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=0}^{2}(\mathbb{L}_{ii}(\pi_i(t_j)))^2\Delta t\right] \\
&= \ln W(t_3) + \sup_{\boldsymbol{\pi}\in\mathcal{P}_{B2}} \mathbb{E}\left[\left\{\sum_{i=1}^{3}\xi_i(t_3)\pi_i(t_3) - \frac{1}{2}\sum_{j=1}^{4}\left(\sum_{i=1}^{3}\sigma_{ij}(t_3)\pi_i(t_3)\right)^2\right\}\Delta t\right. \\
&\quad + \sum_{j=1}^{4}\sum_{i=1}^{3}\sigma_{ij}(t_3)\pi_i(t_3)(B_j(t_4)-B_j(t_3)) \\
&\quad \left. - \frac{1}{2}\sum_{i=1}^{3}(\mathbb{L}_{ii}(\pi_i(t_3)))^2\Delta t - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=0}^{2}(\mathbb{L}_{ii}(\pi_i(t_j)))^2\Delta t\right],
\end{aligned} \tag{4.96}$$

where from (3.27), $0 < W(t_3)$ almost surely has the form

$$\begin{aligned} W(t_3) &= W(t_0)\exp\left(\sum_{k=0}^{2}\left\{\sum_{i=1}^{3}\xi_i(t_k)\pi_i(t_k) - \frac{1}{2}\sum_{j=1}^{4}\left(\sum_{i=1}^{3}\pi_i(t_k)\sigma_{ij}(t_k)\right)^2\right\}\Delta t \right. \\ &\quad \left. + \sum_{k=0}^{2}\sum_{j=1}^{4}\sum_{i=1}^{3}\pi_i(t_k)\sigma_{ij}(t_k)(B_j(t_{k+1}) - B_j(t_k))\right) \end{aligned}$$

and $W(t_0)$ is the insider's initial wealth value. The time-t_3 really good portfolio is the constrained portfolio $\boldsymbol{\pi}(t_3)$ which satisfies the constraints $\mathbf{a}(t_3) \le \boldsymbol{\pi}(t_3) \le \mathbf{b}(t_3)$ and has the largest objective functional value $\bar{J}_{2,t_3}$ (in (4.96)) amongst the set of time-t_3 constrained portfolios listed in Table 4.6. Now if we are at time t_3, then we know the value of $W(t_3)$. We then seek the time-t_3 portfolio $\boldsymbol{\pi}(t_3)$ which maximises $\bar{J}_{2,t_3}$ over the time interval $[t_3, T]$. Since $0 < W(t_3)$, from (4.96) without loss of generality, to compare different values of $\bar{J}_{2,t_3}$ for different constrained portfolios $\boldsymbol{\pi}(t_3)$, we can assume that $W(t_3) = 1$. Then a portfolio which maximises $\bar{J}_{2,t_3}$ with $W(t_3) = 1$ will still maximise $\bar{J}_{2,t_3}$ for some $0 < W(t_3)$. We then substitute each constrained portfolio in Table 4.7 into (4.96). We simulate different values of $\mathbf{B}(t_4)$ and calculate the expectation, hence value of $\bar{J}_{2,t_3}$, in (4.96). After doing this for each constrained portfolio in Table 4.7, we obtained the objective functional values listed in Table 4.7. The time-t_3 constrained portfolio $\boldsymbol{\pi}(t_3) = (0.44, 0.1, 0.46)$ satisfied the constraints $\mathbf{a}(t_3) \le \boldsymbol{\pi}(t_3) \le \mathbf{b}(t_3)$ and had the largest objective functional value of $\bar{J}_{2,t_3} = 13.841$. Note that for each sample path of $\mathbf{B}$, new values of $\mathbf{B}(T_0)$ are not simulated. The values $\mathbf{B}(T_0)$ in Table 4.3 remain the same for the portfolios calculated in the different sample paths of $\mathbf{B}$. Also to speed up the computational time, one actually needs to find only the time-t_3 constrained portfolio with the least number of active constraints (and which has the largest objective functional value).

The time-t_2 really good portfolio was determined as follows: Suppose we are at time t_2. We want to a find the time-t_2 constrained portfolio which satisfies the constraints (4.4) and has the largest objective functional value (3.43). The objective functional value (3.43) is an expectation over different sample paths of $\mathbf{B}$ over the time interval $[t_2, t_4]$ and consequently it is an expectation over different constrained optimal portfolios from time t_2 to time t_3. To evaluate (3.43) we need to calculate constrained optimal portfolios for each sample path of $\mathbf{B}$ over $[t_2, t_4]$. So for each sample path of $\mathbf{B}$ we follow its evolution from time t_2 to t_4. At time t_3 we calculate the unconstrained portfolio $\boldsymbol{\pi}(t_3)$ for that sample path of $\mathbf{B}$. If $\mathbf{a}(t_3) \le \boldsymbol{\pi}(t_3) \le \mathbf{b}(t_3)$, then we use this portfolio in the calculation of $\bar{J}_{2,t_2}$ defined in (4.97) below. Otherwise we use the time-t_3 really good portfolio (which is then the time-t_3 constrained optimal portfolio for that sample path of $\mathbf{B}$) in the calculation of $\bar{J}_{2,t_2}$. From (3.43) the objective

$\begin{pmatrix} 0.86 \\ -0.56 \\ 0.70 \end{pmatrix}$	$\begin{pmatrix} 6.75 \\ 0.70 \\ -6.45 \end{pmatrix}$	$\begin{pmatrix} -0.40 \\ 0.70 \\ 0.70 \end{pmatrix}$	$\begin{pmatrix} 0.70 \\ 0.08 \\ 0.22 \end{pmatrix}$	$\begin{pmatrix} 0.70 \\ -0.40 \\ 0.70 \end{pmatrix}$
$\begin{pmatrix} 0.70 \\ 0.70 \\ -0.40 \end{pmatrix}$	$\begin{pmatrix} 0.96 \\ -0.06 \\ 0.10 \end{pmatrix}$	$\begin{pmatrix} 0.20 \\ 0.70 \\ 0.10 \end{pmatrix}$	$\begin{pmatrix} 0.70 \\ 0.20 \\ 0.10 \end{pmatrix}$	$\begin{pmatrix} 2.93 \\ 0.10 \\ -2.03 \end{pmatrix}$
$\begin{pmatrix} 0.20 \\ 0.10 \\ 0.70 \end{pmatrix}$	$\begin{pmatrix} 0.70 \\ 0.10 \\ 0.20 \end{pmatrix}$	$\begin{pmatrix} 0.80 \\ 0.10 \\ 0.10 \end{pmatrix}$	$\begin{pmatrix} 0.10 \\ 0.85 \\ 0.05 \end{pmatrix}$	$\begin{pmatrix} 0.10 \\ 0.20 \\ 0.70 \end{pmatrix}$
$\begin{pmatrix} 0.10 \\ 0.70 \\ 0.20 \end{pmatrix}$	$\begin{pmatrix} 0.10 \\ 0.80 \\ 0.10 \end{pmatrix}$	$\begin{pmatrix} 0.10 \\ 0.10 \\ 0.80 \end{pmatrix}$		

Table 4.8: All time-t_2 constrained portfolios for Example 6.

$\bar{J}_{2,t_2}$	13.870	13.884	13.891
$\boldsymbol{\pi}(t_2)$	$\begin{pmatrix} 0.2 \\ 0.7 \\ 0.1 \end{pmatrix}$	$\begin{pmatrix} 0.7 \\ 0.2 \\ 0.1 \end{pmatrix}$	$\begin{pmatrix} 0.2 \\ 0.1 \\ 0.7 \end{pmatrix}$
$\bar{J}_{2,t_2}$	13.887	13.886	13.875
$\boldsymbol{\pi}(t_2)$	$\begin{pmatrix} 0.7 \\ 0.1 \\ 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.1 \\ 0.2 \\ 0.7 \end{pmatrix}$	$\begin{pmatrix} 0.1 \\ 0.7 \\ 0.2 \end{pmatrix}$

Table 4.9: Valid time-t_2 constrained portfolios and objective functional values for Example 6.

functional $\bar{J}_{2,t_2}$ has the form

$$\begin{aligned}
\bar{J}_{2,t_2} &= \sup_{\boldsymbol{\pi}\in\mathcal{P}_{B2}} \mathbb{E}\left[U(T,W(T)) - \frac{1}{2}\int_0^T \|\mathbb{L}(\boldsymbol{\pi}(s))\|^2 ds\right] \\
&= \sup_{\boldsymbol{\pi}\in\mathcal{P}_{B2}} \mathbb{E}\left[U(t_4,W(t_4)) - \frac{1}{2}\sum_{i=1}^{3}(\mathbb{L}_{ii}(\pi_i(t_2)))^2\Delta t - \frac{1}{2}\sum_{i=1}^{3}\sum_{j\in\{0,1,3\}}(\mathbb{L}_{ii}(\pi_i(t_j)))^2\Delta t\right] \\
&= \ln W(t_2) + \sup_{\boldsymbol{\pi}\in\mathcal{P}_{B2}} \mathbb{E}\left[\sum_{k=2}^{3}\left\{\sum_{i=1}^{3}\xi_i(t_k)\pi_i(t_k) - \frac{1}{2}\sum_{j=1}^{4}\left(\sum_{i=1}^{3}\pi_i(t_k)\sigma_{ij}(t_k)\right)^2\right\}\Delta t\right. \\
&\quad + \sum_{k=2}^{3}\sum_{j=1}^{4}\sum_{i=1}^{3}\pi_i(t_k)\sigma_{ij}(t_k)(B_j(t_{k+1}) - B_j(t_k)) \\
&\quad \left. -\frac{1}{2}\sum_{i=1}^{3}(\mathbb{L}_{ii}(\pi_i(t_2)))^2\Delta t - \frac{1}{2}\sum_{i=1}^{3}\sum_{j\in\{0,1,3\}}(\mathbb{L}_{ii}(\pi_i(t_j)))^2\Delta t\right],
\end{aligned} \tag{4.97}$$

where $0 < W(t_2)$ almost surely has the form

$$\begin{aligned}
W(t_2) &= W(t_0)\exp\left(\sum_{k=0}^{1}\left\{\sum_{i=1}^{3}\xi_i(t_k)\pi_i(t_k) - \frac{1}{2}\sum_{j=1}^{4}\left(\sum_{i=1}^{3}\pi_i(t_k)\sigma_{ij}(t_k)\right)^2\right\}\Delta t\right. \\
&\quad \left. + \sum_{k=0}^{1}\sum_{j=1}^{4}\sum_{i=1}^{3}\pi_i(t_k)\sigma_{ij}(t_k)(B_j(t_{k+1}) - B_j(t_k))\right).
\end{aligned}$$

Since $0 < W(t_2)$ almost surely, from (4.97) without loss of generality, to compare different values of $\bar{J}_{2,t_2}$, we can assume that $W(t_2) = 1$. Then a portfolio which maximises $\bar{J}_{2,t_2}$ with $W(t_2) = 1$ will still maximise $\bar{J}_{2,t_2}$ for some $0 < W(t_2)$. All time-t_2 constrained portfolios are listed in Table 4.8. Only the portfolios which satisfy the constraints $\mathbf{a}(t_2) \leq \boldsymbol{\pi}(t_2) \leq \mathbf{b}(t_2)$, and their corresponding objective functional values, are listed in Table 4.9. In Table 4.8 there are only six time-t_2 constrained portfolios which satisfy the constraints $\mathbf{a}(t_2) \leq \boldsymbol{\pi}(t_2) \leq \mathbf{b}(t_2)$ (and these are listed in Table 4.9). From Table 4.9 we show below how we found the time-t_2 really good portfolio.

<u>Case 1:</u> $\boldsymbol{\pi}(t_2) = (0.2, 0.7, 0.1)$. Substitute $\boldsymbol{\pi}(t_2) = (0.2, 0.7, 0.1)$ into (4.97). Simulate different sample paths of $\mathbf{B}$ from time t_2 to t_4. For each sample path of $\mathbf{B}$ calculate $\boldsymbol{\pi}(t_3)$ using (4.94) with $\mathcal{C}^*(t_3)$ empty. If $\mathbf{a}(t_3) \not\leq \boldsymbol{\pi}(t_3)$ or $\boldsymbol{\pi}(t_3) \not\leq \mathbf{b}(t_3)$, then use the time-$t_3$ really good portfolio $(0.44, 0.1, 0.46)$ (which is then the time-t_3 constrained optimal portfolio for that sample path of $\mathbf{B}$) in the calculation of $\bar{J}_{2,t_2}$ in (4.97). By calculating the expectation in (4.97), we can evaluate $\bar{J}_{2,t_2}$ for $\boldsymbol{\pi}(t_2) = (0.2, 0.7, 0.1)$ and it has the value $\bar{J}_{2,t_2} = 13.870$.

⋮

Case 6: $\boldsymbol{\pi}(t_2) = (0.1, 0.7, 0.2)$. Substitute $\boldsymbol{\pi}(t_2) = (0.1, 0.7, 0.2)$ into (4.97). Simulate different sample paths of $\mathbf{B}$ from time t_2 to t_4. For each sample path of $\mathbf{B}$ calculate $\boldsymbol{\pi}(t_3)$ using (4.94) with $\mathcal{C}^*(t_3)$ empty. If $\mathbf{a}(t_3) \not\leq \boldsymbol{\pi}(t_3)$ or $\boldsymbol{\pi}(t_3) \not\leq \mathbf{b}(t_3)$, then use the time-$t_3$ really good portfolio $(0.44, 0.1, 0.46)$ in the calculation of $\bar{J}_{2,t_2}$ in (4.97). By calculating the expectation in (4.97), we can evaluate $\bar{J}_{2,t_2}$ for $\boldsymbol{\pi}(t_2) = (0.1, 0.7, 0.2)$ and it has the value $\bar{J}_{2,t_2} = 13.875$.

After doing this for each constrained portfolio in Table 4.9 we obtained the corresponding objective functional values shown in Table 4.9. The time-t_2 constrained portfolio of $\boldsymbol{\pi}(t_2) = (0.2, 0.1, 0.7)$ satisfied the constraints $\mathbf{a}(t_2) \leq \boldsymbol{\pi}(t_2) \leq \mathbf{b}(t_2)$ and had the largest objective functional value of $\bar{J}_{2,t_2} = 13.891$, thus it is the time-t_2 really good portfolio. The times t_0 and t_1 really good portfolios are calculated similarly. Thus for the realised sample paths of the Brownian motions $\mathbf{B}$ in Table 4.3, the constrained optimal portfolios in Table 4.4 are calculated as follows:

- Start at time t_0. Calculate the unconstrained portfolio at time t_0 using (4.94) with $\mathcal{C}^*(t_0)$ empty. See Table 4.5. If $\mathbf{a}(t_0) \not\leq \boldsymbol{\pi}(t_0)$ or $\boldsymbol{\pi}(t_0) \not\leq \mathbf{b}(t_0)$, then the time-$t_0$ constrained optimal portfolio is the time-t_0 really good portfolio.

- Evolve to time t_1. Calculate the unconstrained portfolio at time t_1 using (4.94) with $\mathcal{C}^*(t_1)$ empty. See Table 4.5. If $\mathbf{a}(t_1) \not\leq \boldsymbol{\pi}(t_1)$ or $\boldsymbol{\pi}(t_1) \not\leq \mathbf{b}(t_1)$, then the time-$t_1$ constrained optimal portfolio is the time-t_1 really good portfolio.

- Evolve to time t_2. Calculate the unconstrained portfolio at time t_2 using (4.94) with $\mathcal{C}^*(t_2)$ empty. See Table 4.5. If $\mathbf{a}(t_2) \not\leq \boldsymbol{\pi}(t_2)$ or $\boldsymbol{\pi}(t_2) \not\leq \mathbf{b}(t_2)$, then the time-$t_2$ constrained optimal portfolio is the time-t_2 really good portfolio.

- Evolve to time t_3. Calculate the unconstrained portfolio at time t_3 using (4.94) with $\mathcal{C}^*(t_3)$ empty. See Table 4.5. If $\mathbf{a}(t_3) \not\leq \boldsymbol{\pi}(t_3)$ or $\boldsymbol{\pi}(t_3) \not\leq \mathbf{b}(t_3)$, then the time-$t_3$ constrained optimal portfolio is the time-t_3 really good portfolio.

With respect to how long it took to derive the constrained optimal portfolios listed in Table 4.4, on a 3.00GHz CPU, 512MB RAM personal computer it took 40 seconds to complete. In the calculation of each really good portfolio 500 Brownian motion sample paths were generated.

4.4.7 Example 7

Recall equation (4.30) viz for all $0 \leq t \leq T, i \in \mathcal{N}_S$

$$-\mathbf{D}(t) = -\bar{\boldsymbol{\sigma}}(t)\boldsymbol{\pi}(t) + \mathbf{U}(t)\boldsymbol{\pi}''(t) + \mathbf{V}(t)\boldsymbol{\pi}'(t) + \mathbf{Y}(t)\boldsymbol{\pi}(t). \tag{4.98}$$

In this example we solve for $\boldsymbol{\pi}$ defined in (4.98) (for problem **(P2)** assuming the securities $\mathbf{S}$ are diffusions). We assume the insider filtration $\mathbb{H}$ has the form given in (4.95) above and that

$$U(t,x) \equiv \ln x, \ \ T=1, \ \ T_0=2, \ \ N_S=4, \ \ N_B=2.$$

We assume the coefficient functions of the operators $\mathbb{L}_{ii}, i=1,2,3$ are of the form

$$\bar{\kappa}_i(t) = \ln(2+t) \text{ and } \kappa_i(t) = \exp(0.00001t), \quad i=1,2,3, \ \ 0 \le t \le T$$

and we assume the initial values

$$\boldsymbol{\pi}(0) = \boldsymbol{\pi}_{BM} := (0.11, 0.16, 0.33, 0.40).$$

Let $0 = t_0 < t_1 < t_2 < t_3 < t_4 < t_5 = T$ be an equally spaced partition of the interval $[0,T]$. Then we also assume that the financial market parameters have the values given in Tables 4.10 and 4.11 and that the particular sample paths of the Brownian motions which are realised are those in Table 4.12. With $\mathcal{C}^*(t)$ empty for all $0 \le t \le T$ we use (4.98) to solve for unconstrained portfolios at times $t_j, j=0,\ldots,4$. We solve (4.98) numerically and so we approximate the derivatives $\boldsymbol{\pi}'(t_j), j=1,\ldots,4$ and $\boldsymbol{\pi}''(t_j), j=2,\ldots,4$ respectively as

$$\boldsymbol{\pi}'(t_j) = \frac{\boldsymbol{\pi}(t_j) - \boldsymbol{\pi}(t_{j-1})}{\Delta t} \quad \text{and} \tag{4.99}$$

$$\boldsymbol{\pi}''(t_j) = \frac{\boldsymbol{\pi}(t_j) - 2\boldsymbol{\pi}(t_{j-1}) + \boldsymbol{\pi}(t_{j-2})}{(\Delta t)^2}. \tag{4.100}$$

Substituting (4.99)-(4.100) into (4.98) it reduces to

$$\boldsymbol{\pi}(t_j) = \tilde{\mathbf{U}}(t_j)\left[-(\Delta t)^2\mathbf{D}(t_j) + \Delta t\mathbf{V}(t_j)\boldsymbol{\pi}(t_{j-1}) - \mathbf{U}(t_j)(-2\boldsymbol{\pi}(t_{j-1}) + \boldsymbol{\pi}(t_{j-2}))\right], \tag{4.101}$$

where

$$\begin{aligned}
\mathbf{D}(t_j) &:= \boldsymbol{\xi}(t_j) + \boldsymbol{\sigma}(t_j)\frac{d}{dt}\mathbf{H}(t_j) - \boldsymbol{\lambda}(t_j) + \boldsymbol{\mu}(t_j) - \bar{\boldsymbol{\mu}}(t_j),\\
\mathbf{U}(t_j) &:= (diag(\bar{\boldsymbol{\kappa}}(t_j)))^2,\\
\mathbf{V}(t_j) &:= 2 \times diag(\bar{\boldsymbol{\kappa}}(t_j))diag(\bar{\boldsymbol{\kappa}}'(t_j)),\\
\mathbf{Y}(t_j) &:= diag(\bar{\boldsymbol{\kappa}}(t_j))diag(\boldsymbol{\kappa}'(t_j)) + diag(\bar{\boldsymbol{\kappa}}'(t_j))diag(\boldsymbol{\kappa}(t_j)) - (diag(\boldsymbol{\kappa}(t_j)))^2,\\
\bar{\mathbf{U}}(t_j) &:= -(\Delta t)^2\bar{\boldsymbol{\sigma}}(t_j) + \mathbf{U}(t_j) + \Delta t\mathbf{V}(t_j) + (\Delta t)^2\mathbf{Y}(t_j) \quad \text{and}\\
\tilde{\mathbf{U}}(t_j) &\equiv [\tilde{U}_{ik}(t_j)] \ := \ \bar{\mathbf{U}}^{-1}(t_j).
\end{aligned}$$

We eliminate $\lambda(t_j)$ in (4.101) by substituting (4.101) into the unity weight constraint (4.12) to get that

$$(\Delta t)^2\lambda(t_j) = \hat{U}^{-1}(t_j) - \hat{U}^{-1}(t_j)\sum_{q=1}^{N_S}\sum_{n=1}^{N_S}\tilde{U}_{qn}(t_j)\left[-(\Delta t)^2\bar{D}_n(t_j)\right.$$

$$+\Delta t\sum_{p=1}^{N_S} V_{np}(t_j)\pi_p(t_{j-1}) - \sum_{p=1}^{N_S} U_{np}(t_j)(-2\pi_p(t_{j-1})+\pi_p(t_{j-2}))\Bigg], \tag{4.102}$$

where

$$\begin{aligned}\hat{U}(t_j) &:= \sum_{i=1}^{N_S}\sum_{k=1}^{N_S}\tilde{U}_{ik}(t_j) \quad \text{and}\\ \bar{\mathbf{D}}(t_j) \equiv [\bar{D}_k(t_j)] &:= \boldsymbol{\xi}(t_j)+\boldsymbol{\sigma}(t_j)\frac{d}{dt}\mathbf{H}(t_j)+\boldsymbol{\mu}(t_j)-\bar{\boldsymbol{\mu}}(t_j).\end{aligned}$$

Substituting (4.102) into (4.101) we get for all $t_j \in \{t_0,\ldots,t_4\}, i \in \mathcal{N}_S$ that

$$\pi_i(t_j) = \bar{V}_i(t_j)+C_i(t_j)-(\Delta t)^2\sum_{k\in\mathcal{C}^*}\tilde{U}_{ik}(t_j)\left(\mu_k^*(t_j)-\hat{U}^{-1}(t_j)\sum_{q=1}^{N_S}\sum_{n\in\mathcal{C}^*(t_j)}\tilde{U}_{qn}(t_j)\mu_n^*(t_j)\right), \tag{4.103}$$

where

$$\begin{aligned}\breve{V}_i(t_j) &:= \Delta t\sum_{p=1}^{N_S}V_{ip}(t_j)\pi_p(t_{j-1}),\\ \breve{U}_i(t_j) &:= \sum_{p=1}^{N_S}U_{ip}(t_j)(-2\pi_p(t_{j-1})+\pi_p(t_{j-2})),\\ \bar{V}_i(t_j) &:= \sum_{k=1}^{N_S}\tilde{U}_{ik}(t_j)\Big[\breve{V}_k(t_j)-\breve{U}_k(t_j)\\ &\quad +\hat{U}^{-1}(t_j)\left(1-\sum_{q=1}^{N_S}\sum_{n=1}^{N_S}\tilde{U}_{qn}(t_j)(\breve{V}_n(t_j)-\breve{U}_n(t_j))\right)\Big] \quad \text{and}\\ C_i(t_j) &:= -(\Delta t)^2\sum_{k=1}^{N_S}\tilde{U}_{ik}(t_j)\Bigg(\xi_k(t_j)+\sum_{p=1}^{N_B}\sigma_{kp}(t_j)\frac{d}{dt}H_p(t_j)\\ &\quad -\hat{U}^{-1}(t_j)\sum_{q=1}^{N_S}\sum_{n=1}^{N_S}\tilde{U}_{qn}(t_j)\left(\xi_n(t_j)+\sum_{p=1}^{N_B}\sigma_{np}(t_j)\frac{d}{dt}H_p(t_j)\right)\Bigg).\end{aligned}$$

We eliminate $\boldsymbol{\mu}^*(t_j)$ in (4.103) by substituting (4.103) into (4.23) to get that

$$\boldsymbol{\mu}^*(t_j) = \boldsymbol{\psi}^{-1}(t_j)\left[\bar{\mathbf{c}}(t_j)-\bar{\mathbf{V}}(t_j)-\mathbf{C}(t_j)\right], \tag{4.104}$$

where for all $\alpha_i \in \mathcal{C}^*(t_j)$, $\bar{\mathbf{V}}(t_j) \equiv [\bar{V}_{\alpha_i}(t_j)]$, $\mathbf{C}(t_j) \equiv [C_{\alpha_i}(t_j)]$ and

$$\begin{aligned}\psi_{ik}(t_j) &:= -(\Delta t)^2\left(\tilde{U}_{\alpha_i,\alpha_k}(t_j)-\hat{U}^{-1}(t_j)\sum_{q=1}^{N_S}\sum_{\alpha_l\in\mathcal{C}^*(t_j)}\tilde{U}_{\alpha_i,\alpha_l}(t_j)\tilde{U}_{q,\alpha_l}(t_j)\right),\\ \boldsymbol{\psi}(t_j) &\equiv [\psi_{ik}(t_j)] \quad \text{and}\\ \boldsymbol{\psi}^{-1}(t_j) &:= [\varsigma_{pq}(t_j)].\end{aligned}$$

t	0	0.2	0.4	0.6	0.8
a_1	0.1	0.1	0.1	0.1	0.1
a_2	0.1	0.1	0.1	0.1	0.1
a_3	0.1	0.1	0.1	0.1	0.1
a_4	0.1	0.1	0.1	0.1	0.1
b_1	0.7	0.7	0.7	0.7	0.7
b_2	0.7	0.7	0.7	0.7	0.7
b_3	0.7	0.7	0.7	0.7	0.7
b_4	0.7	0.7	0.7	0.7	0.7
ξ_1	0.0612	0.0636	0.0828	0.1017	0.1088
ξ_2	0.0104	0.0189	0.0471	0.0726	0.0691
ξ_3	0.0761	0.0714	0.0762	0.1037	0.1214
ξ_4	0.0548	0.0753	0.0680	0.0950	0.1058

Table 4.10: Time-t values of financial market parameters for securities $S_i, i \in \{1, 2, 3, 4\}$ for Example 7.

Substituting (4.104) into (4.103) we get that

$$\begin{aligned}\pi_i(t_j) \quad = \quad & \bar{V}_i(t_j) + C_i(t_j) - (\Delta t)^2 \sum_{k=1}^{m(t_j)} \tilde{U}_{ik}(t_j) \left(\sum_{p=1}^{m(t_j)} \varsigma_{kp}(t_j) \left(\bar{c}_p(t_j) - \bar{V}_p(t_j) - C_p(t_j) \right) \right. \\ & \left. - \hat{U}^{-1}(t_j) \sum_{q=1}^{N_S} \sum_{n=1}^{m(t_j)} \tilde{U}_{qn}(t_j) \sum_{p=1}^{m(t_j)} \varsigma_{np}(t_j) \left(\bar{c}_p(t_j) - \bar{V}_p(t_j) - C_p(t_j) \right) \right),\end{aligned} \tag{4.105}$$

where as in (4.25) $m(t_j)$ is the cardinality of $\mathcal{C}^*(t_j)$. From (3.43), (4.105) and the inputs in Tables 4.10, 4.11 and 4.12, we found that the constrained optimal portfolios are those given in Table 4.13. The unconstrained portfolios are those given in Table 4.14. From Table 4.14, time t_2 was the first time that one of the unconstrained portfolios violated the inequality constraints (4.4). Thus from Section 4.3 we need to find all really good portfolios from times t_4 to t_2. Note that the time-t_2 really good portfolio will in fact be the time-t_2 constrained optimal portfolio.

The time-t_4 really good portfolio was determined by considering at most 65 time-t_4 constrained portfolios and their objective functional values (3.43). From

t	$\boldsymbol{\sigma}$
0	$\begin{pmatrix} -0.4920 & -0.3710 \\ -0.2550 & -0.5477 \\ 0.0874 & -0.0743 \\ -0.2264 & -0.0957 \end{pmatrix}$
0.2	$\begin{pmatrix} 0.0940 & -0.5438 \\ -0.3642 & -0.0933 \\ -0.1573 & -0.5671 \\ 0.0257 & -0.5806 \end{pmatrix}$
0.4	$\begin{pmatrix} 0.0050 & -0.1558 \\ -0.2980 & -0.5310 \\ 0.0673 & -0.1561 \\ -0.0506 & -0.0564 \end{pmatrix}$
0.6	$\begin{pmatrix} 0.0404 & -0.5236 \\ 0.0227 & -0.2405 \\ 0.1424 & 0.1812 \\ -0.0190 & 0.1751 \end{pmatrix}$
0.8	$\begin{pmatrix} -0.3762 & -0.0378 \\ -0.4188 & -0.0550 \\ -0.4841 & -0.2338 \\ -0.5257 & -0.1741 \end{pmatrix}$

Table 4.11: Time-t volatility matrices of securities $S_i, i \in \{1, 2, 3, 4\}$ for Example 7.

t	0	0.2	0.4	0.6	0.8	1	2
B_1	0	-0.4658	-0.8529	-0.3567	0.1636	0.4157	0.0939
B_2	0	0.0801	0.1919	0.2658	0.8693	1.5430	1.3297

Table 4.12: Time-t values of the Brownian motions B_1, B_2 for Example 7.

t	0	0.2	0.4	0.6	0.8
$\boldsymbol{\pi}$	$\begin{pmatrix} 0.11 \\ 0.16 \\ 0.33 \\ 0.40 \end{pmatrix}$	$\begin{pmatrix} 0.11 \\ 0.16 \\ 0.33 \\ 0.40 \end{pmatrix}$	$\begin{pmatrix} 0.10 \\ 0.19 \\ 0.32 \\ 0.39 \end{pmatrix}$	$\begin{pmatrix} 0.13 \\ 0.10 \\ 0.35 \\ 0.42 \end{pmatrix}$	$\begin{pmatrix} 0.13 \\ 0.10 \\ 0.34 \\ 0.43 \end{pmatrix}$

Table 4.13: Time-t constrained optimal portfolios for Example 7.

t	0	0.2	0.4	0.6	0.8
$\boldsymbol{\pi}$	$\begin{pmatrix} 0.11 \\ 0.16 \\ 0.33 \\ 0.40 \end{pmatrix}$	$\begin{pmatrix} 0.11 \\ 0.16 \\ 0.33 \\ 0.40 \end{pmatrix}$	$\begin{pmatrix} 0.09 \\ 0.19 \\ 0.33 \\ 0.39 \end{pmatrix}$	$\begin{pmatrix} 0.09 \\ 0.21 \\ 0.32 \\ 0.38 \end{pmatrix}$	$\begin{pmatrix} 0.08 \\ 0.23 \\ 0.31 \\ 0.38 \end{pmatrix}$

Table 4.14: Time-t unconstrained portfolios for Example 7.

(3.43) the objective functional $\bar{J}_{2,t_4}(\boldsymbol{\pi})$ has the form

$$\begin{aligned}
\bar{J}_{2,t_4}(\boldsymbol{\pi}) &= \sup_{\boldsymbol{\pi}\in\mathcal{P}_{B2}} \mathbb{E}\left[U(T, W(T)) - \frac{1}{2}\int_0^T \|\mathbb{L}(\boldsymbol{\pi}(s))\|^2 ds\right] \\
&= \sup_{\boldsymbol{\pi}\in\mathcal{P}_{B2}} \mathbb{E}\left[U(t_5, W(t_5)) - \frac{1}{2}\sum_{i=1}^{4}(\mathbb{L}_{ii}(\pi_i(t_4)))^2\Delta t - \frac{1}{2}\sum_{i=1}^{4}\sum_{j=0}^{3}(\mathbb{L}_{ii}(\pi_i(t_j)))^2\Delta t\right] \\
&= \ln W(t_4) + \sup_{\boldsymbol{\pi}\in\mathcal{P}_{B2}} \mathbb{E}\left[\left\{\sum_{i=1}^{4}\xi_i(t_4)\pi_i(t_4) - \frac{1}{2}\sum_{j=1}^{2}\left(\sum_{i=1}^{4}\sigma_{ij}(t_4)\pi_i(t_4)\right)^2\right\}\Delta t\right. \\
&\quad \left. + \sum_{j=1}^{2}\sum_{i=1}^{4}\sigma_{ij}(t_3)\pi_i(t_4)(B_j(t_5) - B_j(t_4))\right] \\
&\quad -\frac{1}{2}\sum_{i=1}^{4}(\mathbb{L}_{ii}(\pi_i(t_4)))^2\Delta t - \frac{1}{2}\sum_{i=1}^{4}\sum_{j=0}^{3}(\mathbb{L}_{ii}(\pi_i(t_j)))^2\Delta t,
\end{aligned} \tag{4.106}$$

where $0 < W(t_4)$ almost surely has the form

$$\begin{aligned}
W(t_4) &= W(t_0)\exp\left(\sum_{k=0}^{3}\left\{\sum_{i=1}^{4}\xi_i(t_k)\pi_i(t_k) - \frac{1}{2}\sum_{j=1}^{2}\left(\sum_{i=1}^{4}\sigma_{ij}(t_k)\pi_i(t_k)\right)^2\right\}\Delta t\right. \\
&\quad \left. + \sum_{k=0}^{3}\sum_{j=1}^{2}\sum_{i=1}^{4}\sigma_{ij}(t_k)\pi_i(t_k)(B_j(t_{k+1}) - B_j(t_k))\right).
\end{aligned}$$

The time-t_4 really good portfolio is that which satisfies the constraints $\mathbf{a}(t_4) \leq \boldsymbol{\pi}(t_4) \leq \mathbf{b}(t_4)$ and has the largest value of $\bar{J}_{2,t_4}(\boldsymbol{\pi})$ in (4.106). Since $0 < W(t_4)$ almost surely, from (4.106) without loss of generality, to compare different values of $\bar{J}_{2,t_4}(\boldsymbol{\pi})$, we can assume that $W(t_4) = 1$. Then a portfolio which maximises $\bar{J}_{2,t_4}(\boldsymbol{\pi})$ with $W(t_4) = 1$ will still maximise $\bar{J}_{2,t_4}(\boldsymbol{\pi})$ for some $0 < W(t_4)$. To find the time-t_4 really good portfolio we need to calculate $\bar{J}_{2,t_4}(\boldsymbol{\pi})$ for at most 65 combinations of active and inactive inequality constraints (4.4). (Recall that a really good portfolio is clearly that with the least number of active inequality constraints (4.4), satisfies (4.4) and has the largest objective functional value. So the first portfolio which has these properties will be the time t_4 really good portfolio. Knowledge of this speeds up the computation.) There are sixty-four cases (constrained portfolios) to consider, however many of these will be eliminated by the fact that they do not satisfy the constraints $\mathbf{a}(t_4) \leq \boldsymbol{\pi}(t_4) \leq \mathbf{b}(t_4)$. We substitute each time-t_4 constrained portfolio into (4.106). We simulate different sample paths of $\mathbf{B}$ over $[t_4, t_5]$ and calculate the expectation, hence value of $\bar{J}_{2,t_4}(\boldsymbol{\pi})$, in (4.106). After doing this for each time-t_4 constrained portfolio we obtained objective functional values for each portfolio. We found that the constrained portfolio $\boldsymbol{\pi}(t_4) = (0.13, 0.10, 0.34, 0.43)$ was the time-t_4 really good portfolio since it satisfied the constraints (4.4) and had the largest objective functional value of 13.841. For the particular sample paths of $\mathbf{B}$ in Table 4.12, this really good portfolio turned out also to be the time-t_4 constrained optimal portfolio since the time-t_4 unconstrained portfolio didn't satisfy the constraints $\mathbf{a}(t_4) \leq \boldsymbol{\pi}(t_4) \leq \mathbf{b}(t_4)$.

With respect to how long it took to derive the constrained optimal portfolios listed in Table 4.13, on a 3.00GHz CPU, 512MB RAM personal computer it took 100 seconds to complete. In the calculation of each really good portfolio 10 000 Brownian motion sample paths were generated.

Chapter 5

Conclusion

In this thesis constrained portfolio selection problems were solved via the methods of stochastic dynamic programming and the calculus of variations. The difference between the two approaches is that in the former the state variables are required to be Markov processes, whereas in the latter the state variables need not be Markov.

The first constrained portfolio selection problem we solved involved the determination of constrained optimal investment in diffusions. It is a problem similar to that of Merton [101], but in our problem inequality constraints are also imposed on the portfolio security weights. Here, as in [101], we made a distinction between the solving the constrained portfolio selection problem with a money market security unavailable and available for investment and derived the optimality equations (2.41) and (2.66) respectively. We do this since a money market security has zero volatility and this will result in the covariance matrices $\bar{\boldsymbol{\sigma}}(t), 0 \leq t \leq T$ (of the security returns) being singular and the analysis in Section 2.3.3 then cannot be applied (if at least one of the securities has zero volatility). We defined a value functional J and proved that

(i) J is concave in the wealth level and initial security prices if the utility function $U(t, x)$ is concave in its second argument (Propositions 1 and 2),

(ii) for fixed $t \in [0, T]$ J is continuous in its spatial arguments (Corollary 3) and

(iii) J satisfies a homothetic property (Theorem 1).

Since the state variables $\mathbf{S}$ are Markov processes stochastic dynamic programming was used to solve the constrained portfolio selection problems. Since the covariance matrices $\boldsymbol{\sigma}$ are positive definite we used the Karush-Kuhn-Tucker conditions in the derivation of the corresponding Hamilton-Jacobi-Bellman (HJB) equations for J. These HJB equations are second order degenerate parabolic homogeneous partial differential equations with nonhomogeneous boundary conditions. Constrained optimal portfolios are given in feedback form in terms of the solution J of the HJB equations and its partial derivatives. As an example we solved a constrained portfolio selection numerically and showed the results in Section 2.5. We also conducted an analysis of the no-constraining (NC) region of a portfolio.

What we confirmed is that the disadvantage of using dynamic programming to solve (even unconstrained) optimisation problems is that the dimensionality of the (dynamic optimisation) problem increases disproportionately with the number of state variables making it very difficult to solve a practical optimisation problem. (In our problem the state variables are financial securities and our optimisation problem is a portfolio selection problem.) Moreover if the optimisation problems are constrained, then use of the Karush-Kuhn-Tucker (KKT) conditions further adds to the computational time (required to solve the constrained optimisation problems). Use of the KKT conditions to find constrained optimal portfolios involves the calculation of several different values of objective functional $J = J(t, W, \mathbf{S})$ at each time $t \in [0, T]$. By comparing these values of $J(t, W, \mathbf{S})$, a time-t constrained optimal portfolio is that which satisfies the inequality constraints $\mathbf{a}(t) \leq \boldsymbol{\pi}(t) \leq \mathbf{b}(t)$ and has the largest objective functional value $J(t, W, \mathbf{S})$. Due to the number of permutations, it is less computationally expensive to solve several low dimensional HJB equations than one high dimensional HJB equation. Thus for managers who implement a top-down approach in their asset allocation, use of dynamic programming may not be impractical. The first step in their asset allocation process involves an allocation between say domestic and international investments. For this decision there are only two state variables between which to optimise. It is relatively inexpensive to solve the resulting two (spatial) dimensional HJB equation. The next decision may involve optimal allocations between domestic equity, domestic bond, domestic cash and domestic property investments. This decision involves only four state variables and it will not be computationally expensive to solve the resulting four (spatial) dimensional HJB equation. With respect to the NC region analysis, we also found that under certain conditions analytical descriptions of no-constraining regions are possible. See Section 2.6.1.

The second constrained portfolio selection problem we solved involved the determination of constrained optimal investments (for an insider) in a financial market driven by Lévy processes. This work is an extension of the models in ([18], [42], [62]). From [62], by an *insider* in a financial market we mean an investor who possesses more information than the information generated by the disturbances in the financial market itself. An insider may be for example an executive or simply an employee of a company. [62] An *honest* investor can only use the filtration (or information) generated by the market itself if making an investment decision. An insider has a larger filtration available to him and uses this to make investment decisions. In reality insiders do not trade in the absence of market inefficiencies. The market inefficiency of portfolio security weight inequality constraints was considered in Chapters 3-4 since an insider may not be able to trade unconstrained units of some security. An application of this work is to improve the detection of insider trading. This work can also be extended to the pricing of contingent claims in the presence of investment constraints and where the amount of information agents have is important. As in Chapter 2 we also distinguish between solving the portfolio selection problems assuming a money market security is unavailable and available for investment (by the insider). The reasons for this are exactly the same as in the dynamic programming case, viz that (i) the invertibility of the covariance matrices are required for the tractability of the model and (ii) a money market security cannot be explicitly constrained if it is available for investment. If a money market security is available for investment, then it must be treated as a balancing security as is done in [42] in the one-dimensional case. In other words constrained optimal portfolio weights are determined for the risky securities and the money market security weight is defined so that the portfolio weights sum to unity.

Forward integration with respect to Brownian motion and Poisson random measures was defined and an Itô formula for functionals of forward Lévy processes was stated. From [57] a forward stochastic differential equation for the insider wealth process was derived. The Itô formula (for functionals of forward Lévy processes) was then used to derive an analytical formula for the wealth process. We defined the three constrained portfolio selection problems we wished to solve, viz

(i) find optimal portfolios for an insider with a strictly increasing, concave, at least once-differentiable utility function and absence of explicit weight constraints and absence of penalty functions,

(ii) find constrained optimal portfolios for an insider with logarithmic utility in the presence of explicit weight constraints and presence of penalty functions, but a money market security is not available for investment and

(iii) find constrained optimal portfolios for an insider with logarithmic utility in the presence of explicit weight constraints and presence of penalty functions and where a money market security is available for investment.

We considered two financial markets viz (i) where the risky securities are driven by diffusions and (ii) where the risky securities are driven by Lévy processes with jumps. If the securities exhibit jumps, then Malliavin calculus is required to solve the portfolio selection problems. If the securities $\mathbf{S}$ are continuous, then only classical differentiation (and not Malliavin calculus) is required to solve the portfolio selection problems. See Remark 2. We defined Malliavin differentiation and showed the relationship between the Malliavin derivative and forward Poisson integration. To get explicit solutions of constrained optimal portfolios we considered different forms of the ordinary differential penalty functions $\mathbb{L}$. We solved some examples analytically and showed that our constrained optimal portfolios reduced to those derived in ([18], [42], [62]). We also solved two examples numerically - see Sections 4.4.6-4.4.7.

With respect to future research we aim to extend the portfolio selection model in Chapter 2 to determine constrained optimal investments in financial markets driven by multidimensional Lévy processes (with jumps). We also aim to include transaction costs but possibly still keep the portfolio weights as the control variables. (In almost all portfolio selection models including transaction costs, the control variables are the security holdings not security weights). We aim to consider a portfolio selection model in which the prices of the securities depend on economic factors and we want to include exogenous constraints such as liquidity. We also aim to deal correctly with margined derivatives in our portfolio selection model. The difficulty in dealing with margined derivatives (in Chapter 2) is that the expected value of a margin call does not tend to zero as the time partition becomes finer. Now the differential generator $\mathcal{L}_1 = \lim_{\Delta t \to 0} \frac{\mathbb{E}[\Delta J]}{\Delta t}$ is used in equation (2.25). If $\mathbb{E}[\Delta J]$ contains any terms which do not tend to zero as Δt tends to zero (for example an expected margin call), then $\mathcal{L}_1$ will be infinite. A possible alternative is to use the *impulse control* framework [16] to model these margined derivatives and at the same time incorporate fixed transaction costs into our portfolio selection model. We also aim to include in our model stochastic interest rates and differentiate between rates for borrowing and lending.

To improve the portfolio selection model in Chapter 3, we aim to incorporate both fixed and proportional transaction costs. In follow up work we hope to prove that the discrete constrained optimal portfolios (required in Section 4.3) in fact tend to the continuous-time constrained optimal portfolios (derived in Sections 3.7.2, 3.7.3, 3.8.3 and 3.8.4) as the time increment tends to zero. Another possible improvement is to allow the insider to have knowledge of a *distribution* of time-T_0 values of the disturbances $\mathbf{B}$ and $\mathbf{q}$ instead of only time-T_0 values. We then aim to extend this to a term structure of future distributions of the disturbances $\mathbf{B}$ and $\mathbf{q}$.

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Appendix A

Definition of Karush-Kuhn-Tucker (KKT) Optimality Conditions

Consider the maximisation problem

$$
\begin{aligned}
&\max_{\mathbf{x}\in\mathcal{P}} f(\mathbf{x}) \\
\text{subject to}\quad g_1(\mathbf{x}) &\leq b_1 \\
&\vdots \\
g_m(\mathbf{x}) &\leq b_m,
\end{aligned}
\tag{A.1}
$$

where $N \in \mathbb{N}, \mathcal{P} \subseteq \mathbb{R}^N, \mathbf{x} := (x_1, \ldots, x_N) \in \mathcal{P}$, $b_i \in \mathbb{R}, i = 1, \ldots, m \in \mathbb{N}$ and the functions $f : \mathcal{P} \to \mathbb{R}$ and $g_i : \mathcal{P} \to \mathbb{R}$. Necessary conditions for the existence of an optimal solution of (A.1) are given in the following theorem.

Theorem 29 (KKT Necessary Optimality Conditions) *If* $\bar{\mathbf{x}} := (\bar{x}_1, \ldots, \bar{x}_N) \in \mathbb{R}^N$ *is an optimal solution of (A.1), then* $\bar{\mathbf{x}}$ *must satisfy the* m *constraints in (A.1) and there must exist multipliers* $\bar{\boldsymbol{\lambda}} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)$ *satisfying*

$$
\begin{aligned}
\frac{\partial f(\bar{\mathbf{x}})}{\partial x_j} - \sum_{i=1}^{m} \bar{\lambda}_i \frac{\partial g_i(\bar{\mathbf{x}})}{\partial x_j} &= 0, \qquad \text{for } j = 1, \ldots, N, \\
\bar{\lambda}_i [b_i - g_i(\bar{\mathbf{x}})] &= 0, \qquad \text{for } i = 1, \ldots, m, \\
\bar{\boldsymbol{\lambda}} &\geq \mathbf{0}.
\end{aligned}
$$

Proof: See [149]. ■

In the following theorem taken from [149] we give sufficient conditions for $\bar{\mathbf{x}}$ to be an optimal solution of (A.1).

Theorem 30 (KKT Sufficient Optimality Conditions) *If $f(\mathbf{x})$ is a concave function and for each $i \in \mathcal{N}_S$ the function $g_i(\mathbf{x})$ is a convex function, then any point $\bar{\mathbf{x}}$ satisfying the hypotheses of Theorem 29 is an optimal solution of (A.1).*

Appendix B

Derivation of (2.35)

If we write (2.30) in matrix form, then it becomes

$$0 = -\boldsymbol{\lambda} - \boldsymbol{\mu} + \bar{\boldsymbol{\mu}} + WJ_W\boldsymbol{\xi} + W^2 J_{WW}\bar{\boldsymbol{\sigma}}\boldsymbol{\pi}^* + \bar{\boldsymbol{\sigma}}\mathbf{M}W, \tag{B.1}$$

where $\mathbf{M} \equiv [M_i]$. Solving for $\boldsymbol{\pi}^*$ in (B.1) we find that

$$\boldsymbol{\pi}^* = \bar{\boldsymbol{\sigma}}^{-1}\left(\frac{\boldsymbol{\lambda} + \boldsymbol{\mu} - \bar{\boldsymbol{\mu}}}{W^2 J_{WW}} - \frac{J_W \boldsymbol{\xi}}{WJ_{WW}} - \frac{\bar{\boldsymbol{\sigma}}\mathbf{M}}{WJ_{WW}}\right)$$

which we can write in index form as

$$\pi_i^* = \sum_{k=1}^{N_S} \nu_{ki}\left(\frac{\lambda + \mu_k - \bar{\mu}_k}{W^2 J_{WW}} - \frac{J_W \xi_k}{WJ_{WW}} - \frac{\sum_{j=1}^{N_S} \bar{\sigma}_{kj} M_j}{WJ_{WW}}\right), \tag{B.2}$$

where $\bar{\boldsymbol{\sigma}}^{-1} \equiv [\nu_{ij}]$ and $\Gamma := \sum_{i=1}^{N_S}\sum_{j=1}^{N_S} \nu_{ij}$. Substituting (B.2) into (2.31) we find that

$$\frac{\lambda}{W^2 J_{WW}} = \frac{\Upsilon}{\Gamma} - \frac{1}{\Gamma}\sum_{m=1}^{N_S}\sum_{n=1}^{N_S} \nu_{nm}\left(\frac{\mu_n - \bar{\mu}_n}{W^2 J_{WW}} - \frac{J_W \xi_n}{WJ_{WW}} - \frac{\sum_{p=1}^{N_S} \bar{\sigma}_{np} M_p}{WJ_{WW}}\right). \tag{B.3}$$

Substituting (B.3) into (B.2) we obtain (2.35).

Appendix C

Derivation of (2.40)

If we substitute (2.37) into (2.39), then for all $i \in \mathcal{C}^*$ (the active constraints) we find that

$$\begin{aligned}(-1)^{-d_i} c_i &= \hat{G}_i + \frac{1}{W^2 J_{WW}} \left(\sum_{k \in \mathcal{C}^*} \nu_{ki} \mu_k^* - \frac{\nu_i}{\Gamma} \sum_{n \in \mathcal{C}^*} \nu_n \mu_n^* \right) + \hat{M}_i \\ &= \hat{G}_i + \frac{1}{W^2 J_{WW}} \Big[\mu_{\alpha_1}^* (\nu_{\alpha_1 i} - \frac{\nu_{\alpha_1} \nu_i}{\Gamma}) + \ldots \\ &\quad + \mu_{\alpha_j}^* (\nu_{\alpha_j i} - \frac{\nu_{\alpha_j} \nu_i}{\Gamma}) + \ldots + \mu_{\alpha_m}^* (\nu_{\alpha_m i} - \frac{\nu_{\alpha_m} \nu_i}{\Gamma}) \Big] + \hat{M}_i. \end{aligned} \tag{C.1}$$

With $\bar{c}_i := (-1)^{-d_i} c_i$ for all $i \in \mathcal{C}^*$ we can rewrite (C.1) in matrix form as

$$\bar{\mathbf{c}} = \hat{\mathbf{G}} + \frac{1}{W^2 J_{WW}} \boldsymbol{\Psi} \boldsymbol{\mu}^* + \hat{\mathbf{M}}, \tag{C.2}$$

where all vectors are of length m, $\boldsymbol{\Psi}$ is an $m \times m$ matrix with

$$\boldsymbol{\Psi} \equiv [\Psi_{jk}] := \left[\nu_{\alpha_k \alpha_j} - \frac{\nu_{\alpha_k} \nu_{\alpha_j}}{\Gamma} \right], \quad \boldsymbol{\Psi}^{-1} \equiv [\kappa_{jk}],$$

$$\bar{c}_i := (-1)^{-d_i} c_i, \quad \hat{\mathbf{G}} \equiv [\hat{G}_{\alpha_j}] \quad \text{and} \quad \hat{\mathbf{M}} \equiv [\hat{M}_{\alpha_j}].$$

Solving for $\boldsymbol{\mu}^*$ in (C.2) we find that

$$\boldsymbol{\mu}^* = W^2 J_{WW} \boldsymbol{\Psi}^{-1} \left[\bar{\mathbf{c}} - \hat{\mathbf{G}} - \hat{\mathbf{M}} \right].$$

Appendix D

Derivation of (2.41)

Substituting (2.40) into (2.37) we find that

$$\begin{aligned}
\pi_i^* &= \hat{G}_i + \hat{M}_i + \sum_{k,c\in\mathcal{C}^*} \nu_{ki}\kappa_{kc}\Bigg[\bar{c}_c - \hat{G}_c - \frac{J_W}{WJ_{WW}}\left(\frac{\nu_c}{\Gamma}\sum_{n=1}^{N_S}\nu_n\xi_n - \sum_{m=1}^{N_S}\nu_{mc}\xi_m\right) \\
&\quad -\frac{1}{WJ_{WW}}\left(\frac{\nu_c}{\Gamma}\sum_{p=1}^{N_S}M_p - M_c\right)\Bigg] - \frac{\nu_i}{\Gamma}\sum_{n,d\in\mathcal{C}^*}\nu_n\kappa_{nd}\Bigg[\bar{c}_d - \hat{G}_d \\
&\quad -\frac{J_W}{WJ_{WW}}\left(\frac{\nu_d}{\Gamma}\sum_{b=1}^{N_S}\nu_b\xi_b - \sum_{q=1}^{N_S}\nu_{qd}\xi_q\right) - \frac{1}{WJ_{WW}}\left(\frac{\nu_d}{\Gamma}\sum_{e=1}^{N_S}M_e - M_d\right)\Bigg] \\
&= \hat{G}_i + \hat{M}_i + R_i + \sum_{k,c\in\mathcal{C}^*}\nu_{ki}\kappa_{kc}\Bigg[-\frac{J_W}{WJ_{WW}}\left(\frac{\nu_c}{\Gamma}\sum_{n=1}^{N_S}\nu_n\xi_n - \sum_{m=1}^{N_S}\nu_{mc}\xi_m\right) \\
&\quad -\frac{1}{WJ_{WW}}\left(\frac{\nu_c}{\Gamma}\sum_{p=1}^{N_S}M_p - M_c\right)\Bigg] - \frac{\nu_i}{\Gamma}\bar{R} \\
&\quad -\frac{\nu_i}{\Gamma}\sum_{n,d\in\mathcal{C}^*}\nu_n\kappa_{nd}\Bigg[-\frac{J_W}{WJ_{WW}}\left(\frac{\nu_d}{\Gamma}\sum_{b=1}^{N_S}\nu_b\xi_b - \sum_{q=1}^{N_S}\nu_{qd}\xi_q\right) - \frac{1}{WJ_{WW}}\left(\frac{\nu_d}{\Gamma}\sum_{e=1}^{N_S}M_e - M_d\right)\Bigg] \\
&= \hat{G}_i + \hat{M}_i + R_i \underline{- \frac{J_W}{WJ_{WW}}\underbrace{\sum_{k,c\in\mathcal{C}^*}\nu_{ki}\kappa_{kc}\left(\frac{\nu_c}{\Gamma}\sum_{n=1}^{N_S}\nu_n\xi_n - \sum_{m=1}^{N_S}\nu_{mc}\xi_m\right)}_{=:N_i}} \\
&\quad \underline{\underline{-\frac{1}{WJ_{WW}}\underbrace{\sum_{k,c\in\mathcal{C}^*}\nu_{ki}\kappa_{kc}\left(\frac{\nu_c}{\Gamma}\sum_{p=1}^{N_S}M_p - M_c\right)}_{=:O_i}}} - \frac{\nu_i}{\Gamma}\bar{R}
\end{aligned}$$

$$
\begin{aligned}
&\underline{+\frac{J_W}{WJ_{WW}}\frac{\nu_i}{\Gamma}\underbrace{\sum_{n,d\in\mathcal{C}^*}\nu_n\kappa_{nd}\left(\frac{\nu_d}{\Gamma}\sum_{b=1}^{N_S}\nu_b\xi_b-\sum_{q=1}^{N_S}\nu_{qd}\xi_q\right)}_{=:\bar{N}}} \\
&\underline{\underline{+\frac{1}{WJ_{WW}}\frac{\nu_i}{\Gamma}\underbrace{\sum_{n,d\in\mathcal{C}^*}\nu_n\kappa_{nd}\left(\frac{\nu_d}{\Gamma}\sum_{e=1}^{N_S}M_e-M_d\right)}_{=:\bar{O}}}} \\
= \quad & \hat{G}_i+\hat{M}_i+R_i-\underline{\frac{J_W}{WJ_{WW}}N_i}-\underline{\underline{\frac{1}{WJ_{WW}}O_i}}-\frac{\nu_i}{\Gamma}\bar{R}+\underline{\frac{J_W}{WJ_{WW}}\frac{\nu_i}{\Gamma}\bar{N}}+\underline{\underline{\frac{1}{WJ_{WW}}\frac{\nu_i}{\Gamma}\bar{O}}} \\
= \quad & \hat{G}_i+\hat{M}_i+\underbrace{R_i-\frac{\nu_i}{\Gamma}\bar{R}}_{=:\hat{R}_i}-\frac{J_W}{WJ_{WW}}\underbrace{\left(N_i-\frac{\nu_i}{\Gamma}\bar{N}\right)}_{=:\hat{N}_i}-\frac{1}{WJ_{WW}}\underbrace{\left(O_i-\frac{\nu_i}{\Gamma}\bar{O}\right)}_{=:\hat{O}_i} \\
= \quad & C_i+E_i,
\end{aligned}
$$

where some of these variables are defined in (2.42).

Appendix E

Derivation of (2.44)

Substituting C_i and E_i (defined in (2.41) and (2.42) respectively) into (2.43) we find that

$$
\begin{aligned}
0 \quad = \quad & M^e + \sum_{i=1}^{N_S} \xi_i [\hat{R}_i - \frac{J_W \hat{N}_i}{W J_{WW}} - \frac{\hat{O}_i}{W J_{WW}}] W J_W \\
& + \frac{1}{2} \sum_{i=1}^{N_S} \sum_{j=1}^{N_S} \bar{\sigma}_{ij} \Bigg([\hat{R}_i - \frac{J_W \hat{N}_i}{W J_{WW}} - \frac{\hat{O}_i}{W J_{WW}}][\hat{G}_j + \frac{J_W \tilde{N}_j}{W J_{WW}} + \frac{\tilde{O}_j}{W J_{WW}}] \\
& + [\hat{G}_i + \frac{J_W \tilde{N}_i}{W J_{WW}} + \frac{\tilde{O}_i}{W J_{WW}}][\hat{R}_j - \frac{J_W \hat{N}_j}{W J_{WW}} - \frac{\hat{O}_j}{W J_{WW}}] \\
& + [\hat{R}_i - \frac{J_W \hat{N}_i}{W J_{WW}} - \frac{\hat{O}_i}{W J_{WW}}][\hat{R}_j - \frac{J_W \hat{N}_j}{W J_{WW}} - \frac{\hat{O}_j}{W J_{WW}}] \Bigg) W^2 J_{WW} \\
& + \sum_{i=1}^{N_S} \sum_{j=1}^{N_S} \bar{\sigma}_{ij} S_i [\hat{R}_j - \frac{J_W \hat{N}_j}{W J_{WW}} - \frac{\hat{O}_j}{W J_{WW}}] W J_{iW} \\
0 \quad = \quad & M^e + \sum_{i=1}^{N_S} \xi_i \hat{R}_i W J_W - \sum_{i=1}^{N_S} \xi_i \frac{J_W \hat{N}_i}{W J_{WW}} W J_W - \sum_{i=1}^{N_S} \xi_i \frac{\hat{O}_i}{W J_{WW}} W J_W \\
& + \frac{1}{2} \sum_{i=1}^{N_S} \sum_{j=1}^{N_S} \bar{\sigma}_{ij} \Bigg(\hat{R}_i \hat{G}_j + \hat{R}_i \frac{J_W \tilde{N}_j}{W J_{WW}} + \frac{\tilde{O}_j}{W J_{WW}} \hat{R}_i - \frac{J_W \hat{N}_i}{W J_{WW}} \hat{G}_j \\
& - \frac{J_W^2 \hat{N}_i}{W^2 J_{WW}^2} \tilde{N}_j - \frac{J_W}{W^2 J_{WW}^2} \hat{N}_i \tilde{O}_j - \frac{\hat{O}_i}{W J_{WW}} \hat{G}_j - \frac{J_W \hat{O}_i}{W^2 J_{WW}^2} \tilde{N}_j \\
& - \frac{\hat{O}_i}{W^2 J_{WW}^2} \tilde{O}_j + \hat{G}_i \hat{R}_j - \hat{G}_i \frac{J_W \hat{N}_j}{W J_{WW}} - \hat{G}_i \frac{\hat{O}_j}{W J_{WW}} \\
& + \frac{J_W \tilde{N}_i}{W J_{WW}} \hat{R}_j - \frac{J_W^2 \tilde{N}_i}{W^2 J_{WW}^2} \hat{N}_j - \frac{J_W \tilde{N}_i}{W^2 J_{WW}^2} \hat{O}_j + \frac{\tilde{O}_i}{W J_{WW}} \hat{R}_j
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{J_W\tilde{O}_i}{W^2J_{WW}^2}\hat{N}_j-\frac{\tilde{O}_i}{W^2J_{WW}^2}\hat{O}_j+\hat{R}_i\hat{R}_j-\hat{R}_i\frac{J_W\hat{N}_j}{WJ_{WW}}-\hat{R}_i\frac{\hat{O}_j}{WJ_{WW}} \\
& -\frac{J_W\hat{N}_i}{WJ_{WW}}\hat{R}_j+\frac{J_W^2\hat{N}_i}{W^2J_{WW}^2}\hat{N}_j+\frac{J_W\hat{N}_i}{W^2J_{WW}^2}\hat{O}_j \\
& \left.-\frac{\hat{O}_i}{WJ_{WW}}\hat{R}_j+\frac{J_W\hat{O}_i}{W^2J_{WW}^2}\hat{N}_j+\frac{\hat{O}_i}{W^2J_{WW}^2}\hat{O}_j\right)W^2J_{WW} \\
& +\sum_{i=1}^{N_S}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}S_i\hat{R}_jWJ_{iW}-\sum_{i=1}^{N_S}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}S_i\frac{J_W\hat{N}_j}{WJ_{WW}}WJ_{iW}-\sum_{i=1}^{N_S}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}S_i\frac{\hat{O}_j}{WJ_{WW}}WJ_{iW} \\
0 \quad = \quad & M^e+\underline{\sum_{i=1}^{N_S}\xi_i\hat{R}_iWJ_W}-\underline{\underline{\sum_{i=1}^{N_S}\xi_i\frac{J_W\hat{N}_i}{J_{WW}}J_W}}-\underline{\underline{\underline{\sum_{i=1}^{N_S}\xi_i\frac{\hat{O}_i}{J_{WW}}J_W}}} \\
& +\frac{1}{2}\sum_{i=1}^{N_S}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}\left(\underline{\underline{\underline{\hat{R}_i\hat{G}_jW^2J_{WW}}}}+\underline{W\hat{R}_iJ_W\tilde{N}_j}+\tilde{O}_jW\hat{R}_i-\underline{WJ_W\hat{N}_i\hat{G}_j}\right. \\
& -\underline{\underline{\frac{J_W^2\hat{N}_i}{J_{WW}}\tilde{N}_j}}-\underline{\underline{\underline{\frac{J_W}{J_{WW}}\hat{N}_i\tilde{O}_j}}}-W\hat{O}_i\hat{G}_j-\underline{\underline{\underline{\frac{J_W\hat{O}_i}{J_{WW}}\tilde{N}_j}}}-\frac{\hat{O}_i}{J_{WW}}\tilde{O}_j \\
& +\underline{\underline{\underline{\hat{G}_i\hat{R}_jW^2J_{WW}}}}-\underline{\hat{G}_iWJ_W\hat{N}_j}-\hat{G}_iW\hat{O}_j+\underline{WJ_W\tilde{N}_i\hat{R}_j}-\underline{\underline{\frac{J_W^2\tilde{N}_i}{J_{WW}}\hat{N}_j}} \\
& -\underline{\underline{\underline{\frac{J_W\tilde{N}_i}{J_{WW}}\hat{O}_j}}}+W\tilde{O}_i\hat{R}_j-\underline{\underline{\underline{\frac{J_W\tilde{O}_i}{J_{WW}}\hat{N}_j}}}-\frac{\tilde{O}_i}{J_{WW}}\hat{O}_j+\underline{\underline{\underline{\hat{R}_i\hat{R}_jW^2J_{WW}}}} \\
& -\underline{\hat{R}_iWJ_W\hat{N}_j}-\hat{R}_iW\hat{O}_j-\underline{WJ_W\hat{N}_i\hat{R}_j}+\underline{\underline{\frac{J_W^2\hat{N}_i}{J_{WW}}\hat{N}_j}}+\underline{\underline{\underline{\frac{J_W\hat{N}_i}{J_{WW}}\hat{O}_j}}} \\
& \left.-W\hat{O}_i\hat{R}_j+\underline{\underline{\underline{\frac{J_W\hat{O}_i}{J_{WW}}\hat{N}_j}}}+\frac{\hat{O}_i}{J_{WW}}\hat{O}_j\right) \\
& +\sum_{i=1}^{N_S}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}S_i\hat{R}_jWJ_{iW}-\underline{\underline{\sum_{i=1}^{N_S}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}S_i\frac{J_W\hat{N}_j}{J_{WW}}J_{iW}}}-\sum_{i=1}^{N_S}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}S_i\frac{\hat{O}_j}{J_{WW}}J_{iW} \\
0 \quad = \quad & M^e+W\sum_{i,j=1}^{N_S}\bar{\sigma}_{ij}\left(S_i\hat{R}_jJ_{iW}+\frac{1}{2}Q_{ij}^O\right)+WJ_W\sum_{i=1}^{N_S}\left(\xi_i\hat{R}_i+\frac{1}{2}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}Q_{ij}^N\right) \\
& -\frac{1}{J_{WW}}\sum_{i,j=1}^{N_S}\bar{\sigma}_{ij}\left(S_i\hat{O}_jJ_{iW}-\frac{1}{2}P_{ij}^O\right)-\frac{J_W^2}{J_{WW}}\sum_{i=1}^{N_S}\left(\xi_i\hat{N}_i-\frac{1}{2}\sum_{j=1}^{N_S}\bar{\sigma}_{ij}P_{ij}^N\right) \\
& -\frac{J_W}{J_{WW}}\sum_{i=1}^{N_S}\left(\xi_i\hat{O}_i+\sum_{j=1}^{N_S}\bar{\sigma}_{ij}\left(S_i\hat{N}_jJ_{iW}-\frac{1}{2}N_{ij}^O\right)\right)+\frac{1}{2}W^2J_{WW}\sum_{i,j=1}^{N_S}\bar{\sigma}_{ij}G_{ij}^R.
\end{aligned}
$$

Appendix F

Derivation of (3.108)

Let $\boldsymbol{\pi}, \boldsymbol{\theta} \in \mathcal{P}_{L1}$ be two admissible portfolios. Let $\delta > 0$ and let $\boldsymbol{\delta}$ be a $N_S \times 1$ vector with each of its elements equal to δ. For each $i \in \mathcal{N}_S$ let $y_i \in (-\delta, \delta)$ and let $\mathbf{y} := (y_1, \ldots, y_{N_S})$. For all $0 \leq t \leq T, \mathbf{z} \in \mathbb{R}^N, j \in \mathcal{N}_q, \mathbf{y} \in (-\boldsymbol{\delta}, \boldsymbol{\delta})$ let

$$\bar{G}_j(t, \mathbf{z}, \mathbf{y}) \quad := \quad g_{0,j}(t, \mathbf{z}) + \sum_{k=1}^{N_S} \hat{g}_{kj}(t, \mathbf{z})(\pi_k(t) + y_k \theta_k(t)),$$

where $g_{ij}, i \in \mathcal{N}_S, i \in \mathcal{N}_q$ are the jump coefficients defined in (3.14)-(3.15). From Remark 5(iv) for all $j \in \mathcal{N}_q, \mathbf{y} \in (-\boldsymbol{\delta}, \boldsymbol{\delta})$ the function $\ln(1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y})) \in \mathbb{M}_{1,2}$. Thus we have from Definition 17(iv)(a), Remark 5(ii), Definition 17(iv)(e), Lemma 1(iv), Lemma 1(iii) and Lemma 1(v) that[1]

$$\begin{aligned}
&\left[\frac{\partial}{\partial y_i} \mathcal{D}_{t^+,\mathbf{z}}[\ln(1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y}))]\right]_{\mathbf{y}=\mathbf{0}} \\
&= \left[\frac{\partial}{\partial y_i}\left[\ln\left(1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y}) + \mathcal{D}_{t^+,\mathbf{z}}[\bar{G}_j(t, \mathbf{z}, \mathbf{y})]\right) - \ln(1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y}))\right]\right]_{\mathbf{y}=\mathbf{0}} \\
&= \left[\frac{\hat{g}_{ij}(t, \mathbf{z})\theta_i(t) + \frac{\partial}{\partial y_i}\left[\sum_{k=1}^{N_t} \hat{g}_{kj}(t, \mathbf{z})(\mathcal{D}_{t^+,\mathbf{z}}[\pi_k(t)] + y_k \mathcal{D}_{t^+,\mathbf{z}}[\theta_k(t)])\right]}{1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y}) + \mathcal{D}_{t^+,\mathbf{z}}[\bar{G}_j(t, \mathbf{z}, \mathbf{y})]} - \frac{\hat{g}_{ij}(t, \mathbf{z})\theta_i(t)}{1 + \bar{G}_j(t, \mathbf{z}, \mathbf{y})}\right]_{\mathbf{y}=\mathbf{0}} \\
&= \frac{\hat{g}_{ij}(t, \mathbf{z})\theta_i(t) + \mathcal{D}_{t^+,\mathbf{z}}[\hat{g}_{ij}(t, \mathbf{z})\theta_i(t)]}{1 + G_j(t, \mathbf{z}) + \mathcal{D}_{t^+,\mathbf{z}}[G_j(t, \mathbf{z})]} - \frac{\hat{g}_{ij}(t, \mathbf{z})\theta_i(t)}{1 + G_j(t, \mathbf{z})}.
\end{aligned} \tag{F.1}$$

Using Lemma 1(iv) with $f(x) = (1 + x)^{-1}$ and $X = G_j(t, \mathbf{z}), j \in \mathcal{N}_q$, we have that for all $0 \leq t \leq T, \mathbf{z} \in \mathbb{R}^N$

$$(1 + G_j(t, \mathbf{z}) + \mathcal{D}_{t^+,\mathbf{z}}[G_j(t, \mathbf{z})])^{-1} = \mathcal{D}_{t^+,\mathbf{z}}[(1 + G_j(t, \mathbf{z}))^{-1}] + (1 + G_j(t, \mathbf{z}))^{-1}. \tag{F.2}$$

[1] Recall that $\mathcal{D}_{t^+,\mathbf{z}}(\hat{g}_{kj}(t, \mathbf{z})) = 0$ from Lemma 1(v).

Thus from (F.2) equation (F.1) reduces to

$$\begin{aligned}
&\left[\frac{\partial}{\partial y_i}\mathcal{D}_{t^+,\mathbf{z}}[\ln(1+\bar{G}_j(t,\mathbf{z},\mathbf{y}))]\right]_{\mathbf{y}=\mathbf{0}} \\
&= \left(\hat{g}_{ij}(t,\mathbf{z})\theta_i(t) + \mathcal{D}_{t^+,\mathbf{z}}[\hat{g}_{ij}(t,\mathbf{z})\theta_i(t)]\right)\left(\mathcal{D}_{t^+,\mathbf{z}}\left[\frac{1}{1+G_j(t,\mathbf{z})}\right]\right. \\
&\quad \left.+\frac{1}{1+G_j(t,\mathbf{z})}\right) - \frac{\hat{g}_{ij}(t,\mathbf{z})\theta_i(t)}{1+G_j(t,\mathbf{z})} \\
&= \hat{g}_{ij}(t,\mathbf{z})\theta_i(t)\mathcal{D}_{t^+,\mathbf{z}}\left[\frac{1}{1+G_j(t,\mathbf{z})}\right] + \mathcal{D}_{t^+,\mathbf{z}}[\hat{g}_{ij}(t,\mathbf{z})\theta_i(t)]\mathcal{D}_{t^+,\mathbf{z}}\left[\frac{1}{1+G_j(t,\mathbf{z})}\right] \\
&\quad +\mathcal{D}_{t^+,\mathbf{z}}[\hat{g}_{ij}(t,\mathbf{z})\theta_i(t)]\frac{1}{1+G_j(t,\mathbf{z})}.
\end{aligned} \tag{F.3}$$

Using Lemma 1(iii) with $X = \hat{g}_{ij}(t,\mathbf{z})\theta_i(t)$ and $Y = \frac{1}{1+G_j(t,\mathbf{z})}, i \in \mathcal{N}_S, j \in \mathcal{N}_q$ we have that

$$\begin{aligned}
\mathcal{D}_{t^+,\mathbf{z}}\left[\frac{\hat{g}_{ij}(t,\mathbf{z})\theta_i(t)}{1+G_j(t,\mathbf{z})}\right] &= \hat{g}_{ij}(t,\mathbf{z})\theta_i(t)\mathcal{D}_{t^+,\mathbf{z}}\left[\frac{1}{1+G_j(t,\mathbf{z})}\right] + \frac{1}{1+G_j(t,\mathbf{z})}\mathcal{D}_{t^+,\mathbf{z}}[\hat{g}_{ij}(t,\mathbf{z})\theta_i(t)] \\
&\quad +\mathcal{D}_{t^+,\mathbf{z}}[\hat{g}_{ij}(t,\mathbf{z})\theta_i(t)]\mathcal{D}_{t^+,\mathbf{z}}\left[\frac{1}{1+G_j(t,\mathbf{z})}\right].
\end{aligned} \tag{F.4}$$

Substituting (F.4) into (F.3) it reduces to (3.108).

Appendix G

Why (4.34) is solved discretely

Now equations (4.30) and (4.34) are equivalent - (4.34) is (4.30) converted to a system of first order differential equations. For ease of reference we repeat equation (4.30) viz

$$-\mathbf{D}(t) = -\bar{\boldsymbol{\sigma}}(t)\boldsymbol{\pi}(t) + \mathbf{U}(t)\boldsymbol{\pi}''(t) + \mathbf{V}(t)\boldsymbol{\pi}'(t) + \mathbf{Y}(t)\boldsymbol{\pi}(t) \tag{G.1}$$

where the functions in (G.1) are defined in (4.31). We show why it is not possible to solve analytically for $\boldsymbol{\pi}$ (in particular eliminating the multipliers $\lambda, \boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$) in (G.1). For simplicity we also assume that the financial market parameters are constant - the simplest form these can have. In this case (G.1) reduces to

$$-\mathbf{D}(t) = -\bar{\boldsymbol{\sigma}}\boldsymbol{\pi}(t) + (diag(\bar{\boldsymbol{\kappa}}))^2\boldsymbol{\pi}''(t). \tag{G.2}$$

We use *variation of parameters* [136] to solve (G.2). First we solve the homogeneous form of (G.2) viz

$$0 = -\bar{\boldsymbol{\sigma}}\boldsymbol{\pi}(t) + (diag(\bar{\boldsymbol{\kappa}}))^2\boldsymbol{\pi}''(t). \tag{G.3}$$

We convert (G.3) to a system of first order ordinary differential equations by defining

$$\bar{\boldsymbol{\pi}} = \boldsymbol{\pi}'. \tag{G.4}$$

Then (G.3) can be rewritten as

$$\begin{pmatrix} \bar{\boldsymbol{\pi}} \\ \boldsymbol{\pi} \end{pmatrix}' = \begin{pmatrix} & & \mathbf{0} & \mathbf{K}_1 & & \\ 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\pi}} \\ \boldsymbol{\pi} \end{pmatrix} =: \mathbf{K}_2 \begin{pmatrix} \bar{\boldsymbol{\pi}} \\ \boldsymbol{\pi} \end{pmatrix}, \tag{G.5}$$

where $\mathbf{K}_1 := \bar{\boldsymbol{\sigma}} \times (diag(\bar{\boldsymbol{\kappa}}))^{-2}$ and $\mathbf{0}$ is an $N_S \times N_S$ matrix of zeros and the defined matrix $\mathbf{K}_2$ has size $2N_S \times 2N_S$. We look for a solution of (G.5) of the form

$$\begin{pmatrix} \bar{\boldsymbol{\pi}} \\ \boldsymbol{\pi} \end{pmatrix} = \hat{\mathbf{v}} e^{ut}, \tag{G.6}$$

where u and $\hat{\mathbf{v}}$ are a scalar and constant vector respectively. Differentiating (G.6) with respect to t and using (G.5) we have that

$$\begin{pmatrix} \bar{\boldsymbol{\pi}} \\ \boldsymbol{\pi} \end{pmatrix}' = u\hat{\mathbf{v}} e^{ut} = u \begin{pmatrix} \bar{\boldsymbol{\pi}} \\ \boldsymbol{\pi} \end{pmatrix} = \mathbf{K}_2 \begin{pmatrix} \bar{\boldsymbol{\pi}} \\ \boldsymbol{\pi} \end{pmatrix}. \tag{G.7}$$

In (G.7) we have an eigenvalue problem with u and $\hat{\mathbf{v}}$ the eigenvalue and eigenvector respectively. Solving (G.7) we find that the $2N_S$ eigenvalues and eigenvectors are $u_1, \ldots, u_{2N_S}$ and $\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_{2N_S}$ respectively. Assuming the eigenvalues are distinct, the solution of the homogeneous problem (G.3) is then a linear combination of its independent solutions viz

$$\begin{pmatrix} \bar{\boldsymbol{\pi}} \\ \boldsymbol{\pi} \end{pmatrix} = \sum_{i=1}^{2N_S} h_i \hat{\mathbf{v}}_i e^{u_i t},$$

where $\mathbf{h} := (h_1, \ldots, h_{2N_S}) \in \mathbb{R}^{2N_S}$, the ith eigenvector $\hat{\mathbf{v}}_i := \begin{pmatrix} \bar{\mathbf{v}}_i \\ \mathbf{v}_i \end{pmatrix}$ and $\mathbf{v}_i, \bar{\mathbf{v}}_i \in \mathbb{R}^{N_S}$ giving

$$\boldsymbol{\pi}(t) = \sum_{i=1}^{2N_S} h_i \mathbf{v}_i e^{u_i t}.$$

In (G.7) if any of the eigenvalues $u_i, i \in \{1, \ldots, 2N_S\}$ are repeated, resulting in the solutions $e^{u_1 t}, \ldots, e^{u_{2N_S} t}$ not being independent, then we multiply the jth repeated solution by the $(j-1)$st power of the independent variable t. For example, the second (third) repeated solution is multiplied by t (t^2).

Continuing, we now look for a particular solution of (G.2) of the form

$$\boldsymbol{\pi}(t) = \sum_{i=1}^{2N_S} h_i(t) \mathbf{v}_i e^{u_i t}. \tag{G.8}$$

Differentiating (G.8) with respect to t we have that

$$\boldsymbol{\pi}'(t) = \sum_{i=1}^{2N_S} h_i'(t) \mathbf{v}_i e^{u_i t} + \sum_{i=1}^{2N_S} h_i(t) \mathbf{v}_i u_i e^{u_i t}. \tag{G.9}$$

Standard in the variation of parameters is to simplify the calculations by imposing the condition

$$\sum_{i=1}^{2N_S} h_i'(t) \mathbf{v}_i e^{u_i t} = 0. \tag{G.10}$$

Then substituting (G.10) into (G.9) and differentiating (G.9) with respect to t, we have that

$$\boldsymbol{\pi}''(t) = \sum_{i=1}^{2N_S} h_i'(t)\mathbf{v}_i u_i e^{u_i t} + \sum_{i=1}^{2N_S} h_i(t)\mathbf{v}_i u_i^2 e^{u_i t}. \tag{G.11}$$

Substituting (G.8) and (G.11) into (G.2) we get that

$$-(diag(\bar{\boldsymbol{\kappa}}))^{-2}\mathbf{D}(t) \quad = \quad \sum_{i=1}^{2N_S} h_i'(t)\mathbf{v}_i u_i e^{u_i t}, \tag{G.12}$$

since each solution $\mathbf{v}_i e^{u_i t}$ solves the homogeneous equation (G.3). In (G.10) and (G.12) we have a system of $2N_S$ equations for $2N_S$ unknowns $h_i'(t), i = 1, \ldots, 2N_S$. Putting (G.10) and (G.12) into matrix form we get that

$$\mathbf{F}(t)\mathbf{h}'(t) \quad = \quad \begin{pmatrix} -(diag(\bar{\boldsymbol{\kappa}}))^{-2}\mathbf{D}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{and} \tag{G.13}$$

$$\mathbf{h}(t) \quad = \quad \mathbf{h}(0) + \int_0^t \mathbf{F}^{-1}(s) \begin{pmatrix} -(diag(\bar{\boldsymbol{\kappa}}))^{-2}\mathbf{D}(s) \\ 0 \\ \vdots \\ 0 \end{pmatrix} ds. \tag{G.14}$$

In (G.13)-(G.14) the $2N_S$ constants $\mathbf{h}(0)$ are determined by using some set of initial conditions on $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$. The variable $(\mathbf{v}_j)_i$ is the ith element of the column vector $\mathbf{v}_j$ and the $2N_S \times 2N_S$ matrix $\mathbf{F}(t) = [F_{ij}(t)]$ is defined as

$$F_{ij}(t) := \begin{cases} (\mathbf{v}_j)_i e^{u_i t} & \text{if} \quad 1 \le i \le N_S \\ (\mathbf{v}_j)_i u_i e^{u_i t} & \text{if} \quad N_S + 1 \le i \le 2N_S. \end{cases}$$

Thus substituting (G.14) into (G.8) it reduces to

$$\boldsymbol{\pi}(t) = \sum_{i=1}^{2N_S} \left(h_i(0) + \int_0^t v_i^D(s)ds \right) \mathbf{v}_i e^{u_i t}, \tag{G.15}$$

where for all $0 \le t \le T$ the vector

$$(v_1^D(t), \ldots, v_{2N_S}^D(t)) \quad =: \quad \mathbf{v}^D(t) \quad := \quad \mathbf{F}^{-1}(t) \begin{pmatrix} -(diag(\bar{\boldsymbol{\kappa}}))^{-2}\mathbf{D}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

In (G.15) we want to eliminate the multiplier $\lambda(t)$ which is contained in the expression $\mathbf{D}(t)$. Substituting (G.15) into the unity weight constraint (4.12) we get that

$$1 = \sum_{j=1}^{N_S} \sum_{i=1}^{2N_S} \left(h_i(0) + \int_0^t v_i^D(s) ds \right) (\mathbf{v}_i)_j e^{u_i t}. \tag{G.16}$$

We encounter two obstacles if we try to solve for $\lambda(t)$ in (G.16). First, (G.16) is an integral equation, thus in general the function λ cannot be uniquely determined. Even if we differentiate (G.16) with respect to t, the integral term will still be present. Second we need to know at least all values $\lambda(s), 0 \leq s \leq t$ to evaluate the integral in (G.16). In discrete time this is not a problem, but in continuous time it is. This same problem is encountered if we try to eliminate any multipliers $\boldsymbol{\mu}(t)$ or $\bar{\boldsymbol{\mu}}(t)$ from (G.15). This same problem is also encountered if we consider penalty functions of the more general form (4.28). So if any penalty function $\mathbb{L}_{ij}$ is differential and at least one portfolio weight constraint is active, then the only way we have been able to solve for constrained optimal portfolios $\boldsymbol{\pi}(t)$ is to solve the differential equation (G.2) numerically. Solving variational problems in discrete time is quite common - see for example ([39], Chapter 1).

We use a simple first order numerical scheme to solve (G.1). With respect to why we can't use a more sophisticated numerical scheme, like for example the standard Runge-Kutta scheme, to solve (4.34) numerically, the problem is the following. If we employed the Runge-Kutta scheme, then at each time t_j for $i = 1, \ldots, 2N_S$ we would need to calculate the variables

$$\begin{aligned}
k_{1,i}^j &:= \Delta t f_i(t_j, \hat{\pi}_{1,j}, \hat{\pi}_{2,j}, \ldots, \hat{\pi}_{2N_S,j}), \\
k_{2,i}^j &:= \Delta t f_i \left(t_j + \frac{1}{2}\Delta t, \hat{\pi}_{1,j} + \frac{1}{2}k_{1,1}^j, \hat{\pi}_{2,j} + \frac{1}{2}k_{1,2}^j, \ldots, \hat{\pi}_{2N_S,j} + \frac{1}{2}k_{1,2N_S}^j, \right), \\
k_{3,i}^j &:= \Delta t f_i \left(t_j + \frac{1}{2}\Delta t, \hat{\pi}_{1,j} + \frac{1}{2}k_{2,1}^j, \hat{\pi}_{2,j} + \frac{1}{2}k_{2,2}^j, \ldots, \hat{\pi}_{2N_S,j} + \frac{1}{2}k_{2,2N_S}^j, \right), \\
k_{4,i}^j &:= \Delta t f_i \left(t_j + \Delta t, \hat{\pi}_{1,j} + k_{3,1}^j, \hat{\pi}_{2,j} + k_{3,2}^j, \ldots, \hat{\pi}_{2N_S,j} + k_{3,2N_S}^j, \right).
\end{aligned}$$

From the Runge-Kutta scheme we then have that

$$\hat{\pi}_{i,j+1} = \hat{\pi}_{i,j} + \frac{1}{6}[k_{1,i}^j + 2k_{2,i}^j + 2k_{3,i}^j + k_{4,i}^j]. \tag{G.17}$$

The problem is that calculation of the variables $k_{1,i}^j, k_{2,i}^j, k_{3,i}^j, k_{4,i}^j$ involve evaluation of $\mathbf{f}$ at two time values viz $t_j + \frac{1}{2}\Delta t$ and $t_j + \Delta t$. This introduces into the calculation of the variables $\hat{\boldsymbol{\pi}}_{j+1}$, the N_S+1 unknowns $\lambda(t_j+\frac{1}{2}\Delta t)$ and $\boldsymbol{\mu}^*(t_j+\frac{1}{2}\Delta t)$. The system of $3N_S + 1$ equations (4.12), (4.23) and (G.17) is then an underdetermined system for the $4N_S + 2$ unknowns $\hat{\boldsymbol{\pi}}_{j+1}, \lambda_{j+1}, \boldsymbol{\mu}_{j+1}^*, \lambda(t_j + \frac{1}{2}\Delta t)$ and $\boldsymbol{\mu}^*(t_j + \frac{1}{2}\Delta t)$. If the Runge-Kutta numerical scheme or any other scheme can be adjusted so that evaluation of the functions $\mathbf{f}$ take place only at times $\{t_0, \ldots, t_n\}$ of the partition, then we can use this scheme to solve (4.34) more efficiently for a constrained optimal portfolio $\boldsymbol{\pi}(t_{j+1})$.