Computation of Greeks using Malliavin calculus

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Abstract

The valuation of derivatives (in the classical sense) of financial options with respect to different parameters appearing in the underlying asset requires differentiating real-valued functions f of the random price process X_t, namely $f(X_t)$. In many cases, f is not differentiable. This thesis includes an introduction to the Malliavin calculus machinery which is used in the valuation. The main purpose is to derive explicit formulae for Greeks of a wider class of options using Malliavin calculus. The Malliavin calculus deals with the differentiation and integration of fairly general random variables and by using the integration by parts formula it avoids the need to differentiate payoff functions. It also does not require explicit knowledge of the density of the underlying asset. We first review the calculations of Greeks using Malliavin calculus in the Brownian motion case. Then we derive explicit formulae for Greeks in the form of expectations, under the risk neutral probability measure, of the option payoff multiplied by a weight function, namely $\mathbb{E}[f(X_t)\pi]$ for some weight function π, in the jump diffusion case. We also derive some explicit formulae in the case of stochastic volatility and some variants of it which include jumps in the price and variance processes. Furthermore, we obtain an expression for the Malliavin derivative of pure jump Lévy stochastic differential equations in terms of its first variation process. Then we give the necessary and sufficient conditions for a function to serve as a weight function in the pure jump case. Working in the white noise setting, we review the extension of the domain of the Malliavin derivative to the whole L^2 in both the pure diffusion and pure jump cases. Using the Donsker delta function of a pure diffusion process and a pure jump process we derive explicit formulae for Δ. In this way, we can compute Greeks in great generality. All the formulae obtained can then be used to evaluate the Greeks by Monte Carlo methods which are well-established.

Declaration

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Dedication

To my late father.

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Contents

Chapter 1

Introduction

Investors make predictions on how option prices vary over a certain period of time basing on the past market events. In general, the vast number of different market events makes it a difficult task. It is therefore, important to understand which factors contribute to the movement of prices and with what effect. This sensitivity analysis is carried over parameters appearing in models for price dynamics and the so-called Greeks represent a form of measure for price sensitivity to some factors. The name Greek is used because of market practice of using names of Greek letters (real and invented) to represent these risk parameters (see [49]).

A Greek is a derivative (in the classical sense) of a financial quantity, usually an option price, with respect to one of the parameters of the model. Hence, the Greeks measure the stability of the financial quantity under variations of the parameters. Practitioners need to have a well developed intuition of the dependence of their position on the movements and events in the market. Greeks are also useful for hedging and risk-management purposes (see [49]). Thus, there is a need to efficiently compute Greeks.

We define the Greek in mathematical terms as follows. We consider a general Itô diffusion process $\{X_t, 0 \leq t \leq T\}$ given by

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \tag{1.1}$$

where $\{W_t, 0 \leq t \leq T\}$ is a standard Brownian motion with values in $\mathbb{R}^n$. The coefficients $b(x)$ and $\sigma(x)$ are deterministic and are assumed to satisfy the usual conditions, to be stated later on, to ensure the existence and uniqueness of the solution of equation (1.1). Moreover, the diffusion coefficient $\sigma(x)$ is assumed to satisfy a uniform ellipticity condition that will be stated later on.

We first state the pricing formula. We consider a payoff Φ of some financial quantity depending on the prices at a finite number of times, that is,

$$\Phi = \Phi(X_{t_1}, \cdots, X_{t_n})$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is infinitely differentiable and Φ and all its partial derivatives have polynomial growth. Given $0 < t_1 < \ldots < t_n = T$, the option price $u(x)$ is computed, under the risk neutral probability measure, as

$$u(x) = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \mid X_0 = x] \tag{1.2}$$

where $\mathbb{E}$ denotes the expectation, r is the risk free interest rate assumed to be constant and $X_{t_i}, i = 1, \ldots, n$ are the underlying assets depending on some model parameters. We mention that the option price can be numerically computed by using Monte Carlo methods (see [19] Chapter 8).

Then, the Greek is calculated as follows

$$\text{Greek} := \frac{\partial}{\partial \alpha}\mathbb{E}[e^{-rT}\Phi(X_{t_1}(\alpha), \cdots, X_{t_n}(\alpha)) \mid X_0 = x] \tag{1.3}$$

where the parameter α could be the initial price x, the drift coefficient b (if constant), the volatility coefficient σ (if constant) or any of the constant parameters appearing in the model. $X_{t_i}(\alpha), i = 1, \ldots, n$ emphasize the dependence of X_{t_i} on α. Specifically, the most interesting Greeks (in the standard Black-Scholes model) include:

- Delta, denoted by Δ, defined as the derivative of the option price with respect to the initial price

  $$\Delta = \frac{\partial}{\partial x}u(x). \tag{1.4}$$

  Δ plays a key role in the computation of other Greeks as well.

- Gamma, denoted by Γ, defined as the second derivative of the option price with respect to the initial price

  $$\Gamma = \frac{\partial^2}{\partial x^2}u(x). \tag{1.5}$$

  Γ measures the sensitivity of Δ.

- Vega, denoted by $\mathcal{V}$, defined as the derivative of the option price with respect to the volatility

  $$\mathcal{V} = \frac{\partial}{\partial \sigma}u(x). \tag{1.6}$$

  $\mathcal{V}$ is not a Greek letter.

Other Greeks include Rho, denoted by ρ, which is defined as the derivative of the option price with respect to interest rate

$$\rho = \frac{\partial}{\partial r} u(x), \tag{1.7}$$

and Theta, denoted by Θ, which is defined as the derivative of the option price with respect to terminal time

$$\Theta = -\frac{\partial}{\partial T} u(x). \tag{1.8}$$

Θ is expressed as a negative derivative to represent the sensitivity of the price with decreasing maturity. Unlike other factors, however, the movement in remaining maturity is perfectly predictable. Hence, time is not a risk factor.

Remark

If the drift b and the volatility σ are functions of the underlying asset price then $\frac{\partial}{\partial b} u(x)$ and $\frac{\partial}{\partial \sigma} u(x)$ are not defined. We will see later on how this situation is handled (see Chapter 3 Section 3.1).

We want to express the Greek (equation (1.3)) as an expectation without derivatives in order to enable efficient computation. Such a representation is crucial for Monte Carlo evaluation (see [53] page 146). The mathematical challenge arises from payoff functions which tend to be discontinuous, non-differentiable or even more complicated. We are particularly interested in the case where the payoff function Φ is a discontinuous function. A typical example is the digital option, that is, $\Phi(x) = 1_{\{x \geq K\}}$ for some constant $K > 0$.

1.1 Approaches to evaluating Greeks

In this section we present different approaches, with their limitations, that have been used traditionally to compute the Greeks. A detailed review of these approaches can be found in [41] on Chapter 7. We illustrate the different approaches to the case of the Δ of the Black-Scholes price of a call option. The ideas can then be easily extended to other Greeks as well.

1. **Finite difference**

   This method involves generating, independently, two estimate option prices $\tilde{u}(x)$ and $\tilde{u}(x+\varepsilon)$ from the initial price x and $x+\varepsilon$, respectively, for small ε, so that an estimate

$\tilde{\Delta}$ of Δ can be computed as follows:

$$\tilde{\Delta} = \frac{\tilde{u}(x+\varepsilon) - \tilde{u}(x)}{\varepsilon}. \tag{1.9}$$

Repeating this several times and averaging we obtain an estimator converging to

$$\Delta = \frac{u(x+\varepsilon) - u(x)}{\varepsilon} \tag{1.10}$$

where $u(x)$ and $u(x+\varepsilon)$ are the option prices at x and $x+\varepsilon$ respectively. Assuming that we simulate $\tilde{u}(x+\varepsilon)$ and $\tilde{u}(x)$ using common random numbers and that for (almost) all values of the random numbers the output $\tilde{u}(\cdot)$ is continuous in the input x, we have

$$\mathrm{Var}(\tilde{\Delta}) = \frac{\mathrm{Var}(\tilde{u}(x+\varepsilon)) + \mathrm{Var}(\tilde{u}(x))}{\varepsilon^2} = O(\varepsilon^{-2}). \tag{1.11}$$

This is because the underlying assets $X(\varepsilon)$ and X are generated independently of each other. Therefore, the variance of $\tilde{\Delta}$ becomes very large if ε is made small. To get an estimator that converges to Δ we must let ε decrease slowly as n increases resulting in slow overall convergence. A detailed review of finite difference approach is found in [41] on page 378. Glynn [42] has shown that the best possible convergence rate using this approach is $n^{-\frac{1}{4}}$. Replacing the forward difference with the central difference,

$$\tilde{\Delta} = \frac{\tilde{u}(x+\varepsilon) - \tilde{u}(x-\varepsilon)}{2\varepsilon},$$

improves the convergence rate to $n^{-\frac{1}{3}}$. However, using common random numbers, it has been shown in [40] and [42] that one can achieve the convergence rate of $n^{-\frac{1}{2}}$, reported to be the best that can be expected from Monte Carlo methods. The disadvantage of the common random number finite difference method is that it may perform very poorly when the payoff function is discontinuous, as in the case of a digital option and a barrier option (see [53] page 140).

2. **Pathwise method**

   This approach assumes that the payoff function Φ is a continuously differentiable with bounded derivatives and the underlying variable $X_{t_i},\ i = 1, \ldots, n$, is differentiable with respect to underlying parameters. This allows the interchanging of the derivative operator and the expectation operator. Thus, we have

$$\frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})] = \mathbb{E}[e^{-rT}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Phi(X_{t_i})\frac{\partial X_{t_i}}{\partial x}]. \tag{1.12}$$

The pathwise method computes the derivative of the payoff function with respect to the parameter of interest. This method only works for specific payoff functions, hence we cannot generalize the implementation of this approach. However, the method gives unbiased results when applicable (see [41] page 386). This approach cannot be applied to non-differentiable payoff functions as in the case of barrier and digital options, for example.

3. **Likelihood ratio method**

This method was first introduced by Broadie and Glasserman [20]. The method assumes that the law of $X_{t_1}, \ldots, X_{t_n}$ is explicitly known and is given by a density, say, $p(x_1, \ldots, x_n)$ which is also a function of x. Thus, we can write equation (1.2) as

$$\mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})] = \int e^{-rT}\Phi(x_{t_1}, \cdots, x_{t_n})p(x_1, \cdots, x_n)dx_1 \cdots dx_n. \tag{1.13}$$

Computing the derivative of the option price with respect to the initial price x we obtain

$$\frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})] = \int e^{-rT}\Phi(x_{t_1}, \cdots, x_{t_n})\frac{\partial}{\partial x}p(x_1, \cdots, x_n)dx_1 \cdots dx_n. \tag{1.14}$$

If this indeed holds then multiplying and dividing the integrand by $p(x_1, \ldots, x_n)$ yields

$$\begin{aligned}
&\frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})] \\
= &\int e^{-rT}\Phi(x_{t_1}, \cdots, x_{t_n})\frac{\frac{\partial}{\partial x}p(x_1, \cdots, x_n)}{p(x_1, \cdots, x_n)}p(x_1, \cdots, x_n)dx_1 \cdots dx_n \\
= &\int e^{-rT}\Phi(x_{t_1}, \cdots, x_{t_n})\frac{\partial}{\partial x}[\ln p(x_1, \cdots, x_n)]p(x_1, \cdots, x_n)dx_1 \cdots dx_n \\
:= &\ \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \cdot \pi] \qquad (1.15)
\end{aligned}$$

where π is the weighting function defined by

$$\pi = \frac{\partial}{\partial x}\ln p(X_{t_1}, \cdots, X_{t_n}).$$

The importance of this approach is that it provides us with an efficient way of avoiding the derivative of the payoff function Φ. We compute the derivative of the probability density of the underlying variables rather than the derivative of the payoff function. The likelihood ratio method gives a single weight function. It has been proven that when applicable the likelihood ratio method gives a weight function with minimal variance (see [7], [9] and [36]). The expectation in equation (1.15) can now be computed using Monte Carlo methods. However, this analysis is theoretical since, in general, the density function $p(x_1, \ldots, x_n)$ is not explicitly known as in the case of Asian options, for example.

4. **Malliavin calculus**

   To circumvent the drawbacks outlined in the other approaches that we have mentioned above we use Malliavin calculus recently introduced in [35] and [36]. Use of the Malliavin calculus avoids the need to differentiate payoff functions and does not require explicit knowledge of the density of the underlying asset. The main tool we use is the integration by parts formula. Using the Malliavin calculus we can show that all Greeks mentioned above can be represented as the expected value of the payoff function multiplied by a weight function

$$\text{Greek} := \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \cdot \pi] \tag{1.16}$$

   where π is a weight function to be determined (see [8], [9], [15], [35], [36], [43], [44], [57], [67] and the references therein). An important advantage is that the weight function π, which is usually a function of the underlying variable, is independent of the payoff function Φ. In the case of discontinuous payoff functions, Malliavin calculus improves the efficiency of Greek computation. We can, therefore, construct a Monte Carlo algorithm for general options and not specifically for each option.

   The Malliavin calculus approach gives several weight functions (see [8], [9], [57] and [67]). If the density of the underlying variable is known, both the Malliavin calculus and the likelihood method give the same weight function (see [67]). Thus, the Malliavin calculus can be seen as an extension of the likelihood method. The real advantage of using Malliavin calculus is that it is applicable to both complicated and discontinuous payoff functions such as look-back options and digital options. It also works well when dealing with underlying random variables whose density is not explicitly known, as in the case of Asian options, for example. We mention that the Malliavin calculus approach is not reported to have lower variance than the finite difference method for smooth functions like the vanilla options (see [35]).

We will focus (in this thesis) on the Malliavin calculus approach. Our motivation to use Malliavin calculus is: we want a method which is applicable to a wide class of option prices. The Malliavin calculus approach enables us to obtain tractable formulae for Greeks which can be simulated using Monte Carlo methods.

1.2 Background on the Malliavin calculus

The Malliavin calculus was introduced by Paul Malliavin in his celebrated paper "Stochastic calculus of variation and hypoelliptic operators" that was published in 1976 (see [65]). It is an infinite dimensional calculus defined on the Wiener space. The initial application of Malliavin calculus was to prove the results about the smoothness of densities of solutions of stochastic differential equations driven by Brownian motion (see [38], [50], [65], [70] and [83]). This remained the only known application for Malliavin calculus for several years.
In 1984 Ocone [72] obtained an explicit interpretation of the Clark-Haussmann-Ocone formula in terms of the Malliavin derivative. The formula is usually abbreviated as the CHO formula. This result was later applied to finance by Ocone and Karatzas in 1991 (see [55]). The authors prove that the Malliavin derivative can be used to obtain explicit formulae for the replicating portfolios of contingent claims in markets driven by Brownian motion. This led to a huge increase in the interest in the Malliavin calculus both among mathematicians and finance researchers (see [17], [24], [27], [31], [60], [66], [74] and the references therein).
In 1999 Fournie et al. [35] use the Malliavin calculus to compute Greeks. Following their work, much attention has been dedicated to finding more efficient and more general methods of computing the Greeks using Monte Carlo methods (see [7], [8], [9], [10], [11], [15], [43], [44], [57], [58], [67] and the references therein) and also to extending the computations to a wider class of options and models. The above cited papers compute the Greeks, using Malliavin calculus, with price dynamics driven by Brownian motion only.
Over the last decade, there has been an increasing interest in jump type diffusion models for modelling in stochastic finance (see [23] and [81]). This is because there is a growing evidence that models driven by jump processes may be more realistic than those that insist on continuous sample paths. The continuous time models use the normal distribution to fit the log returns of the underlying asset prices whereas the data suggest that log returns of stocks/indices are not normally distributed. The log returns of most financial assets are skewed and have kurtosis higher than that of normal distribution (see [81]). It is therefore natural to want to explore possible extensions of the work of Fournie et al. [35] to cases where the underlying asset prices are modelled by processes with jumps.
Although markets driven by processes with jumps are not in general complete, the Greeks remain important for financial applications. For example, Greeks are useful in model calibration which involves minimizing over model parameters to bring model prices as close as possible to market prices (see [14]).
Malliavin calculus was first extended to jump diffusion models by Bichteler et al. [18]. The

main focus in this monograph is on the existence and smoothness of the density of the solution of a stochastic differential equation with jumps, a study which is not relevant to the study of the Greeks. Carlein and Pardoux [21] define an analogous calculus to the Malliavin calculus on the Poisson space (see also [68]). Nualart and Schoutens [71] develop a theory of chaos expansion for functionals of Lévy processes when the Lévy measure satisfies an exponential moment condition. This has been used as a basis to define Malliavin calculus similar to the Brownian motion case (see [27], [62], [70] and [84]).

The extension of the work of Fournie et al. [35] to models with jumps already exists in the literature (see [6], [25], [26], [32], [34], [48], [66], [76], [78] and the references therein). In [25] the authors use Malliavin calculus for simple Lévy processes to calculate Greeks for a class of "separable" jump diffusions. The general idea in [25] is to take the Malliavin derivative in the direction of the Wiener process on the Wiener-Poisson space (to be defined later on). This enables the authors to stay in the framework of a Malliavin calculus for the Brownian motion case without major changes. El-Khatib and Privault [32] consider a market driven by jumps alone. The authors use a Malliavin calculus defined on a Poisson space to compute Greeks for Asian options but imposing a regularity condition on the payoff. Their approach cannot be used to compute Greeks for European options. Forster et al. [34] use the hypoelliptic condition and the standard Malliavin calculus to obtain Greeks with the weight function given as a Skorohod integral. In this thesis we deal with discontinuous payoff functions and also cases where the volatility is stochastic.

The extension of the work of Fournie et al. [35] to models with jumps still introduces many challenges prompting further research, for example the Malliavin derivative of the pure jump case is not a derivative; it is, instead, a difference operator.

1.3 Aims and structure of the thesis

The aim of this thesis is to provide a broad coverage of the many applications of the Malliavin calculus to the calculations of the Greeks, and to provide the reader with a summary of the necessary background to be able to understand this.

The thesis is organized as follows. Chapter 2 is devoted to a review of some basic definitions and results related to Malliavin calculus as well as to give some important remarks. In Chapter 3 we demonstrate, using some results in Chapter 2, how the Malliavin calculus techniques are applied to calculate Greeks for continuous processes. The chain rule and the integration by parts formula are used extensively to compute Δ, Γ and $\mathcal{V}$ for different types

of payoff functions. In Chapter 4 we review white noise analysis for continuous processes (see [1] and [27]). In particular, we consider the Wick product, the Hermite transform and the Donsker delta function. Then we apply these concepts to calculate the Greeks for pure diffusion processes. In Chapter 5 we define a first variation process of a pure jump Lévy stochastic differential equation and give a representation formula for the stochastic derivative of a pure jump Lévy stochastic differential equation. We also give the necessary and sufficient conditions for a function to serve as a weighting function in the pure jump case. This is an extension to the pure jump case of the work in [10] where the author gives the necessary and sufficient conditions for a function to serve as a weighting function in the pure diffusion case.
Chapter 6 deals with the jump diffusion case. We calculate Greeks for different models. Here we are inspired by ideas in [25] where the authors calculate weight functions for processes driven by a Brownian motion and a Poisson process with deterministic jump sizes. We extend the work to more general Lévy processes. We also approximate a Lévy process by Brownian motion and compute its corresponding Greeks. Greeks for original Lévy processes are then obtained by a limiting argument. In Chapter 7 we review white noise analysis for Lévy processes. As in Chapter 4 we consider the Wick product, the Hermite transform and the Donsker delta function for Lévy processes (see [27]). Using the Donsker delta representation formula, we are able to express the option price in terms of the Donsker delta function. Then we calculate Δ for pure jump Lévy processes in the same way as in Chapter 4.

In addition, we will present some new results as indicated below. The new results are contained in Chapter 3 Section 3.5, Chapter 4 Section 4.8, Chapter 5 Sections 5.5 and 5.6, Chapter 6 Sections 6.5 and 6.6 and in Chapter 7 Section 7.8. We also give interesting examples and proofs that are missing in the literature. We mention that the Greeks for the Heston model are known in the literature. The full derivation of the Greeks was not available. We provide this in Section 3.5. The use of white noise is common but its extension to the computation of Greeks via Malliavin calculus was not available. This is done in Section 4.8. We derive a new representation for the Malliavin derivative of a pure jump Lévy stochastic differential equation (see Section 5.5) in terms of its first variation process and then give the necessary and sufficient conditions for a function to serve as a weighting function (see Section 5.6). In Section 6.5 we compute the Greeks for the Heston model with jumps. This is an extension of results in Section 3.5. The extension to the additional jump case is new at the best of my knowledge. The idea of approximating the Lévy process appears in several papers in the literature (see [4]). In Section 6.6, we compute Greeks of an approximation of a Lévy process and then pass through limit arguments to obtain Greeks for the original Lévy process. This approach can be applied to a wide class of Lévy processes. In Section 7.8 we only compute Δ for a pure jump Lévy process using the Donsker delta function representation defined in the white noise setting. This is a new result. Other Greeks are left for future work.

Chapter 2

Basic Properties of the Malliavin calculus

This chapter is an introduction and a survey of the Malliavin calculus machinery (see [70]). The original construction of Malliavin calculus was given on the Wiener space $\Omega = C_0([0,T])$ (to be defined later on). We review this construction of the Malliavin derivative in the first half of this chapter. We also review the concept of a chain rule as well as an integration by parts formula that will be used in the following chapters to compute Greeks. In the second half of this chapter we review the construction of Malliavin calculus based on the chaos expansion. The definition of Malliavin calculus in this way allows for a useful combination with Hida's white noise calculus. We conclude the chapter by considering the Clark-Haussmann-Ocone formula. The formula is useful in mathematical finance but is not used in this thesis. The material discussed here was taken from [5], [38], [54], [60], [66], [69], [70] and [74].

2.1 Wiener Space

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{\{0\le t\le T\}}, P)$ on which a Brownian motion $W(t,\omega)$ is defined ($t \in [0,T], \omega \in \Omega$). $\mathbb{F} = (\mathcal{F}_t)_{\{0\le t\le T\}}$ is assumed to be the filtration generated by $W(t,\omega)$. Note that $W(t,\omega)$ is a function of two variables. If we fix ω then this becomes a continuous function of one variable t, that is, $W(t,\omega) \in C([0,T])$. So at each state ω the Brownian motion $W(t,\omega)$ associates a continuous function. We write this function as $\omega(t)$, that is,

$$\omega(t) = W(t,\omega). \tag{2.1}$$

The Itô construction of the stochastic integral yields

$$\int_0^T h(t)dW(t,\omega), \quad h \in L^2([0,T]) \tag{2.2}$$

(or more generally, $h \in L^2([0,T] \times \Omega)$). This integral can also be written as

$$\int_0^T h(t)d\omega(t)$$

where we have $\omega(t) \in C([0,T])$ and the integral is with respect to Wiener measure on $C([0,T])$ (see [54] page 125). We note that whereas there is no a priori additive structure on Ω, the identification

$$\Omega \ni \omega \leftrightarrow \omega(t) \in C([0,T])$$

enables us to use the Brownian motion to convert Ω naturally into a vector space. It thus becomes possible to define derivatives with respect to $\omega \in \Omega$ in a similar way to the usual vector space definitions as we will see.

Let $H = L^2([0,T])$ and let

$$W(H) = \{\int_0^T hdW \mid h \in H\}.$$

Let $\mathcal{S}_n$ be the space of all polynomials of degree n in elements of $W(H)$ and let $\mathcal{S}$ be the space of all such polynomials. If $F \in \mathcal{S}$ then there exists n, a polynomial f of degree n and $h_1, \ldots, h_n \in H$ such that

$$F(\omega) = f(W(h_1), \cdots, W(h_n)) \tag{2.3}$$

where $W(h) := \int_0^T h(t)dW_t$. We can orthonormalize the $h_i, i = 1, 2, \ldots, n$ by using the Gram-Schmidt procedure, that is, letting

$$\xi_1 = \frac{h_1}{\| h_1 \|}, \; \xi_2 = \frac{(h_2 - (\xi_1, h_2)\xi_1)}{\| h_2 - (\xi_1, h_2)\xi_1 \|}, \ldots$$

and then write the $h_i, i = 1, 2, \ldots, n$ in terms of the $\xi_i, i = 1, 2, \ldots, n$ and multiply out. This gives

$$F(\omega) = \tilde{f}(W(\xi_1), \cdots, W(\xi_n))$$

where $\tilde{f}$ is another polynomial of degree n and $\xi_1 = \frac{h_1}{\|h_1\|}$. Hence, we can assume where appropriate that the $h_i, i = 1, 2, \ldots, n$ are orthonormal in H.
Define

$$\gamma_n := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-\frac{|x|^2}{2}} dx.$$

Remark
We define
$$\mathbb{E}[f(W(\xi_1),\cdots,W(\xi_n))] := \int_{\mathbb{R}^n} f(x_1,\cdots,x_n)d\gamma_n(x)$$
where $d\gamma_n(x)$ is given by
$$d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{\frac{-|x|^2}{2}} dx. \tag{2.4}$$
This remark is very useful in the setup of the theory.

We note that if $\xi \in L^2([0,T])$ is normal then
$$1 = \parallel \xi \parallel^2 = \int_0^T \mid \xi(t) \mid^2 dt = \mathrm{var}(W(\xi))$$
so
$$W(\xi) \sim N(0,1)$$
and
$$\mathbb{E}[f(W(\xi))] = \int_{\mathbb{R}} f(x)d\gamma_1(x).$$

2.2 Malliavin derivative

Since our state space is a vector space we can define the derivative of a random variable
$$F(\omega) = W(h) = \int_0^T h(t)dW(t,\omega) \tag{2.5}$$
in the direction γ with
$$\gamma(t) = \int_0^t g(s)ds, \quad g \in H$$
as follows.

Definition 2.2.1 *Let $F : \Omega \to \mathbb{R}$ be a random variable of the form (2.5) and let $\gamma \in \Omega$ be of the form*
$$\gamma(t) = \int_0^t g(s)ds, \qquad g \in H. \tag{2.6}$$
Then the directional derivative $D_\gamma F$ of a random variable F at the point $\omega \in \Omega$ in the direction $\gamma \in \Omega$ is given by
$$D_\gamma F(\omega) := \frac{d}{d\varepsilon}[F(\omega + \varepsilon\gamma)]_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{F(\omega+\varepsilon\gamma) - F(\omega)}{\varepsilon} \tag{2.7}$$
if the limit exists.

The set of $\gamma \in \Omega$ which can be written in the form (2.6) for some $g \in H$ is called the Cameron-Martin space and we denote it by $\mathcal{H}$. Note that it is not possible to proceed in the manner we are about to with functions that are not Cameron-Martin functions. However, there are ways of getting around this.

Let

$$F(\omega) = \int_0^T h(t) d\omega(t).$$

Then

$$\begin{aligned}\frac{1}{\varepsilon}\left(F(\omega+\varepsilon\gamma) - F(\omega)\right) &= \frac{1}{\varepsilon}\left(\int_0^T h(t)d(\omega+\varepsilon\gamma)(t) - \int_0^T h(t)d\omega(t)\right) \\ &= \frac{1}{\varepsilon}\left(\varepsilon\int_0^T h(t)d\gamma(t)\right).\end{aligned}$$

Since $d\gamma(t) = g(t)dt$ we have

$$\frac{1}{\varepsilon}(F(\omega+\varepsilon\gamma) - F(\omega)) = \int_0^T h(t)g(t)dt = \langle h, g\rangle_H.$$

Here $\langle \cdot, \cdot \rangle_H$ is the inner product on H. We will write

$$D\left(\int_0^T h(t)dW(t,\omega)\right) = h.$$

We note that this is only valid for a deterministic integrand h. For a general case we refer to Proposition 2.4.2 (to be given later on). From these considerations we give the following definition.

Definition 2.2.2 *The derivative* $D : \mathcal{S} \to L^2([0,T]\times\Omega)$ *of a random variable* $F(\omega) = f(W(h_1), \dots, W(h_n))$ *is defined by*

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n))h_i(t). \tag{2.8}$$

D is called the *Malliavin derivative* on $\mathcal{S}$.

Since $D\left(\int_0^T h(t)dW_t\right) = D(W(h)) = h$,

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n))D(W(h_i)). \tag{2.9}$$

As f has only polynomial growth we have $DF \in L^2(\Omega \times [0,T])$. Sometimes it is convenient to write

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n))h_i(t). \tag{2.10}$$

However, because $h_i \in H$, D_t is not really well-defined, since h_i is only defined up to a set of Lebesgue measure 0. The notation is nevertheless suggestive. We will use both D_t and D (which are operators with different domain and range) where there can be no confusion.

We have the following product rule.

Lemma 2.2.3 *Suppose that $F, G \in \mathcal{S}$. Then*

$$D(FG) = (DF)G + F(DG). \tag{2.11}$$

We have the following integration by parts formula which is found in [70] page 25.

Lemma 2.2.4 *Suppose that $F \in \mathcal{S}$ and $h \in H$. Then*

$$\mathbb{E}\left[\langle DF, h\rangle_H\right] = \mathbb{E}\left[FW(h)\right]. \tag{2.12}$$

Proof

By first dividing by $\| h \|$ if necessary, we may assume that the norm of h is one. There exist orthonormal elements of H, $\xi_1, \ldots, \xi_n$, such that $\xi_1 = h$ and F is a smooth random variable of the form

$$F = f(W(\xi_1), \ldots, W(\xi_n)). \tag{2.13}$$

Let $\gamma_n(x)$ denote the density of the standard normal distribution on $\mathbb{R}^n$. Then, by integration by parts, we have

$$\begin{aligned}
\mathbb{E}\left[\langle DF, h\rangle_H\right] &= \mathbb{E}\left[\langle DF, \xi_1\rangle_H\right] = \mathbb{E}\left[\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f \langle \xi_i, \xi_1\rangle_H\right] = \mathbb{E}\left[\frac{\partial}{\partial x_1} f(W(\xi_1), \cdots, W(\xi_n))\right] \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_1} f(x) d\gamma_n(x) = -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) \frac{\partial}{\partial x_1}\left(e^{-\frac{|x|^2}{2}}\right) dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) x_1 e^{-\frac{|x|^2}{2}} dx = \mathbb{E}\left[FW(\xi_1)\right] = \mathbb{E}\left[FW(h)\right]. \qquad \square
\end{aligned}$$

Applying Lemma 2.2.4 to the product FG where F and G are random variables and using Lemma 2.2.3, we get following result.

Corollary 2.2.5 *Suppose that $F, G \in \mathcal{S}$ and $h \in H$. Then we have*

$$\mathbb{E}\left[G\langle DF, h\rangle_H\right] = \mathbb{E}\left[-F\langle DG, h\rangle_H + FGW(h)\right]. \tag{2.14}$$

Define

$$\| F \|_{1,2} := \| F \|_{L^2(\Omega)} + \| D_t F \|_{L^2([0,T]\times\Omega)} . \tag{2.15}$$

We recall that random variables are elements of L^2 and hence only defined almost surely with respect to the Wiener measure. Given a random variable F, define

$$G(\omega) = F(\omega + k)$$

where k is a fixed continuous function. In which direction can we shift the argument ω of a functional while keeping it well-defined? If k lies in the Cameron-Martin space $\mathcal{H}$ then $G(\omega)$ is well-defined. However, if k lies in the complement, it can be shown (see [38]) that G is not a well-defined function in L^2 and cannot be used to define a directional derivative in the direction k.

Let $\mathbb{D}_{1,2}$ be the closure of $\mathcal{S}$ in the norm $\| \cdot \|_{1,2}$. We will show that the derivative on $\mathcal{S}$ can be extended to a closed operator on $\mathbb{D}_{1,2}$.

Definition 2.2.6 *$\bar{A} : H \to K$ is a closable operator on a normed complete linear space if*

$$F_n \to F,\ \bar{A}F_n \to G_1,\ G_n \to F,\ \bar{A}G_n \to G_2 \quad \textit{implies} \ \ G_1 = G_2.$$

If $\bar{A}$ is closable and we know F on a subset $\mathcal{S}$ of the space then we can extend $\bar{A}$ to an operator A defined on the closure of $\mathcal{S}$ called the closure of $\bar{A}$ by defining $\bar{A}F = G$ whenever there exists a sequence $F_n \to F$ such that $\bar{A}F_n \to G$. The closability implies that G is uniquely defined by the later two conditions. To prove closability, we need only to show that $F_n \to 0$ and $\bar{A}F_n$ converges implies $\bar{A}F_n \to 0$. The following lemma was taken from [70] page 26.

Lemma 2.2.7 *The Malliavin derivative $D : \mathcal{S} \to L^2(\Omega \times [0,T])$ is closable. It has a closed extension to $\mathbb{D}_{1,2}$ (which is also denoted by D).*

Proof
Let $\{F_n, n \geq 1\}$ be a sequence of random variables such that F_n converges to 0 in $L^2(\Omega)$ and

the sequence of derivatives DF_n converge to η in $L^2(\Omega \times [0,T])$. Our aim is to show that $\eta = 0$. For any $h \in H$ and for any bounded random variable $F \in \mathcal{S}$ such that $FW(h)$ is bounded in L^2 we have

$$\begin{aligned}
\mathbb{E}\left[\langle \eta, h\rangle_H F\right] &= \lim_{n\to\infty} \mathbb{E}\left[\langle DF_n, h\rangle_H F\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[-F_n\langle DF, h\rangle_H + F_n F W(h)\right] \quad \text{by} \quad (2.14) \\
&= 0.
\end{aligned}$$

This is because F_n converges to 0 in $L^2(\Omega)$ as n tends to infinity and the random variables $\langle DF, h\rangle_H$ and $FW(h)$ are bounded. This implies that $\eta = 0$ and the proof is complete. □

Define

$$D : \mathbb{D}_{1,2} \to L^2([0,T] \times \Omega)$$

as the closure of our previously defined operator. In general it will not be defined on the whole of $L^2(\Omega)$ and will not be continuous. However, F_n converges in $\mathbb{D}_{1,2}$, if and only if both F_n and DF_n converge and if $F_n \to F$, $DF_n \to G$ then $G = DF$ (see [70] page 26).

Example

$$De^{W(h)} = e^{W(h)}h, \quad h \in H.$$

Let

$$F_n = \sum_{i=0}^{n} \frac{1}{i!}(W(h))^i.$$

Then

$$DF_n = \sum_{i=1}^{n} \frac{i}{i!}(W(h))^{i-1}D(W(h)) = \sum_{i=1}^{n} \frac{1}{(i-1)!}(W(h))^{i-1}D(W(h)) = F_{n-1}D(W(h)) = F_{n-1}h.$$

Note that $F_n \to e^{W(h)}$ in $L^2(\Omega)$. Since D is a closed operator it follows that

$$DF_n \to e^{W(h)}h \quad \text{in} \quad L^2([0,T] \times \Omega).$$

Since both F_n and DF_n converge it follows from the fact that D is closed that $\lim_{n\to\infty} F_n = e^{W(h)}$ is in the domain of D and

$$De^{W(h)} = \lim_{n\to\infty} DF_n = e^{W(h)}h.$$

The chain rule holds for the Malliavin derivative in the following form.

Proposition 2.2.8 *Let $F = (F_1, \ldots, F_n) \in \mathbb{D}_{1,2}$ and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Then $\varphi(F) \in \mathbb{D}_{1,2}$ and*

$$D_t\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F) D_t F_i, \quad t \geq 0 \qquad a.s. \tag{2.16}$$

Proof
The proof follows easily by approximating the random variable F by a sequence of smooth random variables and ρ by $\rho \cdot \gamma_\varepsilon$ where $\{\gamma_\varepsilon\}$ is an approximation of the density (see [70] page 29). We omit the details. □

Example
For some fixed $s \in (0, T]$ put

$$\varphi(W) = W_s^2 = \left(\int_0^T 1_{[0,s]}(t) dW_t \right)^2. \tag{2.17}$$

Then

$$D_t\varphi(W) = 2W_t 1_{[0,s]}(t) = \begin{cases} 2W_t & \text{if } t \leq s \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2.2.9 *Let $u(s,\omega)$ be $\mathcal{F}_s$-adapted and assume that $u(s,\cdot) \in \mathbb{D}_{1,2}$. Then $D_t u(s,\omega) = 0$ for $t > s$.*

Proof
We will show later on (see Lemma 2.7.3) that $D_t(\mathbb{E}[u(s,\omega) \mid \mathcal{F}_s]) = \mathbb{E}[D_t u(s,\omega) \mid \mathcal{F}_s] 1_{[0,s]}(t)$. Therefore we have

$$D_t u(s,\omega) = D_t \mathbb{E}[u(s,\omega) \mid \mathcal{F}_s] = \mathbb{E}[D_t u(s,\omega) \mid \mathcal{F}_s] 1_{[0,s]}(t).$$

Since $s < t$ the result follows. □

The intuition of the corollary is that if $u(s,\omega)$ only depends on the early parts of the paths up to time s then perturbing the paths later on, that is, on the region $t > s$ makes no difference to $u(s,\omega)$.
The following proposition is useful.

Proposition 2.2.10 *Let $\{F_n, n \geq 1\}$ be a sequence of random variables in $\mathbb{D}_{1,2}$. Assume that F_n converges to F in $L^2(\Omega)$ and*

$$\sup_n \mathbb{E}\left(\| DF_n \|^2_{L^2(\Omega)} \right) < \infty.$$

Then F belongs to $\mathbb{D}_{1,2}$ and the sequence of derivatives $\{DF_n,\ n \geq 1\}$ converges to DF in $L^2(\Omega \times [0,T], L^2(\Omega))$.

Proof

The proof can be found in [70] page 29. We omit the details. □

2.3 Skorohod integral

The Malliavin derivative D is a closed linear operator defined on $\mathbb{D}_{1,2}$ and

$$D : \mathbb{D}_{1,2} \to L^2(\Omega \times [0,T]). \tag{2.18}$$

D is an unbounded operator with domain dense in $L^2(\Omega)$ whose domain and range belong to different Hilbert spaces. In standard theory, adjoints of unbounded operators have the same domain and range. Here we show one way of defining an adjoint operator δ of D such that

$$\delta : L^2(\Omega \times [0,T], L^2(\Omega)) \to L^2(\Omega \times [0,T]). \tag{2.19}$$

We shall denote the domain of the adjoint operator δ by $\mathrm{Dom}(\delta)$.

Definition 2.3.1 *Let $u \in L^2(\Omega \times [0,T])$. Then u belongs to the domain $Dom(\delta)$ of δ if for all $F \in \mathbb{D}_{1,2}$ we have*

$$| \, \mathbb{E}\left[\langle DF, u\rangle_{L^2(\Omega)}\right] | = | \, \mathbb{E}\left[\int_0^T D_t F u(t) dt\right] | \leq c \parallel F \parallel_{L^2(\Omega)} \tag{2.20}$$

where c is some constant depending on u. If u belongs to $Dom(\delta)$, then

$$\delta(u) = \int_0^T u_t \delta W_t \tag{2.21}$$

is the element of $L^2(\Omega)$ such that the integration by parts formula holds:

$$\mathbb{E}[\left(\int_0^T D_t F u_t dt\right)] = \mathbb{E}[F\delta(u)] \quad \textit{for all} \quad F \in \mathbb{D}_{1,2}. \tag{2.22}$$

$\delta : L^2(\Omega \times [0,T], L^2(\Omega)) \to L^2(\Omega \times [0,T])$. is linear and densely defined. We call δ the *Skorohod integral.*

Remark

Definition 2.3.1 gives the relationship between the derivative operator and the Skorohod integral.

Lemma 2.3.2 *Suppose $F \in \mathcal{S}$ and $h \in H$. Then*

$$\delta(Fh) = FW(h) - \langle h, DF \rangle_H. \tag{2.23}$$

In addition, setting $F = 1$ one obtains

$$\delta(h) = W(h), \quad h \in H.$$

Proof

The proof can be found in [38] page 11. We recall the integration by parts formula

$$\mathbb{E}\left[G\langle DF, h\rangle_{H)}\right] = \mathbb{E}\left[\langle -FDG, h\rangle_H + FGW(h)\right].$$

This can be extended to all $F, G \in \mathbb{D}_{1,2},\ h \in H$. Let $F \in \mathcal{S},\ G \in \mathcal{S}_0$ and $h \in H$. Then

$$\mathbb{E}\left[\delta(Fh)G\right] = \mathbb{E}\left[\langle Fh, DG\rangle_H\right] = \mathbb{E}\left[F\langle h, DG\rangle_H\right] = \mathbb{E}\left[-G\langle h, DF\rangle_H + FGW(h)\right].$$

Hence, since $\mathcal{S}$ is dense in $L^2(\Omega)$, we have

$$\delta(Fh) = FW(h) - \langle h, DF\rangle_{L^2(\Omega)}.$$ □

The lemma can be extended to all $F \in \mathbb{D}_{1,2}$. An important property of the Skorohod integral δ is that its domain Dom(δ) contains all adapted stochastic processes which belong to $L^2(\Omega \times [0,T])$. For such processes the Skorohod integral δ coincides with the Itô stochastic integral (see [47] page 48). This is given in the following proposition.

Proposition 2.3.3 *If u is an adapted process belonging to $L^2(\Omega \times [0,T])$, then*

$$\delta(u) = \int_0^T u(t)dW_t. \tag{2.24}$$

Proof

The proof follows from the following proposition (Proposition 2.3.4) by setting $F = 1$. □

Further, if the random variable F is $\mathcal{F}_T$-adapted and belongs to $\mathbb{D}_{1,2}$ then, for any u in *Dom*(δ), the random variable Fu will be Skorohod integrable. This yields the following proposition (see [70] page 39).

Proposition 2.3.4 *Let F belongs to $\mathbb{D}_{1,2}$ and $u \in Dom(\delta)$ such that $\mathbb{E}[\int_0^T F^2 u_t^2 dt] < \infty$. Then $Fu \in Dom(\delta)$ and*

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F u_t dt, \tag{2.25}$$

whenever the right hand side belongs to $L^2(\Omega)$. In particular, if u is moreover adapted, we have

$$\delta(Fu) = F\int_0^T u_t dW_t - \int_0^T D_t F u_t dt. \tag{2.26}$$

Proof
For any smooth random variable $G \in \mathcal{S}_0$, using Equation (2.22), it holds that

$$\begin{aligned}
\mathbb{E}\left[\langle Fu, DG\rangle_H\right] &= \mathbb{E}\left[\langle u, FDG\rangle_H\right] \\
&= \mathbb{E}\left[\langle u, D(FG) - GDF\rangle_H\right] \\
&= \mathbb{E}\left[\langle u, D(FG)\rangle_H - \langle u, GDF\rangle_H\right] \\
&= \mathbb{E}\left[(F\delta(u) - \langle u, DF\rangle_H)G\right].
\end{aligned}$$

This implies the desired result. □

Proposition 2.3.4 is a product rule for Skorohod integrals. The rule, however, differs from the corresponding rule in ordinary calculus by the minus sign. We will apply this proposition several times when calculating Greeks.

2.4 SDEs and Malliavin calculus

We consider a 1-dimensional stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x. \tag{2.27}$$

where $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are bounded with bounded partial derivatives and $\sigma(x) \neq 0$ for all $x \in \mathbb{R}$. Its integral form is given by

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s. \tag{2.28}$$

We want to bring D_t under the integral signs and give a representation for $D_r X_t$. We do this in the following.

Proposition 2.4.1 *Let $u(t,\omega)$ be an $\mathcal{F}_t$-adapted process and let $t < T$. Then*

$$D_t\left(\int_0^T u(s,\omega)ds\right) = \int_t^T D_t u(s,\omega)ds. \tag{2.29}$$

Proof

The proposition is an immediate consequence of Corollary 2.2.9. □

For the stochastic integral, we have the following proposition.

Proposition 2.4.2 *Let $u(s,\omega)$ be a stochastic process such that*

$$\mathbb{E}[\int_0^T u^2(s,\omega)ds] < \infty$$

and assume that $u(s,\cdot) \in \mathbb{D}_{1,2}$ for all $s \in [0,T]$, that $D_t u \in Dom(\delta)$ for all $t \in [0,T]$ and that

$$\mathbb{E}[\int_0^T (\delta(D_t u))^2 dt] < \infty.$$

Then $\delta(u) \in \mathbb{D}_{1,2}$ and

$$D_t(\delta(u)) = u(t,\omega) + \int_0^T D_t u(s,\omega)dW_s. \qquad (2.30)$$

If in addition $u(t,\omega)$ is adapted we can write Equation (2.30) as

$$D_t\left(\int_0^T u(s,\omega)dW_s\right) = u(t,\omega) + \int_t^T D_t u(s)dW_s. \qquad (2.31)$$

Proof

The proof is given in [74] page 5.6. Here consider a special case where we restrict ourselves to an adapted process of the form

$$u(t,\omega) = F(\omega)h(t)$$

with $h(t) = h1_{[t,T]}$ and $\mathcal{F}_t$-measurable F. Let $t \leq T$. Then

$$\begin{aligned}
D_t\left(\int_0^T Fh(s)dW_s\right) &= D_t\left(\int_0^t Fh(s)dW_s + \int_t^T Fh(s)dW_s\right) \\
&= 0 + D_t\int_t^T Fh(s)dW_s \quad \text{by Corollary } 2.2.9 \\
&= D_t(FW(h1_{[t,T]})) \\
&= (D_tF)W(h1_{[t,T]}) + FD_t(W(h1_{[t,T]}) \quad \text{by } (2.11) \\
&= \int_t^T D_tFh(s)dW_s + Fh(t) \\
&= u(t) + \int_t^T D_t u(s)dW_s.
\end{aligned}$$

□

Remark

The limits of the integral on the left hand side reduce from $\int_0^T$ to $\int_t^T$ at the end because the integrand is adapted. This follows by Corollary 2.2.9.

We now assume that the solution X_t belongs to $\mathbb{D}_{1,2}$. If we take the Malliavin derivative on both sides of Equation (2.28) we obtain (using Propositions 2.4.1 and 2.4.2), for $r < t$,

$$\begin{aligned} D_r X_t &= D_r\left(\int_0^t b(X_s)ds\right) + D_r\left(\int_0^t \sigma(X_s)dW_s\right) \\ &= \int_r^t D_r b(X_s)ds + \sigma(X_r) + \int_r^t D_r\sigma(X_s)dW_s. \end{aligned}$$

Then applying Proposition 2.2.8 we obtain

$$D_r X_t = \int_r^t b'(X_s)D_r X_s ds + \sigma(X_r) + \int_r^t \sigma'(X_s)D_r X_s dW_s.$$

The primes denote the derivative with respect to the space variable. Fix r and set $Z_t := D_r X_t$ for $r < t$. We obtain, using the Itô formula, the following stochastic differential equation

$$dZ_t = b'(X_t)Z_t dt + \sigma'(X_t)Z_t dW_t, \quad t > r \tag{2.32}$$

with initial condition $Z_r = \sigma(X_r)$. The solution of Equation (2.32) is given by

$$Z_t = \sigma(X_s)\exp\{\int_s^t [b'(X_u) - \frac{1}{2}\left(\sigma'(X_u)\right)^2]du + \int_s^t \sigma'(X_u)dW_u\} \tag{2.33}$$

A fundamental matrix of a linear system of equations is an $n \times n$ matrix whose columns form a linearly independent set of solutions, and so span the solution space. A fundamental matrix is nonsingular for all t. Now let Y_t be a fundamental matrix of the equation called the Variational Equation

$$dY_t = b'(X_t)Y_t dt + \sigma'(X_t)Y_t dW_t, \qquad Y(0) = I. \tag{2.34}$$

Using the Itô formula the exact solution is given by

$$Y_t = \exp\{\int_0^t [b'(X_s) - \frac{1}{2}\left(\sigma'(X_s)\right)^2]ds + \int_0^t \sigma'(X_s)dW_s\}. \tag{2.35}$$

Given any fundamental matrix Y_t, the solution of the equation with initial constant c at $t = r$ is

$$Y_t Y_r^{-1} c. \tag{2.36}$$

But $Z_t = D_r X_t$ is such a solution and so

$$D_r X_t = Y_t Y_r^{-1}\sigma(X_r)1_{r\le t} \tag{2.37}$$

which is equivalent to

$$Y_t = D_r X_t \sigma^{-1}(X_r) Y_r 1_{r \leq t} \tag{2.38}$$

We remark that in one dimension $U_t = \frac{\partial}{\partial x} X_t$ is a solution of the Variational Equation satisfying $U_0 = 1$. Hence in that case we can take

$$Y_t := \frac{\partial}{\partial x} X_t. \tag{2.39}$$

$\{Y_t, 0 \leq t \leq T\}$ is commonly known as the *first variation process* of $\{X_t, 0 \leq t \leq T\}$.

2.5 The integration by parts formula

We use the Malliavin derivative and the relation between the Malliavin derivative and the Skorohod integral to obtain an integration by parts formula which plays an important role in the calculation of Greeks. We give this in the following proposition (see [70] page 330).

Proposition 2.5.1 *Let F, G be two random variables such that $F, G \in \mathbb{D}_{1,2}$. Consider a random variable $u(t,\omega)$ for fixed ω, $u(t,\cdot) \in H$ such that $\langle DF, u\rangle_H \neq 0$ a.s. and $Gu(\langle DF, u\rangle_H) \in Dom(\delta)$. Then for any continuously differentiable f with polynomial growth we have*

$$\mathbb{E}\left[f'(F)G\right] = \mathbb{E}\left[f(F)H(F,G)\right]. \tag{2.40}$$

where $H(F,G) = \delta(Gu(\langle DF, u\rangle_H)^{-1})$.

Proof
An application of the chain rule (see Proposition 2.2.8) gives

$$\langle Df(F), u\rangle_H = \langle f'(F)DF, u\rangle_H = f'(F)\langle DF, u\rangle_H.$$

Since $\langle DF, u\rangle_H \neq 0$ we have

$$f'(F) = \langle Df(F), u\rangle_H (\langle DF, u\rangle_H)^{-1}.$$

Consequently, we have

$$\begin{aligned} \mathbb{E}\left[f'(F)G\right] &= \mathbb{E}\left[\langle Df(F), u\rangle_H G(\langle DF, u\rangle_H)^{-1}\right] \\ &= E\left[\langle Df(F), Gu(\langle DF, u\rangle_H)^{-1}\rangle_H\right]. \end{aligned}$$

Since $Gu(\langle DF, u\rangle_H) \in \mathrm{Dom}(\delta)$, an application of Equation (2.22) gives

$$\mathbb{E}\left[f'(F)G\right] = \mathbb{E}\left[f(F)\delta(Gu(\langle DF, u\rangle_H)^{-1})\right]$$

which proves Equation (2.40). $\square$

Remark

1. If $u = DF$, then Equation (2.40) becomes

$$\mathbb{E}\left[f'(F)G\right] = \mathbb{E}\left[f(F)\delta\left(\frac{GDF}{\| DF \|_H^2}\right)\right].$$

2. If u is a deterministic process then it suffices to assume that $G(\langle DF, u\rangle_H)^{-1} \in \mathbb{D}_{1,2}$ as this implies that $Gu(\langle DF, u\rangle_H)^{-1} \in \text{Dom}(\delta)$.

2.6 Iterated Wiener-Itô integrals

In this section we define the Malliavin derivative via the Wiener-Itô chaos decomposition. See [27], [60] and [74] for more details about the Wiener-Itô chaos decomposition. We mention that one can construct the Malliavin calculus as in the previous sections and use it in concrete applications without mentioning the chaos decomposition. Thus, we restrict ourselves to a short discussion of this topic. We first briefly review the construction of the multiple Wiener-Itô integral with respect to the Brownian motion. Our main reference is [60].

We use T for $[0, T]$ and $H = L^2(T, \mathcal{B}, \mu)$ where μ is a σ-finite measure without atoms and $\mathcal{B}$ is a Borel σ field. For any set $A \in \mathcal{B}$ with $\mu(A) < \infty$, we define

$$W(A) = W(1_A).$$

Then $A \to W(A)$ is a Gaussian measure with independent increments, that is, if $A_1, \dots, A_n$ are disjoint sets with finite measure, the random variables $W(A_1), \dots, W(A_n)$ are independent and for any $A \in \mathcal{B}$ with measure $\mu(A) < \infty$, $W(A)$ has the distribution $N(0, \mu(A))$. We will say that W is an $L^2(\Omega)$-valued Gaussian measure on $(T, \mathcal{B})$. Thus, $W(h)$ can be regarded as the stochastic integral of the function $h \in L^2(T)$ with respect to W. We will write

$$W(h) = \int_T h dW, \quad h \in L^2(T).$$

Let

$$S^n = \{(t_1, t_2, \dots, t_n) \in T^n : \exists i \neq j \;\text{ such that }\; t_i = t_j\} \tag{2.41}$$

be the diagonal set of T^n. We consider the set $\mathcal{E}_n$ of elementary functions of the form

$$f(t_1,\ldots,t_n)=\sum_{i_1,i_2,\ldots,i_n=1}^{k} a_{i_1,i_2,\ldots,i_n}1_{A_{i_1}\times\cdots\times A_{i_n}}(t_1,\ldots,t_n) \tag{2.42}$$

where $k\geq 1$ is finite, $a_{i_1,i_2,\ldots,i_n}\in\mathbb{R}$ and $A_1,A_2,\ldots,A_k$ are pairwise disjoint sets of finite measure. The coefficients $a_{i_1,i_2,\ldots,i_n}$ satisfy the condition

$$a_{i_1,i_2,\ldots,i_n}=0 \quad \text{if} \quad i_p=i_q \quad \text{for some} \quad p\neq q. \tag{2.43}$$

The condition (2.43) implies that the function f vanishes on the set S^n. The collection of elementary functions of the form (2.42)-(2.43) is a vector space. For any elementary function f of the form (2.42)-(2.43), we define

$$I_n(f)=\sum_{i_1,i_2,\ldots,i_n=1}^{k} a_{i_1,i_2,\ldots,i_n}W(A_{i_1})\cdots W(A_{i_n}). \tag{2.44}$$

The value of $I_n(f)$ is well-defined, that is, its definition does not depend on how f is represented by Equations (2.42)-(2.43). In addition, the mapping I_n is linear on the vector space of elementary functions of the form (2.42)-(2.43).
The connection between Wiener-Itô chaos expansion and Malliavin calculus is best explained through the symmetric expansion.

Definition 2.6.1 *The symmetrization $\hat{f}(t_1,t_2,\ldots,t_n)$ of a function $f(t_1,t_2,\ldots,t_n)$ is defined by*

$$\hat{f}(t_1,t_2,\ldots,t_n)=\frac{1}{n!}\sum_{\sigma} f(t_{\sigma(1)},t_{\sigma(2)},\ldots,t_{\sigma(n)}) \tag{2.45}$$

where the summation is taken over all the permutations σ of the set $\{1,2,\ldots,n\}$.

We say that f is symmetric if $\hat{f}=f$. Since Lebesgue measure is symmetric we have

$$\int_{T^n} \mid f(t_{\sigma(1)},t_{\sigma(2)},\ldots,t_{\sigma(n)})\mid^2 dt_1dt_2\cdots dt_n=\int_{T^n}\mid f(t_1,t_2,\ldots,t_n)\mid^2 dt_1dt_2\cdots dt_n \tag{2.46}$$

for any permutation σ. Therefore, by the triangle inequality, we have

$$\parallel \hat{f}\parallel_{L^2(T^n)}\leq \frac{1}{n!}\sum_{\sigma}\parallel f\parallel_{L^2(T^n)}=\frac{1}{n!}n!\parallel f\parallel_{L^2(T^n)}=\parallel f\parallel_{L^2(T^n)}. \tag{2.47}$$

Thus , we have

$$\parallel \hat{f}\parallel_{L^2(T^n)}\leq\parallel f\parallel_{L^2(T^n)}. \tag{2.48}$$

This leads to the following lemma.

Lemma 2.6.2 *If f is an elementary function of the form (2.42)-(2.43) then $I_n(f) = I_n(\hat{f})$.*

Proof

The proof can be found in [60] on page 170. We omit the details. $\square$

Lemma 2.6.3 *If f is an elementary function of the form (2.42)-(2.43) then $\mathbb{E}[I_n(f)] = 0$ and*

$$\mathbb{E}[I_n(f)^2] = n! \int_{T^n} | \hat{f}(t_1, t_2, \dots, t_n) |^2 \, dt_1 dt_2 \cdots dt_n. \tag{2.49}$$

Proof

Let f be defined by Equations (2.42)-(2.43). Then $I_n(f)$ is given in Equation (2.44). Since the sets $A_1, A_2, \dots, A_n$ are pairwise disjoint the corresponding product has expectation 0. Hence, we have $\mathbb{E}[I_n(f)] = 0$.

By Lemma 2.6.2 we have $I_n(f) = I_n(\hat{f})$. Hence, we may assume that f is symmetric, that is,

$$a_{i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(n)}} = a_{i_1, i_2, \dots, i_n}$$

for any permutation σ. Thus, we can write $I_n(f)$ as

$$I_n(f) = n! \sum_{i_1, \dots, i_n = 1}^{k} a_{i_1, i_2, \dots, i_n} W(A_{i_1}) \cdots W(A_{i_n})$$

and

$$\mathbb{E}[I_n(f)^2] = (n!)^2 \sum_{i_1, \dots, i_n}^{k} \sum_{j_1, \dots, j_n}^{k} a_{i_1, \dots, i_n} a_{j_1, \dots, j_n} \mathbb{E}[W(A_{i_1}) \cdots W(A_{i_n}) W(A_{j_1}) \cdots W(A_{j_n})].$$

We note that, for fixed set of indices $i_1 < \dots < i_n$, we have, by Itô isometry,

$$\mathbb{E}[W(A_{i_1}) \cdots W(A_{i_n}) W(A_{j_1}) \cdots W(A_{j_n})] = \begin{cases} \prod_{p=1}^{n} \mu(A_{i_p}), & \text{if } j_1 = i_1, \dots, j_n = i_n \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\mathbb{E}[I_n(f)^2] = (n!)^2 \sum_{i_1 < \dots < i_n} a_{i_1, \dots, i_n}^2 \prod_{p=1}^{n} \mu(A_{i_p}) = n! \int_{T^n} f(t_1, \dots, t_n)^2 dt_1 \cdots dt_n, \tag{2.50}$$

which proves Equation (2.49) since f is assumed to be symmetric. $\square$

Lemma 2.6.4 *Let f be a function in $L^2(T^n)$. Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of elementary functions satisfying (2.42)-(2.43) such that*

$$\lim_{k \to \infty} \int_{T^n} | f(t_1, t_2, \dots, t_n) - f_k(t_1, t_2, \dots, t_n) |^2 \, dt_1 dt_2 \cdots dt_n = 0. \tag{2.51}$$

Proof

The proof is immediate because the usual set of elementary functions is dense in $L^2(T^n)$ and the set S^n has Lebesgue measure zero. □

Now suppose $f \in L^2(T^n)$. Choose a sequence $\{f_k\}_{k=1}^{\infty}$ of elementary functions converging to f in $L^2(T^n)$ which exists by Lemma 2.6.4. Then, by the linearity of I_n and Lemma 2.6.3 we have

$$\mathbb{E}[(I_n(f_k) - I_n(f_l))^2] = n! \parallel \hat{f}_k - \hat{f}_l \parallel^2_{L^2(T^n)} \leq n! \parallel f_k - f_l \parallel^2_{L^2(T^n)} \to 0, \quad \text{as} \quad k, l \to \infty. \quad (2.52)$$

The inequality in (2.52) is justified by (2.48). Hence, the sequence $\{I_n(f_k)\}_{k=1}^{\infty}$ is Cauchy in $L^2(\Omega)$. Define

$$I_n(f) := \lim_{k\to\infty} I_n(f_k) \quad \text{in} \quad L^2(\Omega). \quad (2.53)$$

The value of $I_n(f)$ is well-defined, namely, it does not depend on the choice of the sequence $\{f_k\}_{k=1}^{\infty}$ used in Equation (2.53).

Definition 2.6.5 *Let $f \in L^2(T^n)$. The limit $I_n(f)$ in Equation (2.53) is called the multiple Wiener-Itô integral of f and is denoted by*

$$\int_{T^n} f(t_1, t_2, \ldots, t_n) dW(t_1) dW(t_2) \cdots dW(t_n).$$

The Lemmas 2.6.2 and 2.6.3 can be extended to functions in $L^2(T^n)$ using symmetry, the approximations in Lemma 2.6.4 and the definition of the multiple Wiener-Itô integral. We state this in the following theorem (see [60] page 172).

Theorem 2.6.6 *Let $f \in L^2(T^n)$, $\quad n \geq 1$. Then we have*

1. *$I_n(f) = I_n(\hat{f})$ where $\hat{f}$ is the symmetrization of f.*

2. *$\mathbb{E}[I_n(f)] = 0$.*

3. *$\mathbb{E}[I_n(f)^2] = n! \parallel \hat{f} \parallel^2$ where $\parallel \cdot \parallel$ is the norm on $L^2(T^n)$.*

Next we give the relationship between a multiple Wiener-Itô integral and an iterated Itô integral.

Theorem 2.6.7 *Let $f \in L^2(T^n)$, $n \geq 2$. Then*

$$
\begin{aligned}
I_n(f_n) &:= \int_{T^n} f(t_1, t_2, \ldots, t_n) dW(t_1) dW(t_2) \cdots dW(t_n) \\
&= n! \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1, t_2, \ldots, t_n) dW(t_n) \right) dW(t_{n-1}) \cdots dW(t_1) := n! J_n(f_n)
\end{aligned}
$$

where $\hat{f}$ is the symmetrization of f and
$J_n(f_n) = \int_0^T \cdots \int_0^{t_{n-2}} \left(\int_0^{t_{n-1}} \hat{f}(t_1, t_2, \ldots, t_n) dW(t_n) \right) dW(t_{n-1}) \cdots dW(t_1)$.

Note that in each step the corresponding integrand is adapted because of the limits of the preceding integrals.

Proof

The equality is clear if f is any elementary function of the form (2.42)-(2.43). In the general case, the equality follows by a density argument taking into account that the iterated stochastic Itô integral satisfies the same Itô isometry property as the multiple stochastic integral. Details can be found in [60] on page 173. □

Finally, we give a result on the orthogonality of $I_n(f)$ and $I_m(g)$ in the Hilbert space $L^2(\Omega)$ when $n \neq m$ (see [60] page 175).

Theorem 2.6.8 *Let $n \neq m$. Then*

$$\mathbb{E}[I_n(f) I_m(g)] = 0 \tag{2.54}$$

for any $f \in L^2(T^n)$ and $g \in L^2(T^m)$.

We now state the following theorem without proof. A detailed proof can be found in [70] page 6.

Theorem 2.6.9 *The space $L^2(\Omega)$ can be decomposed into the orthogonal direct sum*

$$L^2(\Omega) = K_0 \oplus K_1 \oplus K_2 \oplus \cdots \oplus K_n \oplus \cdots,$$

where $\oplus$ denotes the direct sum, K_n consists of linear combinations of multiple Wiener-Itô integrals of order n. Each function F in $L^2(\Omega)$ can be uniquely represented by

$$F = \sum_{n=0}^{\infty} I_n(f_n), \qquad f_n \in \hat{L}^2(T^n) \tag{2.55}$$

where $\hat{L}^2(T^n)$ denotes the symmetrization of the space $L^2(T^n)$ and the following equality holds:

$$\| F \|^2_{L^2(\Omega)} = \sum_{n=0}^{\infty} n! \| f_n \|^2_{\hat{L}^2([0,T])^n} . \tag{2.56}$$

For $n = 0$ we adopt the convention that $I_0(f_0) = f_0$ when f_0 is constant.

Corollary 2.6.10 *For each $F \in L^2(\Omega)$ there exist $\hat{f}_n \in \hat{L}^2(T^n)$ such that*

$$F = \sum_{n=0}^{\infty} I_n(\hat{f}_n).$$

Moreover, the functions $\hat{f}_n$ are uniquely defined on $\hat{L}^2(T^n)$.

2.7 Malliavin derivative via chaos expansion

The operators D_t and δ can be represented in terms of the Wiener-Itô chaos expansion. The following theorem is found in [70] page 33.

Theorem 2.7.1 *Let $F \in \mathbb{D}_{1,2}$ and let $F = \sum_{n=0}^{\infty} I_n(\hat{f}_n)$, $\hat{f}_n \in \hat{L}^2(T^n)$. Write $\hat{f}_n(\cdot, t) = \hat{f}_n(x_1, \ldots, x_{n-1}, t)$ so that $\hat{f}_n$ is a symmetric function of the $(x_1, x_2, \ldots, x_n)$. Then*

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(\hat{f}_n(\cdot, t)). \tag{2.57}$$

Also

$$D_t I_n(\hat{f}_n) = n I_{n-1}(\hat{f}_n(\cdot, t)). \tag{2.58}$$

In addition, $F \in \mathbb{D}_{1,2}$ if and only if

$$\sum_{n=1}^{\infty} n n! \| \hat{f}_n \|^2_{L^2(T^n)} < \infty. \tag{2.59}$$

Remark

$D_t F$ is obtained simply by removing one of the stochastic integrals, letting the variable t to be free and multiplying by the factor n.

Proof

Suppose first that $F = I_n(\hat{f}_n)$ where $\hat{f}_n$ is a symmetric and elementary function of the form (2.42)-(2.43). Then

$$D_t F = \sum_{j=1}^{n} \sum_{i_1,\ldots,i_n=1}^{k} a_{i_1,\ldots,i_n} W(A_{i_1}) \cdots 1_{A_{i_j}}(t) \cdots W(A_{i_n}) = n I_{n-1}(\hat{f}_n(\cdot, t)) \tag{2.60}$$

where we have used the symmetry in the second equality. Let $F \in \mathbb{D}_{1,2}$, $F = \sum_{n=0}^{\infty} I_n(\hat{f}_n)$, $\hat{f}_n \in \hat{L}^2(T^n)$. We consider the sequence of partial sums $F_N = \sum_{n=0}^{N} I_n(\hat{f}_n)$, $N \geq 0$. Then F_N converges to F in $L^2(\Omega)$ as $N \to \infty$. This implies that the sequence DF_N converges to DF in $L^2(\Omega \times T^n)$ as $N \to \infty$.
On the other hand, for each $n \geq 1$, $I_n(\hat{f}_n)$ is the limit in $L^2(\Omega)$ of the sequence $\{I_n(\hat{f}_n^k)\}_{k\geq 1}$, where $\hat{f}_n^k$ is an elementary function and the sequence $\hat{f}_n^k$ converges to $\hat{f}_n$ in $\hat{L}^2(T^n)$ as $k \to \infty$. Hence, the sequence $D(I_n(\hat{f}_n^k))$ converges to $D(I_n(\hat{f}_n))$ in $L^2(\Omega \times T^n)$ as $k \to \infty$. Moreover, the sequence $I_{n-1}(\hat{f}_n^k(\cdot, t))$ converges to $I_{n-1}(\hat{f}_n(\cdot, t))$ in $L^2(\Omega \times T^n)$ as $k \to \infty$. Finally, using the closability of the operator D and Equation (2.60) the result follows. □

Next we will compute the derivative of a conditional expectation with respect to a σ-field generated by Gaussian stochastic integrals. Let $A \in \mathcal{B}$. We will denote by $\mathcal{F}_A$ the σ-field (completed with respect to the probability P) generated by random variables $\{W(B), B \subset A, B \in \mathcal{B}_0\}$. We need the following lemma (see [70] page 33).

Lemma 2.7.2 *Suppose F is a sequence of integrable random variable with a representation in (2.55). Let $A \in \mathcal{B}$. Then*

$$\mathbb{E}[F \mid \mathcal{F}_A] = \sum_{n=0}^{\infty} I_n(\hat{f}_n 1_A^{\otimes n}). \tag{2.61}$$

where $1_A^{\otimes n} = 1_A \otimes \cdots \otimes 1_A$.

Proof
It suffices to assume that $F = I_n(\hat{f}_n)$ where $\hat{f}_n$ is a function in $L^2(\Omega)$. By density arguments we set

$$\hat{f}_n = 1_{B_1 \times \cdots \times B_n}$$

where $B_1, \ldots, B_n$ are pairwise disjoint sets of finite measure. Then we have

$$\begin{aligned}\mathbb{E}[F \mid \mathcal{F}_A] &= \mathbb{E}[W(B_1) \cdots W(B_n) \mid \mathcal{F}_A] = \mathbb{E}[\prod_{i=1}^{n}(W(B_i \cap A) + W(B_i \cap A^c)) \mid \mathcal{F}_A] \\ &= I_n(1_{B_1 \cap A \times \cdots \times B_n \cap A})\end{aligned}$$ □

Lemma 2.7.3 *Suppose $F \in \mathbb{D}_{1,2}$ and let $A \in \mathcal{B}$. Then the conditional expectation $\mathbb{E}[F \mid \mathcal{F}_A]$ also belongs to the space $\mathbb{D}_{1,2}$ and we have*

$$D_t(\mathbb{E}[F \mid \mathcal{F}_A]) = \mathbb{E}[D_t F \mid \mathcal{F}_A] 1_A(t) \qquad a.e \ in \quad [0, T] \times \Omega. \tag{2.62}$$

Proof

By Theorem 2.7.1 and Lemma 2.7.2 we have

$$D_t\mathbb{E}[F \mid \mathcal{F}_A] \;=\; \sum_{n=1}^{\infty} nI_{n-1}(\hat{f}_n(\cdot,t)1_A^{\otimes(n-1)})1_A(t) = \mathbb{E}[D_tF \mid \mathcal{F}_A]1_A(t). \qquad \square$$

We note that $D_tI(\hat{f}) = \hat{f}(t)$. Suppose that $F \in \mathbb{D}_{1,2}$ with a Wiener chaos expansion $F = \sum_{n=0}^{\infty} I_n(\hat{f}_n)$. Then applying Theorem 2.7.1 k times we obtain

$$\begin{aligned} D^k_{t_1,\ldots,t_k}F &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)I_{n-k}(\hat{f}_n(\cdot,t_1,\ldots,t_n)) \\ &= k!\hat{f}_k(t_1,\ldots,t_k) + (k+1)!I_1(\hat{f}_{k+1}(t_1,\ldots,t_k,\cdot) + \ldots \end{aligned} \tag{2.63}$$

We can write Equation (2.63) as

$$D^k_{t_1,\ldots,t_k}F = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} I_{n-k}(\hat{f}_n(\cdot,t_1,\ldots,t_n)).$$

Using Equation (2.56) we see that the L^2-norm of this is given by

$$\| D^k_{t_1,\ldots,t_k}F \|^2_{L^2([0,T]^k)} = \sum_{n=k}^{\infty} \frac{(n!)^2}{((n-k)!)^2}(n-k)! \| \hat{f}_n \|^2_{L^2([0,T]^k)} = \sum_{n=k}^{\infty} \frac{(n!)^2}{(n-k)!} \| \hat{f}_n \|^2_{L^2(T^n)} .$$

We recall that

$$\mathbb{E}[I_n(\hat{f}_n)] = 0 \quad \text{for all} \quad n \geq 1.$$

Hence, by taking the expectation on both sides of Equation (2.63) we obtain the following result:

$$\hat{f}_k = \frac{1}{k!}\mathbb{E}[D^k_{t_1,\ldots,t_k}F] \quad \text{for every} \quad k \geq 0. \tag{2.64}$$

This is called the Stroock formula (see [30]). The formula is given in [70] page 35 as an exercise.

2.8 Skorohod integral via chaos expansion

In this section we define the Skorohod integral in terms of the Wiener-Itô chaos expansion. Let $u(t,\omega), \omega \in \Omega,\ t \in [0,T]$ be a stochastic process such that

$$u(t,\cdot) \;\text{ is }\; \mathcal{F}_t \;\text{ measurable for all }\; t \in [0,T] \tag{2.65}$$

and

$$\mathbb{E}[u^2(t,\omega)] < \infty \quad \text{for all} \quad t \in [0,T]. \tag{2.66}$$

Then, for each $t \in [0,T]$, we can apply the Wiener Itô chaos expansion to the random variable $\omega \to u(t,\omega)$ and obtain

$$u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)) \tag{2.67}$$

where for each $n \geq 1$, $f_n \in L^2(S^{n+1})$ is a symmetric function in the first n variables. The following theorem was taken from [70] page 41.

Theorem 2.8.1 *Let $u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$. Then $u(t,\omega)$ belongs to Dom(δ) if and only if the series*

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\hat{f}_n) \tag{2.68}$$

converges in $L^2(\Omega)$.

We can write Equation (2.68) without symmetrization because, for each n,
$I_{n+1}(f_n) = I_{n+1}(\hat{f}_n)$. However, we need the symmetrization in order to compute the L^2-norm of the stochastic integrals.

Proof

Let $n \geq 1$ and $g \in L^2(T^n)$ a symmetric function. We have that

$$\begin{aligned}
\mathbb{E}\left[\langle u, D\left(I_n(g)\right)\rangle_H\right] &= \sum_{m=0}^{\infty} \int_{T^n} \mathbb{E}\left[I_m(f_m(\cdot,t)) n I_{n-1}(g(\cdot,t))\right] dt \\
&= n \int_{T^n} \mathbb{E}\left[I_{n-1}(f_{n-1}(\cdot,t)) I_{n-1}(g(\cdot,t))\right] dt \\
&= n(n-1)! \int_{T^n} \langle f_{n-1}(\cdot,t), g(\cdot,t)\rangle_{L^2(T^{n-1})} dt \\
&= n! \langle f_n, g\rangle_{L^2(T^n)} \\
&= n! \langle \hat{f}_{n-1}, g\rangle_{L^2(T^n)} \\
&= E\left[I_n(\hat{f}_{n-1} I_n(g)\right].
\end{aligned}$$

Suppose first that $u \in \text{Dom}(\delta)$. Then using Equation (2.22) and the above calculation we obtain that for all $n \geq 1$ and $g \in L^2(T^n)$ symmetric,

$$\mathbb{E}\left[\delta(u) I_n(g)\right] = \mathbb{E}\left[\langle u, D(I_n(g))\rangle_H\right] = \mathbb{E}\left[\langle I_n(\hat{f}_{n-1}), I_n(g)\rangle_H\right].$$

This implies that $I_n(g)$ is the projection of $\delta(u)$ in the n-th Wiener chaos. Hence, the series (2.68) converges in $L^2(\Omega)$ to its sum which is equal to $\delta(u)$.
Conversely, we assume that the series (2.68) converges in $L^2(\Omega)$ and we denote its sum by V. Let $F_N = \sum_{n=0}^{N} I_n(g_n)$, where $g_n \in L^2(T^n)$ are symmetric and $N \geq 1$. Using the above calculation we obtain that for all $N \geq 1$

$$\mathbb{E}\left[\int_{T^n} u_t D_t F_N dt\right] = \sum_{n=1}^{N} \mathbb{E}\left[I_n(\hat{f}_{n-1}) I_n(g_n)\right].$$

In particular,

$$| \mathbb{E}\left[\int_{T^n} u_t D_t F_N dt\right] | \leq \| V \|_{L^2(\Omega)} \| F_N \|_{L^2(\Omega)} .$$

Let $F \in \mathbb{D}_{1,2}$, $F = \sum_{n=0}^{\infty} I_n(g_n)$, where $g_n \in L^2(T^n)$ are symmetric. Then F_N converges to F in $L^2(\Omega)$ as $N \to \infty$ and DF_N converges to DF in $L^2(T^n \times \Omega)$ as $N \to \infty$. Therefore

$$| \mathbb{E}\left[\int_{T^n} u_t D_t F dt\right] | \leq \| V \|_{L^2(\Omega)} \| F_N \|_{L^2(\Omega)},$$

which implies that $u \in \mathrm{Dom}(\delta)$. □

2.9 The Clark-Haussmann-Ocone formula

Suppose that $W = \{W_t,\ t \in [0,T]\}$ is a 1-dimensional Brownian motion. The Itô representation theorem states that any $F \in L^2(\Omega)$ can be written as

$$F = \mathbb{E}[F] + \int_0^T \phi(t) dW_t \tag{2.69}$$

where ϕ is an adapted process in $L^2([0,T] \times \Omega)$. If, in addition, $F \in \mathbb{D}_{1,2}$ it turns out that the process ϕ can be expressed as a Malliavin derivative of F. This is called the Clark-Ocone representation formula (see [70] page 46).

Theorem 2.9.1 *Let $F \in \mathbb{D}_{1,2}$. Then we have*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F \mid \mathcal{F}_t] dW_t \quad a.s. \tag{2.70}$$

Proof
Suppose that F can be written in the form (2.69) with $\phi \in L^2([0,T] \times \Omega)$. Then, for any

$\varphi \in L^2([0,T] \times \Omega)$, using Itô isometry and that the expected value of an Itô integral is 0, we have

$$\mathbb{E}\left[\delta(\varphi)F\right] = \mathbb{E}\left[\int_0^T \varphi(t)dW_t \left(\mathbb{E}[F] + \int_0^T \phi(t)dW_t\right)\right] = \int_0^T \mathbb{E}[\varphi(t)\phi(t)]dt. \quad (2.71)$$

On the other hand, using integration by parts formula and taking into account that φ is adapted, we obtain

$$\mathbb{E}\left[\delta(\varphi)F\right] = \mathbb{E}\left[\int_0^T \varphi(t)D_tF dt\right] = \int_0^T \mathbb{E}\left[\varphi(t)\mathbb{E}\left[D_tF \mid \mathcal{F}_t\right]\right]dt. \quad (2.72)$$

Comparing Equations (2.71) and (2.72) we get

$$\phi(t) = \mathbb{E}\left[D_tF \mid \mathcal{F}_t\right]$$

which proves Equation (2.70). □

Theorem 2.9.1 shows that the Malliavin derivative D_t provides an identification of the integrator in the martingale representation theorem in a Brownian motion framework, which plays a central role in financial mathematics, in particular, to obtain replicating hedging strategies for options (see [14]). Therefore, the hedging portfolio is naturally related to the Malliavin derivative D_t of the terminal payoff.

Chapter 3

Application of Malliavin calculus to the Calculations of Greeks for Continuous Processes

The main focus of this thesis is to prove some extensions of the work of Fournie et al. [35] and [36] to the case where the market is driven by a Lévy process. Before proceeding to this, we briefly review the applications of Malliavin calculus to compute Greeks of different types of option prices in the Brownian motion case. We use the chain rule and the integration by parts formula. Most of the results we obtain are known in the literature but the results are sometimes only quoted (see [8] and [37]). Here, we give explicit calculations of these results. In particular we calculate Δ, Γ and $\mathcal{V}$ ($\mathcal{V}$=Vega) for path independent (European call option) and path dependent (Asian option) options. We also compute Δ for barrier and look-back options and for the Heston model.

It turns out that the Greeks we obtain are expressed as expectations of the payoff function multiplied by a random variable which is a function of the underlying process. This is crucial for simulation by the Monte Carlo method. We mention that all the results below follow from results in [35] and [36]. Instead of considering the payoff as the element of the L^2 space as in [35], we will consider it as a function of polynomial growth. This allows us to handle more cases, for example the digital option.

Let $X_t \in \mathbb{R}^n$ be an Itô-diffusion process given by the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x \tag{3.1}$$

where $\{W_t, 0 \leq t \leq T\}$ is a n-dimensional standard Brownian motion. The coefficients $b(x)$ and $\sigma(x)$ are deterministic and are assumed to be bounded with bounded partial derivatives. In addition, the coefficients $b(x)$ and $\sigma(x)$ satisfy Lipschitz condition and linear growth condition, that is, there exists a constant $C < \infty$ such that

$$| b(x) - b(y) | + | \sigma(x) - \sigma(y) | \leq C | x - y |, \tag{3.2}$$

$$| b(x) | + | \sigma(x) | \leq C(1 + | x |). \tag{3.3}$$

These conditions ensure the existence of a unique strong solution to Equation (3.1).
Let Φ denote the payoff function of some financial quantity. We consider a payoff depending on the prices at a finite number of times, that is,

$$\Phi = \Phi(X_{t_1}, \cdots, X_{t_n})$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is of polynomial growth (to be defined later on) with bounded derivatives and $0 < t_1 < \cdots < t_n = T$. Examples of payoff functions include the European call option $\Phi(x) = (x - K)^+$ where K is a strike price and the digital option $\Phi(x) = 1_{\{x \geq K\}}$. Here $n = 1$ and $t_1 = T$ the expiry date of the option.
Given $0 < t_1 < \ldots < t_n = T$ we will assume that the option price $u(x)$ is computed, under the risk neutral probability measure, as

$$u(x) = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \mid X_0 = x] \tag{3.4}$$

where r is the risk free interest rate assumed to be constant and x is the initial stock price. In particular, when $n = 1$ and $t_1 = T$ we have $u(x) = \mathbb{E}[e^{-rT}\Phi(X_T) \mid X_0 = x]$. We want to compute the following expression

$$\frac{\partial}{\partial \alpha}\mathbb{E}[e^{-rT}\Phi(X_{t_1}(\alpha), \cdots, X_{t_n}(\alpha)) \mid X_0 = x], \quad \alpha = x, \sigma. \tag{3.5}$$

There are several ways of doing this (see Chapter 1). Here we present the Malliavin calculus approach. At several places, we will require the matrix σ to satisfy the following condition:

$$\exists \eta > 0 \quad \xi^T \sigma^T(x)\sigma(x)\xi > \eta | \xi |^2 \quad \text{for all } \ \xi, x \in \mathbb{R}^n \ \text{ with } \ \xi \neq 0. \tag{3.6}$$

where ξ^T denotes the transpose of ξ. This is called the *uniform ellipticity condition.*
As stated in [35], the condition (3.6) ensures that $\sigma^{-1}(X_t)Y_t$ belongs to $L^2(\Omega \times [0,T])$ where $\{Y_t, 0 \leq t \leq T\}$ is the first variation process (see Chapter 2 Section 2.4) of $\{X_t, 0 \leq t \leq T\}$. In addition, if $\bar{b}$ is a bounded function then $\sigma^{-1}(X_t)\bar{b}(X_t)$ belongs to $L^2(\Omega \times [0,T])$ and $\sigma^{-1}\bar{b}$

is a bounded function.

The weight function obtained when computing Greeks using the integration by parts formula should not degenerate with probability one, otherwise the computation will not be valid. To avoid this degeneracy we introduce the set Υ_n (see [35]) defined by

$$\Upsilon_n = \{a \in L^2([0,T]) \mid \int_0^{t_i} a(t)dt = 1 \text{ for all } i = 1, \dots, n\}. \tag{3.7}$$

We need the following lemma.

Lemma 3.0.2 *Let $a \in \Upsilon_n$ and $X_{t_i} \in \mathbb{D}_{1,2}$. Then*

$$\int_0^T D_t X_{t_i} a(t)\sigma^{-1}(t)Y(t)dt = Y(t_i), \quad i = 1, \dots, n.$$

Proof
We have

$$\begin{aligned}
\int_0^T D_t X_{t_i} a(t)\sigma^{-1}(t)Y(t)dt &= \int_0^T Y(t_i)Y(t)^{-1}\sigma(t)1_{t\le t_i}a(t)\sigma^{-1}(t)Y(t)dt \text{ by } (2.37) \\
&= \int_0^T Y(t_i)Y(t)^{-1}\sigma(t)\sigma^{-1}(t)Y(t)a(t)1_{t\le t_i}dt \\
&= \int_0^T Y(t_i)a(t)1_{t\le t_i}dt \\
&= \int_0^{t_i} Y(t_i)a(t)dt \\
&= Y(t_i)\int_0^{t_i} a(t)dt \\
&= Y(t_i)
\end{aligned}$$

□

We also need the following definition.

Definition 3.0.3 *A function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is of polynomial growth if there exist constants $C > 0, \beta \ge 0$ such that*

$$\mid \Phi(x) \mid \le C(1+ \mid x \mid)^\beta \quad \textit{for all} \quad x \in \mathbb{R}^n. \tag{3.8}$$

The following proposition is a modification of one of the main results in [35].

Proposition 3.0.4 *Assume that b and σ (in Equation (3.1)) are continuously differentiable with bounded partial derivatives and that the matrix σ satisfies the uniform ellipticity condition (see (3.6)). Then, for any $\Phi : \mathbb{R} \to \mathbb{R}$ of polynomial growth and $a \in \Upsilon_n$, we have*

$$\Delta = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\pi \mid X_0 = x] \tag{3.9}$$

where $\pi = \int_0^T a(t)(\sigma^{-1}(X_t)Y_t)^T dW_t$.

This is called the *Bismut-Elworthy-Li formula.*

Proof

We first assume that Φ is continuously differentiable with bounded partial derivatives. We show that we can compute Δ by calculating the derivative of the payoff function with respect to x inside the the expectation operator. We have

$$\frac{e^{-rT}\Phi(X_{t_1}^{x+h}, \cdots, X_{t_n}^{x+h}) - e^{-rT}\Phi(X_{t_1}^{x}, \cdots, X_{t_n}^{x})}{\| h \|} - \frac{\langle e^{-rT}\sum_{i=1}^n \frac{\partial}{\partial x_i}\Phi(X_{t_1}, \cdots, X_{t_n})Y_{t_i}, h\rangle}{\| h \|} \to 0 \text{ a.s.} \tag{3.10}$$

as $\| h \| \to 0$ where $Y_{t_i} = \frac{\partial X_{t_i}}{\partial x}$ and X_t^{x+h} denote the solution X_t starting from $x + h$, that is, $X_0^{x+h} = x + h$. The term $\frac{\langle e^{-rT}\sum_{i=1}^n \frac{\partial}{\partial x_i}\Phi(X_{t_1}, \cdots, X_{t_n})Y_{t_i}, h\rangle}{\| h \|}$ is bounded above by some number independent of h since the payoff function Φ has bounded partial derivatives. Let M be the bound of the partial derivatives of Φ. Then, we can show that

$$\frac{\| e^{-rT}\Phi(X_{t_1}^{x+h}, \cdots, X_{t_n}^{x+h}) - e^{-rT}\Phi(X_{t_1}^{x}, \cdots, X_{t_n}^{x}) \|}{\| h \|} \leq M \sum_{i=1}^n \frac{\| X_{t_i}^{x+h} - X_{t_i}^{x} \|}{\| h \|}. \tag{3.11}$$

Using the result that $\sum_{i=1}^n \frac{\| X_{t_i}^{x+h} - X_{t_i}^{x} \|}{\| h \|}$ is bounded (see [79] page 256) leads to the boundedness of $\frac{e^{-rT}\Phi(X_{t_1}^{x+h}, \cdots, X_{t_n}^{x+h}) - e^{-rT}\Phi(X_{t_1}^{x}, \cdots, X_{t_n}^{x})}{\| h \|}$. This, in turn, tells us that

$$\frac{\| e^{-rT}\Phi(X_{t_1}^{x+h}, \cdots, X_{t_n}^{x+h}) - e^{-rT}\Phi(X_{t_1}^{x}, \cdots, X_{t_n}^{x}) \|}{\| h \|} - \frac{\langle e^{-rT}\sum_{i=1}^n \frac{\partial}{\partial x_i}\Phi(X_{t_1}, \cdots, X_{t_n})Y_{t_i}, h\rangle}{\| h \|}$$

is bounded. Since it converges to zero a.s., the dominated convergence theorem says that it converges also to zero in $L^1(\Omega)$. We therefore conclude that

$$\frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \mid X_0 = x] = \mathbb{E}[e^{-rT}\sum_{i=1}^n \frac{\partial}{\partial x_i}\Phi(X_{t_1}, \cdots, X_{t_n})Y_{t_i} \mid X_0 = x]. \tag{3.12}$$

We now compute Δ as follows:

$$\begin{aligned} \Delta &= \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \mid X_0 = x] = \mathbb{E}[e^{-rT}\sum_{i=1}^n \frac{\partial}{\partial x_i}\Phi(X_{t_1}, \cdots, X_{t_n})\frac{\partial X_{t_i}}{\partial x} \mid X_0 = x] \\ &= \mathbb{E}[e^{-rT}\sum_{i=1}^n \frac{\partial}{\partial x_i}\Phi(X_{t_1}, \cdots, X_{t_n})Y_{t_i} \mid X_0 = x] \qquad (3.13) \end{aligned}$$

where Y_{t_i} is the first variation process associated with X_{t_i}. Assume that X_{t_i} belongs to $\mathbb{D}_{1,2}$. From Lemma 3.0.2 we have

$$Y_{t_i} = \int_0^T D_t X_{t_i} a(t)\sigma^{-1}(X_t)Y_t dt$$

for any $a(t) \in \Upsilon_n$. Therefore, we have

$$\begin{aligned}
\Delta &= \mathbb{E}[\int_0^T e^{-rT} \sum_{i=1}^n \frac{\partial}{\partial x_i}\Phi(X_{t_1}, \cdots, X_{t_n}) D_t X_{t_i} a(t)\sigma^{-1}(X_t)Y_t dt \mid X_0 = x] \\
&= \mathbb{E}[\int_0^T D_t(e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}))a(t)\sigma^{-1}(X_t)Y_t dt \mid X_0 = x] \ \text{ by } \ (2.16) \\
&= \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \int_0^T a(t)(\sigma^{-1}(X_t)Y_t)^T dW_t \mid X_0 = x] \qquad (3.14)
\end{aligned}$$

by (2.22) and adaptedness of $a(t)(\sigma^{-1}(X_t)Y_t)^T$.
We now consider the general case. We approximate Φ by a sequence $\{\Phi_k\}_{k\in\mathbb{N}}$ of functions each with bounded derivatives and compact support. Let $u_k(x) = \mathbb{E}[e^{-rT}\Phi_k(X_{t_1}, \cdots, X_{t_n}) \mid X_0 = x]$ be the price associated with the payoff function Φ_n. We need to first show that $u_k(x)$ converges to $u(x)$. Since $X_{t_1}, \ldots, X_{t_n}$ satisfy Equation (3.1), we let $p_{t_1,\ldots,t_n}(x, x_1, \ldots, x_n)$ be the transition density of $X_{t_1}, \ldots, X_{t_n}$. Then

$$\begin{aligned}
\mid u_k(x) - u(x) \mid &= \mid e^{-rT} \int_{\mathbb{R}^n} [\Phi_k(x_1, \ldots, x_n) - \Phi(x_1, \ldots, x_n)] p_{t_1,\ldots,t_n}(x, x_1, \ldots, x_n) dx_1 \cdots dx_n \mid \\
&\leq \left(e^{-2rT} \int_{\mathbb{R}^n} \mid \Phi_k(x_1, \ldots, x_n) - \Phi(x_1, \ldots, x_n) \mid^2 dx_1 \cdots dx_n\right)^{\frac{1}{2}} \\
&\quad \left(\int_{\mathbb{R}^n} p_{t_1,\ldots,t_n}(x, x_1, \ldots, x_n)^2 dx_1 \ldots dx_n\right)^{\frac{1}{2}}
\end{aligned}$$

The dominated convergence theorem implies that $\mid \Phi_k(x_1, \ldots, x_n) - \Phi(x_1, \ldots, x_n) \mid^2$ converges to 0 as $k \to \infty$. Since b and σ are continuously differentiable with bounded partial derivatives, the transition density is bounded by

$$\mid p_t(x, y) \mid \leq \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-y)^2}{2\sigma^2 t}}. \qquad (3.15)$$

We note that

$$\mid p_{t_1,\ldots,t_n}(x, x_1, \ldots, x_n) \mid \;=\; p_{t_1}(x, x_1) p_{t_2-t_1}(x_1, x_2) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n) < \infty.$$

Therefore we have

$$\mid u_k(x) - u(x) \mid \to 0 \ \text{ for all } \ x. \qquad (3.16)$$

Hence $u_k(x)$ converges to $u(x)$.
From the above calculations we have

$$\frac{\partial}{\partial x}u_k(x) = \mathbb{E}[e^{-rT}\Phi_k(X_{t_1}, \cdots, X_{t_n})\pi \mid X_0 = x]$$

where $\pi = \int_0^T a(s)(\sigma^{-1}(X_s)Y_s)^T dW_s$. Furthermore, let

$$h(x) := \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\pi \mid X_0 = x].$$

By the Cauchy-Schartz inequality we have

$$\mid \frac{\partial}{\partial x}u_k(x) - h(x) \mid \leq \varepsilon_k(x)\psi(x)$$

where

$$\xi_k(x) = \left(\mathbb{E}\left[e^{-rT}\Phi_k(X_{t_1}, \cdots, X_{t_n}) - e^{-rt}\Phi(X_{t_1}, \cdots, X_{t_n})\right]^2\right)^{\frac{1}{2}}$$

and

$$\psi(x) = \left(\mathbb{E}\left[\int_0^T a(t)(\sigma^{-1}(X_t)Y_t)^T dW_t\right]^2\right)^{\frac{1}{2}}.$$

By a continuity argument of the expectation operator, this proves that

$$\sup_{x \in K} \mid \frac{\partial}{\partial x}u_k(x) - h(x) \mid \leq \varepsilon_k(\hat{x})\psi(\hat{x}) \quad \text{for some} \quad \hat{x} \in K$$

where K is an arbitrary compact subset of $\mathbb{R}^n$ which provides

$$\frac{\partial}{\partial x}u_k(x) \to h(x) \quad \text{uniformly on compact subset of} \quad \mathbb{R}^n. \tag{3.17}$$

From (3.16) and (3.17) we conclude that the function $u(x)$ is continuously differentiable and that

$$\frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \mid X_0 = x] = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\pi \mid X_0 = x]. \qquad \square$$

Remark

The application of formula (3.9) does not require the payoff function to be differentiable nor to know the density function of X_t but we do need to know X_t. The weight function π in formula (3.9) is independent of the payoff function and is not unique (it depends on the choice of $a(t)$).

Next we consider Γ which is defined as the second derivative of the option price with respect to the initial price x. It is actually the derivative of Δ with respect to the initial price x.

Proposition 3.0.5 *Assume that the matrix* σ *is uniformly elliptic. Let* $u = a(t)(\sigma^{-1}(X_t)Y_t)^T$ *and* $\delta(u) = \int_0^T a(t)(\sigma^{-1}(X_t)Y_t)^T dW_t$ *for any* $a \in \Upsilon_n$*. Let* $\Phi : \mathbb{R} \to \mathbb{R}$ *be a function of polynomial growth. Then we have*

$$\Gamma = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \left(\delta(u)\delta(u) + \frac{\partial}{\partial x}(\delta(u))\right) \mid X_0 = x]. \tag{3.18}$$

Proof

We first assume that Φ is a continuously differentiable function with bounded derivatives. By definition of Γ we have

$$\Gamma = \frac{\partial^2}{\partial x^2}\mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \mid X_0 = x] = \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\delta(u) \mid X_0 = x]. \tag{3.19}$$

We note that $\Phi(X_{t_1}, \cdots, X_{t_n})\delta(u)$ is a function of both $X_{t_1}, \ldots, X_{t_n}$ and x. Thus, we have

$$\begin{aligned}
\Gamma &= \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\delta(u) \mid X_0 = x] \\
&= \mathbb{E}[e^{-rT}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\Phi(X_{t_1}, \cdots, X_{t_n})\right)\delta(u)\frac{\partial X_{t_i}}{\partial x} \mid X_0 = x] \\
&+ \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\frac{\partial}{\partial x}(\delta(u)) \mid X_0 = x] \\
&= \mathbb{E}[e^{-rT}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(\Phi(X_{t_1}, \cdots, X_{t_n})\delta(u))Y_{t_i} \mid X_0 = x] \\
&+ \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\frac{\partial}{\partial x}(\delta(u)) \mid X_0 = x].
\end{aligned}$$

The first term on the right hand side is similar to the one in the computation of Δ. We proceed in the same way as from Equation (3.13) to Equation (3.14) in the proof of Proposition 3.0.4 and we obtain

$$\begin{aligned}
\Gamma &= \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\delta(u)\delta(u) \mid X_0 = x] + \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\frac{\partial}{\partial x}(\delta(u)) \mid X_0 = x] \\
&= \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\left(\delta(u)\delta(u) + \frac{\partial}{\partial x}(\delta(u))\right) \mid X_0 = x].
\end{aligned}$$

The result can be extended to the general case by a density argument. We omit the details. □

Remark

We can interchange the partial derivative with respect to x and the Skorohod integration to get another expression for Γ.

3.1 Generalized Greeks

The definitions of ρ and $\mathcal{V}$ given in Chapter 1 are the common ones for the Black-Scholes model. We, however, need to develop a more robust framework for the definitions of ρ and $\mathcal{V}$ since the drift and the volatility terms are functions of the underlying asset price. ρ and $\mathcal{V}$ quantify the impact of small perturbation in a specified direction on the drift and the volatility respectively (see [9]).

We consider the stochastic differential equation given in Equation (3.1). Let $\bar{b} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ be a bounded function and let $\bar{\sigma} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be another bounded function. We assume, for every $\varepsilon \in [0,1]$, that $\bar{b}(\cdot)$, $(b+\varepsilon\bar{b})(\cdot)$, $\bar{\sigma}(\cdot)$ and $(\sigma+\varepsilon\bar{\sigma})(\cdot)$ are continuously differentiable with bounded derivatives. Moreover, we assume that $(\sigma+\varepsilon\bar{\sigma})(\cdot)$ satisfies the uniform ellipticity condition:

$$\exists\eta > 0,\;\; \xi^T(\sigma+\varepsilon\bar{\sigma})^T(x)(\sigma+\varepsilon\bar{\sigma})(x)\xi > \eta \mid \xi \mid^2 \quad \text{for all}\;\; \xi, x \in \mathbb{R}^n \;\; \text{with} \;\; \xi \neq 0.$$

As in [35] we introduce the following set.

$$\tilde{\Upsilon}_n = \{a \in L^2([0,T]) \mid \int_{t_{i-1}}^{t_i} a(t)dt = 1 \quad \text{for} \;\; i = 1,\ldots,n\}. \tag{3.20}$$

We need the following theorem (see [56]).

Theorem 3.1.1 *Let $\{W_t, 0 \le t \le T\}$ be a Brownian motion under probability measure P, $\{\mathcal{F}_t\}_{0\le t\le T}$ be the filtration generated by $\{W_t, 0 \le t \le T\}$, $\{\theta_t, 0 \le t \le T\}$ be an $\mathcal{F}_t$-adapted process satisfying*

$$\mathbb{E}[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)] < \infty,$$

and let

$$W_t^\theta = W_t + \int_0^t \theta_s ds$$

and

$$Z_t^\theta = \exp(\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds), \quad t \in [0,T]$$

and define the probability measure Q by

$$Q(F) = \int_F Z_T^\theta dP, \quad F \in \mathcal{F}_t.$$

Then the process W_t^θ is a Brownian motion under the probability measure Q and more generally we have

$$\mathbb{E}[F(W^\theta)] = \mathbb{E}[F(W)Z_T^\theta] \quad F \in L^1(\Omega, P).$$

This is called the *Girsanov Theorem.* We can show that $\mathbb{E}[Z_T^\theta] = 1$ (see [60] page 137).
We define a perturbed process $\{X_t^\varepsilon,\ 0 \le t \le T\}$ with respect to the property under investigation. For ρ we define the perturbed process $\{X_t^\varepsilon,\ 0 \le t \le T\}$ by

$$dX_t^\varepsilon = [b(X_t^\varepsilon) + \varepsilon \bar{b}(X_t^\varepsilon)]dt + \sigma(X_t^\varepsilon)dW_t, \quad X_0^\varepsilon = x \tag{3.21}$$

where ε is a small parameter. We also define the perturbed option price $u^\varepsilon(x)$:

$$u^\varepsilon(x) = \mathbb{E}[e^{-rT}\Phi(X^\varepsilon(\cdot)) \mid X_0^\varepsilon = x]. \tag{3.22}$$

Equations (3.21) and (3.22) imply the impact of a structural change in the drift and the price. We, therefore, define a generalized ρ as follows.

Definition 3.1.2 *ρ is defined as the partial derivative of the perturbed option price with respect to ε in the direction $\bar{b}$:*

$$\rho := \frac{\partial}{\partial \varepsilon} u^\varepsilon(x) \mid_{\varepsilon=0} . \tag{3.23}$$

The following proposition gives the derivative of the perturbed option price $u^\varepsilon(x)$ with respect to ε at $\varepsilon = 0$ [35].

Proposition 3.1.3 *Assume that the matrix σ is uniformly elliptic. Then, for any $\Phi : \mathbb{R} \to \mathbb{R}$ of polynomial growth, the function $\varepsilon \to u^\varepsilon(x)$ is differentiable at $\varepsilon = 0$ for any $x \in \mathbb{R}^n$ and*

$$\frac{\partial}{\partial \varepsilon} u^\varepsilon(x) \mid_{\varepsilon=0} = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \int_0^T (\sigma^{-1}(X_t)\bar{b}(X_t))^T dW_t \mid X_0 = x]. \tag{3.24}$$

Proof
Let

$$\widetilde{M}_T^\varepsilon = \exp\{\varepsilon \int_0^T (\sigma^{-1}(X_t^\varepsilon)\bar{b}(X_t^\varepsilon))^T dW_t^\varepsilon - \frac{\varepsilon^2}{2}\int_0^T \| \sigma^{-1}(X_t^\varepsilon)\bar{b}(X_t^\varepsilon) \|^2 dt\}$$

and define the probability measure Q, equivalent to P, by

$$\frac{dQ}{dP} = \widetilde{M}_T^\varepsilon.$$

Therefore we can write Equation (3.21) as

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sigma(X_t^\varepsilon)dW_t^\varepsilon, \quad X_0^\varepsilon = x \tag{3.25}$$

with $dW_t^\varepsilon = dW_t + \varepsilon\sigma^{-1}(X_t^\varepsilon)\bar{b}(X_t^\varepsilon)dt$. By Theorem 3.1.1 W_t^ε is a standard Brownian motion under the probability measure Q. The joint distribution of $(X^\varepsilon, W^\varepsilon)$ under Q^ε coincide with the joint distribution of (X, W) under P. Therefore, we have

$$u^\varepsilon(x) = \mathbb{E}^Q[e^{-rT}\widetilde{M}_T^\varepsilon \Phi(X_{t_1}^\varepsilon, \cdots, X_{t_n}^\varepsilon) \mid X_0^\varepsilon = x] = \mathbb{E}^P[e^{-rT}M_T^\varepsilon \Phi(X_{t_1}, \cdots, X_{t_n}) \mid X_0 = x]$$

where $M_T^\varepsilon = \exp\{\varepsilon \int_0^T (\sigma^{-1}(X_t)\bar{b}(X_t))^T dW_t - \frac{\varepsilon^2}{2}\int_0^T \| \sigma^{-1}(X_t)\bar{b}(X_t) \|^2 dt\}$. The upper index Q in $\mathbb{E}^Q$ indicates that the expectation is taken with respect to the measure Q.
Let

$$Y_t = \varepsilon \int_0^t (\sigma^{-1}(X_s)\bar{b}(X_s))^T dW_s - \frac{\varepsilon^2}{2}\int_0^t \| \sigma^{-1}(X_s)\bar{b}(X_s) \|^2 ds$$

so that

$$M_t = e^{Y_t}.$$

Using Itô formula we have

$$M_t^\varepsilon = 1 + \varepsilon \int_0^t M_s^\varepsilon \sigma^{-1}(X_s)\bar{b}(X_s) dW_s. \tag{3.26}$$

Rearranging this and setting $t = T$ we have, for $\varepsilon \neq 0$,

$$\frac{M_T^\varepsilon - 1}{\varepsilon} = \int_0^T M_t^\varepsilon (\sigma^{-1}(X_t)\bar{b}(X_t))^T dW_t.$$

As $\varepsilon \to 0$, we have

$$\frac{M_T^\varepsilon - 1}{\varepsilon} \to \int_0^T (\sigma^{-1}(X_t)\bar{b}(X_t))^T dW_t \text{ in } L^2$$

which follows by the dominated convergence theorem. By an application of Cauchy-Schwartz inequality we have

$$\begin{aligned} & \left| \frac{u^\varepsilon(x) - u(x)}{\varepsilon} - \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \int_0^T (\sigma^{-1}(X_t)\bar{b}(X_t))^T dW_t \mid X_0^\varepsilon = x] \right| \\ = & \left| \frac{\mathbb{E}[e^{-rT} M_T^\varepsilon \Phi(X_{t_1}, \cdots, X_{t_n})] - \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})]}{\varepsilon} \right. \\ - & \left. \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \int_0^T (\sigma^{-1}(X_t)\bar{b}(X_t))^T dW_t \mid X_0^\varepsilon = x] \right| \\ \leq & K\mathbb{E}[\left(\frac{M_T^\varepsilon - 1}{\varepsilon} - \int_0^T (\sigma^{-1}(X_t)\bar{b}(X_t))^T dW_t \right)^2] \end{aligned}$$

where K is a constant independent of ε. Therefore, we have

$$\frac{\partial}{\partial \varepsilon} u^\varepsilon(x) \mid_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{u^\varepsilon(x) - u(x)}{\varepsilon} = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \int_0^T (\sigma^{-1}(X_t)\bar{b}(X_t))^T dW_t \mid X_0 = x]. \qquad \square$$

Next we consider the derivative of the option price with respect to the volatility σ. We define the perturbed process $\{X_t^\varepsilon, \ 0 \leq t \leq T\}$ by

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + [\sigma(X_t^\varepsilon) + \varepsilon \bar{\sigma}(X_t^\varepsilon)]dW_t \quad X_0^\varepsilon = x. \tag{3.27}$$

The first variation process Z_t^ε associated with the process X_t^ε, that is, the derivative of X_t^ε with respect to the parameter ε ($Z_t^\varepsilon := \frac{\partial}{\partial\varepsilon} X_t^\varepsilon$) is given by

$$dZ_t^\varepsilon = b'(X_t^\varepsilon)Z_t^\varepsilon dt + \bar{\sigma}(X_t^\varepsilon)dW_t + \sum_{i=1}^{n}(\frac{\partial}{\partial\varepsilon_i}\sigma + \varepsilon\frac{\partial}{\partial\varepsilon_i}\bar{\sigma})(X_t^\varepsilon)Z_t^\varepsilon dW_t^i, \quad Z_0^\varepsilon = 0$$

where 0 is the column vector in $\mathbb{R}^n$. We will use the notation X_t, Y_t and Z_t for X_t^0, Y_t^0 and Z_t^0 respectively. The generalized $\mathcal{V}$ is defined as follows.

Definition 3.1.4 *$\mathcal{V}$ is defined as the partial derivative of the perturbed option price with respect to ε in the direction $\bar{\sigma}$:*

$$\mathcal{V} := \frac{\partial}{\partial\varepsilon} u^\varepsilon(x) \mid_{\varepsilon=0} . \tag{3.28}$$

The following proposition is found in [35].

Proposition 3.1.5 *Assume that the matrix σ is uniformly elliptic. Let $\beta_{t_i} = Z_{t_i} Y_{t_i}^{-1}$, $0 \le t_i \le T$ such that $\sigma^{-1}(X_t)Y_t\tilde{\beta}_a(T) \in Dom(\delta)$ for all $t \in [0,T]$. Then, for any $\Phi : \mathbb{R} \to \mathbb{R}$ of polynomial growth, we have*

$$\frac{\partial}{\partial\varepsilon} u^\varepsilon(x) \mid_{\varepsilon=0} = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\delta\left(\sigma^{-1}(X)Y\tilde{\beta}_a(T)\right) \mid X_0 = x] \tag{3.29}$$

where $\tilde{\beta}_a(t) = \sum_{i=1}^n a(t)(\beta_{t_i} - \beta_{t_{i-1}})1_{\{t_{i-1} \le t \le t_i\}}$ for any $a \in \tilde{\Upsilon}_n$ and $\delta\left(\sigma^{-1}(X)Y\tilde{\beta}_a(T)\right)$ is the Skorohod integral of $\sigma^{-1}(X_t)Y_t\tilde{\beta}_a(T)$, $0 \le t \le T$.

Proof

We proceed as in the proof of Proposition 3.0.4. We first assume that Φ is continuously differentiable function with bounded derivatives. We first prove that the derivative of $u^\varepsilon(x)$ with respect to ε is obtained by differentiating inside the expectation operator. Considering ε as a degenerate process, we can apply Theorem 39 page 250 in [79] which ensures that we can choose versions of $\{X_t^\varepsilon, 0 \le t \le T\}$ which are continuously differentiable with respect to ε for each $(t,\omega) \in [0,T] \times \Omega$. Since Φ is assumed to be continuously differentiable, we prove by the same arguments that we have in the sense of L^1 norm:

$$\begin{aligned}\frac{\partial}{\partial\varepsilon} u^\varepsilon(x) &= \mathbb{E}[e^{-rT}\sum_{i=1}^{n}\frac{\partial}{\partial\varepsilon_i}\Phi(X_{t_1}^\varepsilon, \cdots, X_{t_n}^\varepsilon)\frac{\partial}{\partial\varepsilon}X_{t_i}^\varepsilon \mid X_0^\varepsilon = x] \\ &= \mathbb{E}[e^{-rT}\sum_{i=1}^{n}\frac{\partial}{\partial\varepsilon_i}\Phi(X_{t_1}^\varepsilon, \cdots, X_{t_n}^\varepsilon)Z_{t_i} \mid X_0^\varepsilon = x]. \end{aligned} \tag{3.30}$$

Using

$$D_t X_{t_i} = Y_{t_i} Y_t^{-1} \sigma(X_t) 1_{t \le t_i} \tag{3.31}$$

we get

$$\begin{aligned}
\int_0^T D_t X_{t_i} \sigma^{-1}(X_t) Y_t 1_{t \le t_i} \tilde{\beta}_a(T) dt &= \int_0^T Y_{t_i} Y_t^{-1} \sigma(X_t) \sigma^{-1}(X_t) Y_t \tilde{\beta}_a(T) 1_{t \le t_i} dt \\
&= \int_0^T Y_{t_i} \tilde{\beta}_a(T) 1_{t \le t_i} dt = Y_{t_i} \int_0^{t_i} \tilde{\beta}_a(T) dt \\
&= Y_{t_i} \sum_{j=1}^{i} \left(\int_0^{t_j} a(t)(\beta_{t_j} - \beta_{t_{j-1}}) 1_{t_{j-1} \le t \le t_j} dt \right) \\
&= Y_{t_i} \sum_{j=1}^{i} \left(\int_{t_{j-1}}^{t_j} a(t)(\beta_{t_j} - \beta_{t_{j-1}}) dt \right) \\
&= Y_{t_i} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} a(t) dt (\beta_{t_j} - \beta_{t_{j-1}}) \\
&= Y_{t_i} \beta_{t_i} \\
&= Z_{t_i}
\end{aligned} \tag{3.32}$$

where the second last equality follows because $\int_{t_{j-1}}^{t_j} a(t)dt = 1$ and $\beta_{t_0} = 0$.
Hence

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon} u^{\varepsilon}(x) &= \mathbb{E}[e^{-rT} \sum_{i=1}^{n} \frac{\partial}{\partial \varepsilon_i} \Phi(X_{t_1}^{\varepsilon}, \cdots, X_{t_n}^{\varepsilon}) Z_{t_i} \mid X_0^{\varepsilon} = x] \\
&= \mathbb{E}[e^{-rT} \int_0^T \sum_{i=1}^{n} \frac{\partial}{\partial \varepsilon_i} \Phi(X_{t_1}^{\varepsilon}, \cdots, X_{t_n}^{\varepsilon}) D_t X_{t_i} \sigma^{-1}(X_t) Y_t \tilde{\beta}_t dt \mid X_0^{\varepsilon} = x] \\
&= \mathbb{E}[\int_0^T D_t \left(e^{-rT} \Phi(X_{t_1}^{\varepsilon}, \cdots, X_{t_n}^{\varepsilon}) \right) \sigma^{-1}(X_t) Y_t \tilde{\beta}_t dt \mid X_0^{\varepsilon} = x] \\
&= \mathbb{E}[e^{-rT} \Phi(X_{t_1}, \cdots, X_{t_n}) \delta \left(\sigma^{-1}(X_t) Y \tilde{\beta}_t \right) \mid X_0 = x]
\end{aligned}$$

where the third and last equalities follow by Equation (2.16) and the integration by parts formula, respectively.
The general case follows by using approximations. □

Remark

The weight function is independent of the payoff function.

3.2 Greeks for European Options

In the following sections we explicitly calculate Greeks for different options. We will consider an underlying asset S described by a geometric Brownian motion, under the risk neutral probability,

$$S_t = S_0 + \int_0^t rS_\tau d\tau + \int_0^t \sigma S_\tau dW_\tau, \quad S_0 = x \tag{3.33}$$

where S_t is the price of the underlying asset, S_0 is the initial value, r is the riskless interest rate, σ is the volatility and $\{W_t, 0 \leq t \leq T\}$ is the standard Brownian motion. This model describes the stock prices or stock indices. We will assume that the coefficients r and $\sigma > 0$ are constants. The solution at time t of the differential equation (3.33) is given by

$$S_t = x \exp((r - \frac{\sigma^2}{2})t + \sigma W_t). \tag{3.34}$$

Then

$$\frac{\partial S_t}{\partial x} = \frac{S_t}{x}. \tag{3.35}$$

We consider a path independent payoff function of the form $\Phi(S_T)$ for some T where S_T is the stock price at time T. The payoff function Φ only depends on the price of the stock at final time T. Then, the option price $u(x)$ of a European option at time $t = 0$ is given by

$$u(x) = \mathbb{E}[e^{-rT}\Phi(S_T) \mid S_0 = x]. \tag{3.36}$$

We note that the density of S_T is known.

Lemma 3.2.1 *Let S_τ be given by Equation (3.34). Then we have*

$$D_t S_\tau = \sigma S_\tau D_t W_\tau = \sigma S_\tau 1_{t<\tau}. \tag{3.37}$$

Proof

We have

$$\begin{aligned}
D_t S_\tau &= D_t \left(x \exp((r - \frac{\sigma^2}{2})\tau + \sigma W_\tau) \right) \\
&= x \exp((r - \frac{\sigma^2}{2})\tau) D_t(\exp(\sigma W_\tau)) \\
&= x \exp((r - \frac{\sigma^2}{2})\tau) D_t(\exp \left(\int_0^\tau \sigma dW_u \right)) \\
&= x \exp((r - \frac{\sigma^2}{2})\tau) \exp(\sigma W_\tau) D_t \int_0^\tau \sigma dW_u \\
&= \sigma S_\tau 1_{t<\tau}
\end{aligned}$$

□

Proposition 3.2.2 *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth. Then Δ is given by*

$$\Delta = \frac{e^{-rT}}{\sigma x T}\mathbb{E}[\Phi(S_T)W_T]. \tag{3.38}$$

Proof

Assuming first that Φ is continuously differentiable with bounded derivatives, we have

$$\begin{aligned} \Delta &= \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(S_T)] = \mathbb{E}[e^{-rT}\Phi'(S_T)\frac{\partial S_T}{\partial x}] \\ &= \mathbb{E}[e^{-rT}\Phi'(S_T)\frac{S_T}{x}] = \frac{e^{-rT}}{x}\mathbb{E}[\Phi'(S_T)S_T]. \end{aligned} \tag{3.39}$$

From Lemma 3.2.1 we have

$$D_t S_T = \sigma S_T 1_{t<T}$$

where D_t is the Malliavin derivative. We can write

$$\int_0^T D_t S_T dt = \int_0^T \sigma S_T 1_{t<T} dt = \sigma S_T T. \tag{3.40}$$

Rearranging this we have

$$S_T = \frac{1}{\sigma T}\int_0^T D_t S_T dt, \quad \sigma > 0. \tag{3.41}$$

Substituting for S_T in Equation (3.39) we obtain

$$\begin{aligned} \Delta &= \frac{e^{-rT}}{x}\mathbb{E}[\Phi'(S_T)\frac{1}{\sigma T}\int_0^T D_t S_T dt] \\ &= \frac{e^{-rT}}{\sigma x T}\mathbb{E}[\int_0^T D_t(\Phi(S_T))dt] \quad \text{by} \quad (2.16) \\ &= \frac{e^{-rT}}{\sigma x T}\mathbb{E}[\Phi(S_T)\delta(1)] \quad \text{by} \quad (2.22) \end{aligned}$$

where $\delta(1)$ is the Skorohod integral of 1. The desired result follows by using Equation (2.24). The general case follows by using approximations. □

Remark

In all the calculations that follows we will first assume that Φ is continuously differentiable with bounded derivatives and then pass to the limit by a density argument. We will not explicitly state this.

Proposition 3.2.3 *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth. Then Γ is given by*

$$\Gamma = \frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi(S_T)\left\{\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right\}]. \tag{3.42}$$

Proof

Assuming Φ is continuously differentiable, we have

$$\begin{aligned}\Gamma &= \frac{\partial}{\partial x}\Delta = \frac{\partial}{\partial x}\left(\frac{e^{-rT}}{\sigma x T}\mathbb{E}[\Phi(S_T)W_T]\right) \\ &= -\frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi(S_T)W_T] + \frac{e^{-rT}}{\sigma x T}\mathbb{E}[\Phi'(S_T)\frac{\partial S_T}{\partial x}W_T] \\ &= -\frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi(S_T)W_T] + \frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi'(S_T)S_T W_T]. \qquad (3.43)\end{aligned}$$

Substituting for S_T in the second term on the right hand side of Equation (3.43) and assuming that Φ' is continuously differentiable we proceed as in the proof of Δ (Proposition 3.38) and obtain

$$\begin{aligned}\Gamma &= -\frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi(S_T)W_T] + \frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi'(S_T)\frac{1}{\sigma T}\int_0^T D_t S_T dt W_T] \\ &= -\frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi(S_T)W_T] + \frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\int_0^T D_t(\Phi(S_T))\frac{1}{\sigma T}W_T dt] \quad \text{by} \quad (2.16) \\ &= -\frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi(S_T)W_T] + \frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi(S_T)\frac{1}{\sigma T}\delta(W_T)] \quad \text{by} \quad (2.22). \qquad (3.44)\end{aligned}$$

Using Equation (2.26) with $F = W_T$ and $u = 1$ we calculate $\delta(W_T)$ as follows:

$$\delta(W_T) = W_T\int_0^T dW_t - \int_0^T D_t W_T dt = W_T^2 - T. \qquad (3.45)$$

Therefore, we have

$$\begin{aligned}\Gamma &= -\frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi(S_T)W_T] + \frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi(S_T)\frac{1}{\sigma T}\{W_T^2 - T\}] \\ &= \frac{e^{-rT}}{\sigma x^2 T}\mathbb{E}[\Phi(S_T)\{\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\}].\end{aligned}$$

□

Proposition 3.2.4 *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth. Then $\mathcal{V}$ is given by*

$$\mathcal{V} = e^{-rT}\mathbb{E}[\Phi(S_T)\left\{\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right\}]. \qquad (3.46)$$

Proof

We have, from Equation (3.34),

$$\frac{\partial S_T}{\partial \sigma} = S_T(W_T - \sigma T). \qquad (3.47)$$

By definition of $\mathcal{V}$ and Equation (3.47) we have

$$\mathcal{V} = \frac{\partial}{\partial \sigma}\mathbb{E}[e^{-rT}\Phi(S_T)] = \mathbb{E}[e^{-rT}\Phi'(S_T)\frac{\partial S_T}{\partial \sigma}] = e^{-rT}\mathbb{E}[\Phi'(S_T)S_T(W_T - \sigma T)]. \qquad (3.48)$$

Substituting for S_T (Equation (3.41)) in Equation (3.48) we have

$$\begin{aligned}
\mathcal{V} &= E^{-rT}\mathbb{E}[\Phi'(S_T)\frac{1}{\sigma T}\int_0^T D_tS_Tdt(W_T-\sigma T)] \\
&= \frac{e^{-rT}}{\sigma T}\mathbb{E}[\int_0^T \Phi'(S_T)D_tS_T(W_T-\sigma T)dt] \\
&= \frac{e^{-rT}}{\sigma T}\mathbb{E}[\int_0^T D_t(\Phi(S_T))(W_T-\sigma T)dt] \quad \text{by} \quad (2.16) \\
&= \frac{e^{-rT}}{\sigma T}\mathbb{E}[\Phi(S_T)\delta(W_T-\sigma T)] \quad \text{by} \quad (2.22).
\end{aligned}$$

We can write

$$\delta\left(W_T-\sigma T\right)=\delta\left(W_T\right)-\sigma T\delta(1)$$

because the Skorohod integral is linear. From Equation (3.45) we have

$$\delta\left(W_T\right)=W_T^2-T$$

and we also have

$$\delta(1)=\int_0^T 1dW_t=W_T.$$

Hence

$$\delta\left(W_T-\sigma T\right)=W_T^2-T-\sigma TW_T.$$

Therefore, we have

$$\mathcal{V} \;=\; e^{-rT}\mathbb{E}[\Phi(S_T)\left\{\frac{W_T^2}{\sigma T}-\frac{1}{\sigma}-W_T\right\}]. \qquad \square$$

Comparing Γ and $\mathcal{V}$ we note that

$$\Gamma=\frac{\mathcal{V}}{\sigma x^2T}.$$

In the Black-Scholes context we have the relation

$$\rho=xT\Delta-T\mathbb{E}[e^{-rT}\Phi(S_T)]$$

from which

$$\rho=\mathbb{E}[e^{-rT}\Phi(S_T)\left(\frac{W_T}{\sigma}-T\right)].$$

The formulae given above for Δ, Γ, $\mathcal{V}$ and ρ can be computed numerically using Monte Carlo simulation procedures.

3.3 Greeks for Exotic options

Here, we consider a path dependent option. Let the underlying asset $\{S_t, 0 \le t \le T\}$ be given by Equation (3.33). An example of a path dependent option is an Asian option. An Asian option is an option whose payoff function depends on the average price of the underlying asset over a certain period of time as opposed to at maturity. It is also known as average option. Thus, the payoff is a function of the average of the stock price $\frac{1}{T}\int_0^T S_t dt$, that is,

$$\text{Payoff} = \Phi\left(\frac{1}{T}\int_0^T S_t dt\right) \quad \text{for some deterministic function } \Phi \text{ of one variable.}$$

Note that there is no closed formula for the density function of the random variable $\frac{1}{T}\int_0^T S_t dt$ in contrast to the European call option case. The option price at time $t = 0$ is given by

$$u(x) = \mathbb{E}[e^{-rT}\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)]. \tag{3.49}$$

Proposition 3.3.1 *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth. Then Δ is given by*

$$\Delta = \frac{2e^{-rT}}{\sigma^2 x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\left\{\frac{S_T - S_0}{\int_0^T S_t dt} - \mu\right\}] \tag{3.50}$$

where $\mu = r - \frac{\sigma^2}{2}$.

Proof

Assuming first that Φ is continuously differentiable with bounded derivatives, we compute Δ as follows:

$$\begin{aligned}\Delta &= \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)] = e^{-rT}\mathbb{E}[\Phi'\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{1}{T}\int_0^T \frac{\partial S_t}{\partial x}dt] \\ &= \frac{e^{-rT}}{x}\mathbb{E}[\Phi'\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{1}{T}\int_0^T S_t dt]. \end{aligned} \tag{3.51}$$

Using Proposition 2.5.1 with $F = \frac{1}{T}\int_0^T S_u du$, $G = \frac{1}{T}\int_0^T S_u du$, $u = S_t$ and $f = \Phi$ we obtain

$$\Delta = \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\frac{1}{T}\int_0^T S_u du \cdot S_t}{\int_0^T D_u(\frac{1}{T}\int_0^T S_\tau d\tau)S_u du}\right)]$$

We note that

$$D_u\left(\frac{1}{T}\int_0^T S_\tau d\tau\right) = \frac{1}{T}\int_0^T D_u S_\tau d\tau = \frac{1}{T}\int_0^T \sigma S_\tau D_u W_\tau d\tau = \frac{\sigma}{T}\int_0^T S_\tau 1_{u<\tau} d\tau = \frac{\sigma}{T}\int_u^T S_\tau d\tau, \tag{3.52}$$

Therefore, we have

$$\Delta = \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\frac{1}{T}\int_0^T S_u du \cdot S_t}{\int_0^T \frac{\sigma}{T}\int_u^T S_\tau d\tau S_u du}\right)].$$

We also note that

$$\begin{aligned}
\int_0^T \frac{\sigma}{T}\left(\int_u^T S_\tau d\tau\right) S_u du &= \frac{\sigma}{T}\int_0^T S_\tau \left(\int_0^\tau S_u du\right) d\tau \\
&= \frac{\sigma}{T}\int_0^T \frac{1}{2} d\left(\int_0^\tau S_u du\right)^2 d\tau \\
&= \frac{\sigma}{T}\frac{1}{2}\left(\int_0^T S_u du\right)^2. \qquad (3.53)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\Delta &= \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\int_0^T S_u du \cdot S_t}{\frac{\sigma}{2}(\int_0^T S_u du)^2}\right)] \\
&= \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{2}{\sigma}\delta\left(\frac{S_t}{\int_0^T S_u du}\right)]. \qquad (3.54)
\end{aligned}$$

Using Proposition 2.3.4 with $F = \frac{1}{\int_0^T S_\tau d\tau}$ so that $D_u F = -\frac{\sigma\int_u^T S_\tau d\tau}{\left(\int_0^T S_\tau d\tau\right)^2}$ and $u = S_t$, direct computation shows that

$$\delta\left(\frac{S_t}{\int_0^T S_t dt}\right) = \frac{\int_0^T S_t dW_t}{\int_0^T S_t dt} + \frac{\sigma}{2}. \qquad (3.55)$$

Therefore, we have

$$\Delta = \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\left\{\frac{2\int_0^T S_t dW_t}{\sigma\int_0^T S_t dt} + 1\right\}]. \qquad (3.56)$$

Recall that

$$S_t = S_0 + \int_0^t r S_\tau d\tau + \int_0^t \sigma S_\tau dW_\tau, \quad S_0 = x$$

Rearranging this equation and putting $S_0 = x$ we have

$$\frac{\int_0^T S_t dW_t}{\int_0^T S_t dt} = \frac{S_T - x}{\sigma\int_0^T S_t dt} - \frac{r}{\sigma}.$$

Therefore, Equation (3.56) becomes

$$\begin{aligned}
\Delta &= \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\left\{\frac{2(S_T - x)}{\sigma^2\int_0^T S_t dt} - \frac{2r}{\sigma^2} + 1\right\}] \\
&= \frac{2e^{-rT}}{\sigma^2 x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\left\{\frac{S_T - x}{\int_0^T S_t dt} - r + \frac{\sigma^2}{2}\right\}] \\
&= \frac{2e^{-rT}}{\sigma^2 x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\left\{\frac{S_T - x}{\int_0^T S_t dt} - \mu\right\}].
\end{aligned}$$

The general case follows by using approximations. □

The following two propositions give different formulae for Δ from the one given in Proposition 3.3.1 (see [67]).

Proposition 3.3.2 *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth. Then Δ is given by*

$$\Delta = \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\left\{\frac{\int_0^T S_t dt}{\int_0^T tS_t dt}\frac{W_T}{\sigma} - 1\right\}]. \tag{3.57}$$

Proof

We have seen from the proof of Proposition 3.3.1 (Equation (3.51) that

$$\Delta = \frac{e^{-rT}}{x}\mathbb{E}[\Phi'\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{1}{T}\int_0^T S_t dt].$$

Using Proposition 2.5.1 with $F = \frac{1}{T}\int_0^T S_t dt$, $G = \frac{1}{T}\int_0^T S_t dt$, and $u = 1$ we obtain

$$\Delta = \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\frac{1}{T}\int_0^T S_t dt}{\int_0^T D_t(\frac{1}{T}\int_0^T S_\tau d\tau)dt}\right)] \tag{3.58}$$

We note that

$$\int_0^T D_t\left(\int_0^T S_\tau d\tau\right)dt = \sigma\int_0^T dt\int_t^T S_\tau d\tau = \sigma\int_0^T S_\tau d\tau\int_0^\tau dt = \sigma\int_0^T \tau S_\tau d\tau. \tag{3.59}$$

Therefore, we have

$$\Delta = \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\int_0^T S_t dt}{\sigma\int_0^T tS_t dt}\right)]. \tag{3.60}$$

Let $F = \int_0^T S_t dt$ and $u = \frac{1}{\sigma\int_0^T tS_t dt}$. By an application of Equation (2.26) we obtain

$$\begin{aligned}
\delta\left(\frac{\int_0^T S_t dt}{\sigma\int_0^T tS_t dt}\right) &= \frac{\int_0^T S_t dt \cdot W_T}{\sigma\int_0^T tS_t dt} - \int_0^T D_t\left(\int_0^T S_\tau d\tau\right)\frac{1}{\sigma\int_0^T tS_t dt}dt \\
&= \frac{\int_0^T S_t dt \cdot W_T}{\sigma\int_0^T tS_t dt} - \frac{\int_0^T \sigma t S_t dt}{\int_0^T \sigma t S_t dt} \\
&= \frac{\int_0^T S_t dt \cdot W_T}{\sigma\int_0^T tS_t dt} - 1.
\end{aligned}$$

Therefore,

$$\Delta = \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\left\{\frac{\int_0^T S_t dt}{\int_0^T tS_t dt}\frac{W_T}{\sigma} - 1\right\}]. \qquad \square$$

Proposition 3.3.3 *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth. Then Δ is given by*

$$\Delta = \frac{e^{-rT}}{x}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\left\{\frac{1}{\langle S\rangle}\left(\frac{W_T}{\sigma}+\frac{\langle S^2\rangle}{\langle S\rangle}\right)-1\right\}]$$

where $\langle S\rangle = \frac{\int_0^T tS_t dt}{\int_0^T S_t dt}$ and $\langle S^2\rangle = \frac{\int_0^T t^2 S_t dt}{\int_0^T S_t dt}$.

Proof

The proof follows by letting $F = \frac{\int_0^T S_t dt}{\int_0^T tS_t dt}$ and $u = \frac{1}{\sigma}$ in Equation (3.51) and proceed as in the proof of Proposition 3.3.1. □

Proposition 3.3.4 *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth. Then $\mathcal{V}$ is given by*

$$\mathcal{V} = e^{-rT}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\left\{\frac{\int_0^T\int_0^T S_t W_t dt dW_\tau}{\sigma\int_0^T tS_t dt}+\frac{\int_0^T t^2 S_t dt\int_0^T S_t W_t dt}{(\int_0^T tS_t dt)^2}-W_T\right\}]. \quad (3.61)$$

Proof

Assume first that Φ is continuously differentiable with bounded derivatives. By definition of $\mathcal{V}$ we have

$$\begin{aligned}
\mathcal{V} &= \frac{\partial}{\partial\sigma}\mathbb{E}[e^{-rT}\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)] \\
&= e^{-rT}\mathbb{E}[\Phi'\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{1}{T}\int_0^T\frac{\partial S_t}{\partial\sigma}dt] \\
&= e^{-rT}\mathbb{E}[\Phi'\left(\frac{1}{T}\int_0^T S_t dt\right)\frac{1}{T}\int_0^T S_t(W_t-\sigma t)dt].
\end{aligned}$$

Applying Proposition 2.5.1 with $F = \frac{1}{T}\int_0^T S_t dt$, $G = \frac{1}{T}\int_0^T S_t(W_t-\sigma t)dt$ and $u = 1$ we obtain

$$\begin{aligned}
\mathcal{V} &= e^{-rT}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\frac{1}{T}\int_0^T S_t(W_t-\sigma t)dt}{\int_0^T D_t(\frac{1}{T}\int_0^T S_\tau d\tau)dt}\right)] \\
&= e^{-rT}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\int_0^T S_t W_t dt-\sigma\int_0^T tS_t dt}{\sigma\int_0^T tS_t dt}\right)] \text{ by (3.59)} \\
&= e^{-rT}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\delta\left(\frac{\int_0^T S_t W_t dt}{\sigma\int_0^T tS_t dt}-1\right)].
\end{aligned}$$

We can write

$$\delta\left(\frac{\int_0^T S_t W_t dt}{\sigma\int_0^T tS_t dt}-1\right) = \delta\left(\frac{\int_0^T S_t W_t dt}{\sigma\int_0^T tS_t dt}\right)-\delta(1) = \delta\left(\frac{\int_0^T S_t W_t dt}{\sigma\int_0^T tS_t dt}\right)-W_T.$$

Choosing $F = \frac{1}{\sigma \int_0^T tS_t dt}$, $u = \int_0^T S_t W_t dt$ and using Proposition 2.3.4 we obtain

$$\begin{aligned}
\delta\left(\frac{\int_0^T S_t W_t dt}{\sigma \int_0^T tS_t dt}\right) &= \frac{\int_0^T \int_0^T S_t W_t dt dW_\tau}{\sigma \int_0^T tS_t dt} - \int_0^T D_t \left(\frac{1}{\sigma \int_0^T \tau S_\tau d\tau}\right) \int_0^T S_t W_t dt dt \\
&= \frac{\int_0^T \int_0^T S_t W_t dt dW_\tau}{\sigma \int_0^T tS_t dt} + \frac{\int_0^T D_t(\sigma \int_0^T \tau S_\tau d\tau) \int_0^T S_t W_t dt dt}{(\sigma^2 \int_0^T tS_t dt)^2} \\
&= \frac{\int_0^T \int_0^T S_t W_t dt dW_\tau}{\sigma \int_0^T tS_t dt} + \frac{\sigma^2 \int_0^T t^2 S_t dt \int_0^T S_t W_t dt}{\sigma(\int_0^T tS_t dt)^2} \\
&= \frac{\int_0^T \int_0^T S_t W_t dt dW_\tau}{\sigma \int_0^T tS_t dt} + \frac{\int_0^T t^2 S_t dt \int_0^T S_t W_t dt}{(\int_0^T tS_t dt)^2}.
\end{aligned}$$

Therefore

$$\mathcal{V} = e^{-rT}\mathbb{E}[\Phi\left(\frac{1}{T}\int_0^T S_t dt\right)\left\{\frac{\int_0^T \int_0^T S_t W_t dt dW_\tau}{\sigma \int_0^T tS_t dt} + \frac{\int_0^T t^2 S_t dt \int_0^T S_t W_t dt}{(\int_0^T tS_t dt)^2} - W_T\right\}].$$

The general case follows by using approximations. □

Remark

While there are other choices of u_t in the integration by parts formula, the choice of a constant minimizes the variance of the estimate (see [11]).

3.4 Greeks for Barriers and Look-back options

The ideas presented here are taken from [15], [43] and [58]. Here we consider a one-dimensional risky asset $\{S_t, 0 \le t \le T\}$ whose dynamics is given by

$$S_t = S_0 + \int_0^t rS_s ds + \int_0^t \sigma(S_s)S_s dW_s, \quad S_0 = x \tag{3.62}$$

where r is the interest rate, $\sigma(\cdot)$ is the volatility and $\{W_t, 0 \le t \le T\}$ is the standard Brownian motion. The Barrier and Look-back European options have the payoff function of the form

$$\text{Payoff} = \Phi(\max_{s\in I} S_s, \min_{s\in I} S_s, S_T) \quad \text{for some deterministic function } \Phi \text{ of three variables}$$

where $I \subset [0, T]$ is the set of all times when the extrema are monitored. The price of the option at time 0, under the risk neutral probability measure, is given by

$$u(x) = \mathbb{E}[e^{-rT}\Phi(\max_{s\in I} S_s, \min_{s\in I} S_s, S_T)].$$

We want to compute

$$\Delta = \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(\max_{s\in I} S_s, \min_{s\in I} S_s, S_T)] \text{ and } \Gamma = \frac{\partial^2}{\partial x^2}\mathbb{E}[e^{-rT}\Phi(\max_{s\in I} S_s, \min_{s\in I} S_s, S_T)].$$

The difficulty comes from the lack of differentiability of the minimum and maximum processes. The minimum and maximum processes are in general not differentiable and we cannot use the integration by parts formula directly. The law of the minimum and maximum processes is known, hence we can use the likelihood ratio to compute the Greeks. However, the calculations are cumbersome.

We show how Malliavin calculus techniques can be used to compute Δ and Γ. We use a localization procedure. This procedure was first introduced in [70]. We consider a transformation of S_t which avoids some possible degeneracy problems of the matrix σ namely, $X_t = A(S_t)$ where A is a strictly increasing function

$$A(y) = \int_1^y \frac{du}{u\sigma(u)}, \quad \sigma(y) \neq 0$$

and $A^{-1}(\cdot)$ exists so that

$$\begin{aligned} dX_t &= d\left(\int_1^{S_t} \frac{du}{u\sigma(u)}\right) \\ &= \frac{1}{S_t\sigma(S_t)}dS_t + \frac{1}{2}\frac{d}{du}\left(\frac{1}{u\sigma(u)}\right)_{u=S_t} \sigma(S_t)^2(S_t)^2dt \\ &= \frac{1}{S_t\sigma(S_t)}(rS_tdt + \sigma(S_t)S_tdW_t) - \frac{(u\sigma(u))'}{2} \,|_{u=S_t}\, dt \\ &= [\frac{r}{\sigma(S_t)} - \frac{(u\sigma(u))'}{2} \,|_{u=S_t}]dt + dW_t. \end{aligned} \tag{3.63}$$

Writing Equation (3.63) in integral form, we have

$$X_t = x + \int_0^t h(X_s)ds + W_t \tag{3.64}$$

where $x = A(S_0)$ and $h(u) = [\frac{r}{\sigma(y)} - \frac{(y\sigma(y))'}{2}]_{y=A^{-1}(u)}$. This corresponds to the usual logarithm in the Black-Scholes model with $\sigma = 1$. We assume that h is bounded and continuously differentiable. Let $M_t = \max_{s\le t,\ s\in I} X_s$ and $m_t = \min_{s\le t,\ s\in I} X_s$ be the maximum and the minimum of the asset respectively. Thus, we have

$$\text{Payoff} = \Phi(M_T, m_T, X_T).$$

We need the following condition (see [43]).

Assumption 1 *There exists* $a > 0$ *such that the function* $\Phi(M, m, z)$ *does not depend on the variables* (M, m) *for any* (M, m, z) *such that* $0 \leq M - x < a$ *or* $0 \leq x - m < a$.

As it is discussed in [43], the assumption appears to be a necessary condition to enable a representation of Greeks to take the form $\mathbb{E}[e^{-rT}\Phi(\cdot)\pi]$ for an appropriate random variable π. There are many payoff functions that satisfy Assumption 1.

Example

We consider the following single barrier options:

- Up and out Barrier: $\Phi(M, m, z) = 1_{M<U}\Phi(z)$ with $x < U$. The assumption is satisfied when $a = U - x$.
- Down and in Barrier option: $\Phi(M, m, z) = 1_{m\leq D}\Phi(z)$ with $D < x$. The assumption is satisfied when $a = x - D$.

Definition 3.4.1 *An increasing adapted right-continuous process* $\{Y_t, 0 \leq t \leq T\}$ *is called a dominating process for* X_t *if, for any* t,

$$\mid X_t - x \mid \leq Y_t. \tag{3.65}$$

We need the following proposition

Proposition 3.4.2 *1. For any* t, *we have* $\mid X_t - x \mid \leq Y_t$.

2. Let $p, q > 1$ *such that There exists a positive function* $\alpha : \mathbb{N} \to [0, T]$ *with* $\lim_{q\to\infty} \alpha(q) = \infty$, *such that, for any* $q \geq 1$, *we have*

$$E[Y_t^q] \leq C_q t^{\alpha(q)} \quad \textit{for all} \quad t \in [0, T].$$

In particular one has $Y_0 = 0$.

3. For any $p \geq 1$, *choose an infinitely continuously differentiable and bounded function* $\psi : [0, \infty) \to [0, 1]$ *with*

$$\psi(x) = \begin{cases} 1 & \textit{if} \;\; x \leq \frac{a}{2} \\ 0 & \textit{if} \;\; x \geq a. \end{cases}$$

where a *is the real positive number appearing in Assumption 1. The random variable* $\psi(Y_t)$ *belongs to* $\mathbb{D}_{1,2}$ *for each* t. *In addition, we have*

$$\mid \mathbb{E}[D_t\psi(Y_t)] \mid \leq C_p \quad \textit{for all} \quad p \geq 1 \tag{3.66}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof

The proof follows the same arguments in [15]. We omit the details. □

Remark

We call $\psi(Y_t)$ a localizing process.

The computation of Greeks consists of calculating the derivative of the minimum and the maximum of $(S_t)_{t\in I}$. It has been shown in [68] that if the random variable $U_t \in \mathbb{D}_{1,2}$ for any $t \in [0,T]$ then, under some additional conditions, the random variables

$$\min_{s\le T,\ s\in I} U_s \quad \text{and} \quad \max_{s\le T,\ s\in I} U_s$$

also belong to $\mathbb{D}_{1,2}$. We have the next lemma.

Lemma 3.4.3 *Assume that the random variables M, m and S_T belong to $\mathbb{D}_{1,2}$. Then their derivatives are given, for $t < T$, by*

$$D_t m = m 1_{t\le\tau_m} \quad \textit{and} \quad D_t M = M 1_{t\le\tau_M}$$

where $\tau^m = \inf\{t : m_t = \min_{s\le t,\ s\in I} X_s\}$ and $\tau^M = \inf\{t : M_t = \max_{s\le t,\ s\in I} X_s\}$.

Proof

The proof is given in [68]. We omit the details. □

The following theorem gives the integration by parts formula for Δ (see [43]).

Theorem 3.4.4 *Let Assumption 1 hold. Let Y be a dominating process satisfying Equation (3.66). Then, for any $\Phi : \mathbb{R} \to \mathbb{R}$ of polynomial growth, we have*

$$\Delta = \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(M,m,X_T)] = \mathbb{E}[e^{-rT}\Phi(M,m,X_T)\pi] \tag{3.67}$$

with

$$\pi = \delta\left(\frac{\psi(Y)}{\int_0^T \psi(Y_t)dt}\right)$$

where ψ is as given in Proposition 3.4.2

Proof

We first assume that the payoff function Φ is continuously differentiable with bounded derivatives. Therefore, we have

$$\Delta \;=\; \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(M,m,X_T)] = \mathbb{E}[e^{-rT}\left(\Phi_1' \cdot M + \Phi_2' \cdot m + \Phi_3' \cdot X_T\right)] \tag{3.68}$$

where Φ_1' denote the derivative of Φ with respect to the first argument, Φ_2' denote the derivative of Φ with respect to the second argument and Φ_3' denote the derivative of Φ with respect to the third argument. We have omitted the arguments of Φ for simplicity and the dot denotes the product.

We find a representation for

$$\Phi_1' \cdot M + \Phi_2' \cdot m + \Phi_3' \cdot X_T.$$

We do this as follows. Using Proposition 2.2.8 and Lemma 3.4.3 we calculate the Malliavin derivative of Φ as

$$D_t\Phi = \left(M\Phi_1' 1_{\{t<\tau_M\}} + m\Phi_2' 1_{\{t<\tau_m\}} + X_T\Phi_3'\right) \tag{3.69}$$

Multiplying both sides of Equation (3.69) by $\psi(Y_t)$ we obtain

$$D_t\Phi\psi(Y_t) = \left(M\Phi_1' 1_{\{t<\tau_M\}} + m\Phi_2' 1_{\{t<\tau_m\}} + X_T\Phi_3'\right)\psi(Y_t). \tag{3.70}$$

To apply an integration by parts formula on the representation in (3.70) we have to remove terms of the types $1_{\{t<\tau_M\}}$ and $1_{\{t<\tau_m\}}$ and this can be done using the localizing process $\psi(Y_t)$. To this end, for $t \in [0,T]$, we have

$$\Phi_1' 1_{\{t<\tau_M\}}\psi(Y_t) = \Phi_1'\psi(Y_t) \tag{3.71}$$

and

$$\Phi_2' 1_{\{t<\tau_m\}}\psi(Y_t) = \Phi_2'\psi(Y_t) \tag{3.72}$$

Here we consider Equation (3.71). If $M < a+x$ then Φ does not depend on M (by Assumption 1) and hence Equation (3.71) reduces to $0 = 0$. On the other hand, if $M \geq a+x$ and if t is such that $\psi(Y_t) \neq 0$, we have $Y_t < a$; therefore using (3.65) we have $\max_{s\leq t}(X_s - x) < a < \max_{s\leq T}(X_s - x)$ that is, $t \leq \tau_M$ and the proof of Equation (3.71) is complete. A similar argument can be used to prove Equation (3.72) (see [15]).

Substituting Equations (3.71) and (3.72) into Equation (3.70) we obtain, for $t \in [0,T]$,

$$D_t\Phi\psi(Y_t) = \left(M\Phi_1' + m\Phi_2' + X_T\Phi_3'\right)\psi(Y_t). \tag{3.73}$$

We can write

$$\int_0^T D_t\Phi\psi(Y_t)dt = \int_0^T (M\Phi_1' + m\Phi_2' + X_T\Phi_3')\,\psi(Y_t)dt$$

which is equivalent to

$$\int_0^T D_t\Phi \cdot (\psi(Y_t))dt = (M\Phi_1' + m\Phi_2' + X_T\Phi_3') \int_0^T \psi(Y_t)dt. \tag{3.74}$$

Therefore, we have

$$\int_0^T D_t\Phi \cdot \left(\frac{\psi(Y_t)}{\int_0^T \psi(Y_t)dt}\right) dt = \Phi_1' \cdot M + \Phi_2' \cdot m + \Phi_3' \cdot X_T \tag{3.75}$$

Substituting Equation (3.75) into Equation (3.68) we get

$$\Delta = \mathbb{E}[e^{-rT} \int_0^T D_t\Phi \cdot \left(\frac{\psi(Y_t)}{\int_0^T \psi(Y_t)dt}\right) dt]. \tag{3.76}$$

The desired result follows by an application of Equation (2.22). The general case follows by a density argument. □

Following the same procedure as above we deduce that

$$\Gamma \quad = \quad \frac{\partial^2}{\partial x^2}\mathbb{E}[e^{-rT}\Phi(M, m, X_T)] = \mathbb{E}[e^{-rT}\Phi(M, m, X_T)\pi] \tag{3.77}$$

with

$$\pi = \delta\left(\delta\left(\frac{\psi(Y)}{\int_0^T \psi(Y_t)dt}\right)\frac{\psi(Y)}{\int_0^T \psi(Y_t)dt}\right) - \delta\left(\frac{\psi(Y)}{\int_0^T \psi(Y_t)dt}\right).$$

Remark

The computation of Greeks for Barrier options and Look-back options gives an idea of how to handle discontinuous payoff functions.

3.5 Greeks for the Heston model

We mention that the calculations of Greeks using Malliavin calculus introduced in [35] focussed on models with deterministic volatility. Here, we use Malliavin calculus to compute Greeks for models with stochastic volatility where the underlying asset is driven by a Brownian motion. In particular, we consider the Heston model.

Heston (1993) assumed that the stock price process is described by a stochastic differential equation:

$$dS_t = S_t(bdt + \sqrt{v_t}dW_t^{(1)}) \tag{3.78}$$

where $\{W_t^{(1)}, 0 \leq t \leq T\}$ denotes a Brownian motion, S_t is the stock price at time t, b is the drift coefficient and $\sqrt{v_t}$ is the volatility. In addition, Heston proposed that the variance be driven by a mean-reverting stochastic process of the form

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dW_t^{(2)} \tag{3.79}$$

where $\{W_t^{(2)}, 0 \leq t \leq T\}$ denotes a Brownian motion, θ is the long-run mean, κ is the rate of mean reversion and ν is called volatility of variance (often called volatility of volatility). We assume that the dynamics described by Equations (3.78) and (3.79) are, under a risk neutral measure, chosen by the market and that the risk neutral measure is given by P. $W_t^{(1)}$ and $W_t^{(2)}$ are two correlated standard Brownian motions. We have

$$dW_t^{(1)}dW_t^{(2)} = \rho dt, \quad \rho \in (-1, 1) \tag{3.80}$$

where ρ here represents the correlation coefficient between the two standard Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$. The Heston model is identical to the Black-Scholes model except that the volatility is allowed to be stochastic. Hence, this model is a generalization of the Black-Scholes model to the case of stochastic volatility. The Heston model is described in detail in [39] and is popular in industry because of its quasi-closed form solution for European options in terms of Fourier-transforms. We will not use the quasi-closed form solution; instead we investigate how the Malliavin calculus can be used to compute Greeks for the Heston model. We observe that the Heston model does not satisfy the standard assumptions of a stochastic differential equation. The square root $\sqrt{v_t}$ is neither a differentiable function of v_t at $v_t = 0$ nor even Lipschitz continuous. Hence, we cannot directly apply Malliavin calculus approach to compute Greeks for the Heston model.

To apply Malliavin calculus in the context of the Heston model, we need v_t to satisfy certain conditions as we will show below. To ensure existence and uniqueness of a solution of Equations (3.78) and (3.79), b, κ, θ and ν are assumed to be strictly positive constants, and in addition we require that $2\kappa\theta \geq \nu^2$ to ensure that the variance process is always positive, that is,

$$P(\{v_t > 0, \ \forall\ t > 0\}) = 1.$$

The condition $2\kappa\theta \geq \nu^2$ is called the Novikov condition. Assuming now that $v_0 > 0$ and that the Novikov condition holds, we consider the square root process

$$\sigma_t := \sqrt{v_t}. \tag{3.81}$$

By an application of the Itô formula, we have

$$\begin{aligned} d\sigma_t &= \frac{\partial}{\partial t}(\sqrt{v_t})dt + \frac{\partial}{\partial v}(\sqrt{v_t})dv_t + \frac{1}{2}\nu^2 v_t \frac{\partial^2}{\partial v^2}(\sqrt{v_t})dt \\ &= \{\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\sigma_t} - \frac{1}{2}\kappa\sigma_t\}dt + \frac{\nu}{2}dW_t^{(2)}. \end{aligned} \tag{3.82}$$

The Novikov condition implies in particular that

$$\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right) \geq 0. \tag{3.83}$$

It is not apparent that σ_t, the solution of the stochastic differential equation in (3.82), admits a unique strong solution, but the Yamada-Wanatabe lemma (see [56] Chapter 5 Proposition 2.18) implies, under the Novikov condition, that v_t is a unique solution of the stochastic differential equation (3.79). If σ_t is a solution of Equation (3.82) then we can use Itô formula to show that σ_t^2 is a solution of Equation (3.79) satisfying the condition $v_0 > 0$. As σ_t^2 is a unique solution of Equation (3.79) we conclude uniqueness of the solution of stochastic differential equation (3.82) up to a sign. However, if σ_t satisfies Equation (3.82) it is obvious that $-\sigma_t$ does not and therefore, for $v_0 > 0$, we have uniqueness of the solution σ_t of stochastic differential equation (3.82).

Assuming that the volatility σ_t is Malliavin differentiable, that is, $\sigma_t \in \mathbb{D}_{1,2}$ we calculate Δ as follows. We first construct the Heston stochastic volatility model with correlation ρ from two independent Brownian motions, which consists of a stock S_t and a variance process v_t satisfying Equations (3.78) and (3.79) (see [33]). Given an arbitrary $W^{(2)}$ there exists Z which is independent of $W^{(1)}$ and $W^{(2)}$. Then we can express $W^{(1)}$ as follows

$$W_t^{(1)} = \rho W_t^{(2)} + \sqrt{1-\rho^2}Z_t. \tag{3.84}$$

It is convenient in the sequel to think of the dynamics described by Equations (3.78) and (3.79) as driven by Z_t and $W_t^{(2)}$ respectively rather than $W_t^{(1)}$ and $W_t^{(2)}$) respectively. Substituting Equation (3.84) into Equation (3.78) we obtain

$$dS_t = S_t\left(bdt + \sqrt{v_t}(\rho dW_t^{(2)} + \sqrt{1-\rho^2}dZ_t)\right) \tag{3.85}$$

with respect to some initial condition $S_0 = x$. In the following we will work with the logarithmic price $\log S_t$ rather than the actual price to ensure that the stock price is always positive. Let $X_t = \log S_t$. By an application of Itô formula, we obtain

$$dX_t = (b - \frac{v_t}{2})dt + \rho\sqrt{v_t}dW_t^{(2)} + \sqrt{v_t}\sqrt{1-\rho^2}dZ_t. \tag{3.86}$$

We can write the Equations (3.82) and (3.86) in integral forms as

$$X_t = \log x + \int_0^t \left(b - \frac{1}{2}v_s\right) ds + \rho \int_0^t \sqrt{v_s} dW_s^{(2)} + \sqrt{1-\rho^2} \int_0^t \sqrt{v_s} dZ_s. \tag{3.87}$$

$$\sigma_t = \sigma_0 + \int_0^t \left(\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\sigma_s} - \frac{1}{2}\kappa\sigma_s\right) ds + \int_0^t \frac{\nu}{2} dW_s^{(2)}. \tag{3.88}$$

It can be proved (see [22] page 17) that for any parameter choice with $2\kappa\theta > \nu^2$ and for any $T > 0$ we have

$$\sup_{0\le t\le T} \mathbb{E}[\sigma_t^{-2}] < \infty. \tag{3.89}$$

The two Equations (3.87) and (3.88) can be thought of as a single two dimensional stochastic differential equation of the form

$$\begin{aligned} \begin{pmatrix} X_t \\ \sigma_t \end{pmatrix} &= \begin{pmatrix} \log x \\ \sigma_0 \end{pmatrix} + \int_0^t \begin{pmatrix} (b - \frac{1}{2}\sigma_s^2) \\ \{\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\sigma_s} - \frac{1}{2}\kappa\sigma_s\} \end{pmatrix} ds \\ &+ \int_0^t \begin{pmatrix} \sqrt{1-\rho^2}\sigma_s & \rho\sigma_s \\ 0 & \frac{\nu}{2} \end{pmatrix} \begin{pmatrix} dZ_s \\ dW_s^{(2)} \end{pmatrix}. \end{aligned}$$

We mention that the exact value of the joint density of X_t and σ_t is unknown.
The inverse matrix of

$$\begin{pmatrix} \sqrt{1-\rho^2}\sigma_s & \rho\sigma_s \\ 0 & \frac{\nu}{2} \end{pmatrix}$$

is calculated as follows

$$\frac{2}{\nu\sqrt{1-\rho^2}\sigma_s} \begin{pmatrix} \frac{\nu}{2} & -\rho\sigma_s \\ 0 & \sqrt{1-\rho^2}\sigma_s \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}\sigma_s} & -\frac{2\rho}{\nu\sqrt{1-\rho^2}} \\ 0 & \frac{2}{\nu} \end{pmatrix}.$$

We calculate the first variation process Y_t of $\begin{pmatrix} X_t \\ \sigma_t \end{pmatrix}$ as follows

$$Y_t := \frac{\partial}{\partial x} \begin{pmatrix} X_t \\ \sigma_t \end{pmatrix} = \begin{pmatrix} \frac{1}{x} \\ 0 \end{pmatrix}.$$

An application of Proposition 3.0.4 with $a \in \Upsilon_n$ gives

$$\begin{aligned} \Delta &= \mathbb{E}[e^{-rT}\Phi(X_T) \int_0^T a(s)(\sigma(X_t)^{-1}Y_t)^T dW_t] \\ &= \mathbb{E}[e^{-rT}\Phi(X_T) \int_0^T a(s) \left(\begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}\sigma_s} & -\frac{2\rho}{\nu\sqrt{1-\rho^2}} \\ 0 & \frac{2}{\nu} \end{pmatrix} \begin{pmatrix} \frac{1}{x} \\ 0 \end{pmatrix} \right)^T \begin{pmatrix} dZ_s \\ dW_s^{(2)} \end{pmatrix}]. \end{aligned}$$

Choosing $a(s) = \frac{1}{T}$ and making use of the matrix property that $(AB)^T = B^T A^T$, for A and B matrices, we have

$$\begin{aligned}
\Delta &= \mathbb{E}[e^{-rT}\Phi(X_T)\int_0^T \frac{1}{T}\begin{pmatrix} \frac{1}{x} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}\sigma_s} & 0 \\ -\frac{2\rho}{\nu\sqrt{1-\rho^2}} & \frac{2}{\nu} \end{pmatrix}\begin{pmatrix} dZ_s \\ dW_s^{(2)} \end{pmatrix}] \\
&= \mathbb{E}[e^{-rT}\Phi(X_T)\int_0^T \frac{1}{T}\begin{pmatrix} \frac{1}{x\sqrt{1-\rho^2}\sigma_s} & 0 \end{pmatrix}\begin{pmatrix} dZ_s \\ dW_s^{(2)} \end{pmatrix}] \\
&= \mathbb{E}[e^{-rT}\Phi(X_T)\int_0^T \frac{1}{xT\sqrt{1-\rho^2}\sigma_s} dZ_s].
\end{aligned}$$

Thus we have proved the following result.

Theorem 3.5.1 *Let the stock price and the variance be given by Equations (3.78) and (3.79) respectively. Then, for any $\Phi : \mathbb{R} \to \mathbb{R}$ of polynomial growth, we have*

$$\Delta = \mathbb{E}[e^{-rT}\Phi(X_T)\int_0^T \frac{1}{xT\sqrt{1-\rho^2}\sigma_s} dZ_s]. \tag{3.90}$$

Remark

This result has also been obtained in [22] using an approach different from Proposition 3.0.4.

We now show, following the construction in [2], that σ_t is Malliavin differentiable. First we define an approximating sequence $\Phi_N(y)$. Let $\Phi_N(y),\ N > 0$ be a continuously differentiable function satisfying

$$\Phi_N(y) = \begin{cases} 1 & \text{if } y \geq 2N \\ 0 & \text{if } y < N. \end{cases}$$

In addition, $\Phi_N(y) \leq 1$ for all $y \in \mathbb{R}$. Then

$$\Phi_N'(y) = \begin{cases} 0 & \text{if } y > 2N \\ 0 & \text{if } y < N. \end{cases}$$

Furthermore, we define the function

$$\Lambda_N(y) := \Phi_N(y)\frac{1}{y} \quad \text{with} \quad \Lambda_N(0) := 0.$$

We assume that the function $\Lambda_N(y)$ is bounded and continuously differentiable and it satisfies

$$\Lambda_N'(y) = \Phi_N'(y)\frac{1}{y} - \Phi_N(y)\frac{1}{y^2}. \tag{3.91}$$

In particular, we have

$$\Lambda_N'(y) = \begin{cases} \frac{-1}{y^2} \text{ if } y \geq 2N \\ 0 \text{ if } y < N. \end{cases}$$

Then, we define the approximating sequence σ_t^N of σ_t as the solution of the stochastic differential equation

$$d\sigma_t^N = [\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right) \Lambda_N(\sigma_t^N) - \frac{\kappa}{2}\sigma_t^N]dt + \frac{\nu}{2}dW_t \tag{3.92}$$

with $\sigma_0^N = \sigma_0$ for all $N > 0$.

Proposition 3.5.2 *For each $t \in [0, T]$ the sequence σ_t^N converges to σ_t in $L^2(\Omega)$.*

Proof

The proof can be found in [2]. We omit the details. □

Proposition 3.5.3 *Assuming that $2\kappa\theta \geq \nu^2$ we have $\sigma_t \in \mathbb{D}_{1,2}$ and for $r < t$ we have*

$$D_r\sigma_t = \frac{\nu}{2} \exp\left(\int_r^t [-\frac{\kappa}{2} - \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right) \frac{1}{\sigma_s^2}]ds\right). \tag{3.93}$$

Proof

We give an outline of the proof. By Proposition 3.5.2 the sequence σ_t^N converges to σ_t in $L^2(\Omega)$ for each $t \in [0, T]$. This convergence is pointwise and hence we conclude by using the properties of the function $\Lambda_N(x)$ that

$$D_r\sigma_t^N = \frac{\nu}{2} \exp\left(\int_r^t [-\frac{\kappa}{2} - \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right) \Lambda_N'(\sigma_s^N)]ds\right)$$

converges pointwise to

$$G := D_r\sigma_t = \frac{\nu}{2} \exp\left(\int_r^t [-\frac{\kappa}{2} - \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right) \frac{1}{\sigma_s^2}]ds\right).$$

Using the Novikov condition, we see that the exponent in $D_r\sigma_t^N$ is negative for all choices of N and therefore $\mid D_r\sigma_t^N \mid \leq \frac{\nu}{2}$ for all N. By means of the bounded convergence theorem we conclude that $D_r\sigma_t^N$ converges to G in $L^2(\Omega)$. Using Lemma 1.2.3 in [70] we have that $\sigma_t \in \mathbb{D}_{1,2}$ and the result follows. □

Remark

All the representations of Greeks we have obtained in this chapter can be evaluated by Monte Carlo methods.

Chapter 4

Application of white noise calculus for Gaussian Processes to the Calculation of Greeks

The theory of white noise analysis was first introduced in [45] and was originally applied in quantum physics. Subsequently new applications have been found in stochastic differential equations (see [47]). More recently, the white noise analysis has been applied to finance (see [1], [17], [27], [31], [45], [46], [59], [61], [74] and the references therein).
The Malliavin calculus has been presented in the context of analysis on the Wiener space $\Omega = C_0([0,T])$, the space of all real continuous functions on $[0,T]$. In order to calculate the Malliavin derivative of a random variable we need to show first that the random variable belongs to the domain of the Malliavin derivative. This, however, is restrictive as some interesting options do not satisfy this condition, for example, the digital option. Nonetheless, it is possible to obtain some extension of the domain of the Malliavin derivative to the whole L^2. This extension appears in the literature in the framework of white noise analysis (see [1]) and Wiener setting (see [85] and [86]). The major application in the above papers is to calculate the replicating portfolio of a given contingent claim.
The goal of this chapter is to derive explicit formulae for Greeks using Malliavin calculus approach. We mention that similar results were obtained in [28] in the jump diffusion case. The authors in [28] obtain a functional representation formula for functionals of jump diffusions in terms of the Fourier transform from which they compute Greeks using the method related to the likelihood method. The key idea in our case is to express the payoff function in terms of the Donsker delta function (to be defined later on) and then use the Wick chain

rule to compute Greeks. The results of this chapter give a generalization of the computation of Greeks. The approach is advantageous because we can handle discontinuous and path-dependent payoffs. In addition, we do not require the knowledge of the density of the underlying asset.

In the following section we review white noise concepts for Brownian motion as developed in [1], [17], [27], [47] and [74]. We explore a possible generalization of the computation of Greeks using both Hida-Malliavin derivative and Hermite transforms. For general background information about white noise we refer to [61].

4.1 Basic concepts of Gaussian white noise analysis

Let $L^2(\mathbb{R})$ denotes the set of measurable functions satisfying

$$\| f \|_{L^2(\mathbb{R})}^2 := \int_{\mathbb{R}} f^2(t)dt < \infty. \tag{4.1}$$

We will work with the probability space $\Omega = S'(\mathbb{R})$, which is the space of tempered distributions, equipped with its Borel σ-algebra $\mathcal{F} = \mathcal{B}(\Omega)$. The space $S'(\mathbb{R})$ is the dual of the Schwartz space $S(\mathbb{R})$ of test functions, that is, the rapidly decreasing smooth functions on $\mathbb{R}$. By the Bochner-Minlos theorem (see [47] page 14) there exists a probability measure μ on Ω such that

$$\int_{\Omega} e^{i\langle \omega, f\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|f\|_{L^2(\mathbb{R})}^2}, \qquad f \in S(\mathbb{R}) \tag{4.2}$$

where $i = \sqrt{-1}$ and $\langle \omega, f\rangle = \omega(f)$ denotes the action of $\omega \in \Omega = S'(\mathbb{R})$ applied to $f \in S(\mathbb{R})$. The measure μ is called the *White noise probability measure*. The triple $(\Omega, \mathcal{B}(\Omega), \mu)$ is called the white noise probability space.

Lemma 4.1.1 *Let $f \in S(\mathbb{R})$. Then*

$$\mathbb{E}[\langle \cdot, f\rangle] = 0, \quad f \in S(\mathbb{R}). \tag{4.3}$$

Moreover, we have the Itô isometry

$$\mathbb{E}[\langle \cdot, f\rangle^2] = \| f \|_{L^2(\mathbb{R})}^2 \quad \text{for all} \;\; f \in S(\mathbb{R}). \tag{4.4}$$

Proof

Letting $f(y) = tg(y), t \in \mathbb{R}$ in Equation (4.2) we get

$$\int_{\Omega} e^{i\langle \omega, tg\rangle} d\mu(\omega) = e^{-\frac{1}{2}t^2\|g\|_{L^2(\mathbb{R})}^2}.$$

Performing Taylor expansions on both sides we have

$$\int_{\Omega}(1+i\langle\omega,tg\rangle-\frac{1}{2}\langle\omega,tg\rangle^2+\cdots)d\mu(\omega)=1-\frac{1}{2}t^2\parallel g\parallel^2_{L^2(\mathbb{R})}+\frac{1}{4}t^4\parallel g\parallel^4_{L^2(\mathbb{R})}+\cdots$$

which is equivalent to

$$\begin{aligned}\int_{\Omega}d\mu(\omega)+it\int_{\Omega}\langle\omega,g\rangle d\mu(\omega)-\frac{1}{2}t^2\int_{\Omega}\langle\omega,g\rangle^2d\mu(\omega)+\cdots &= 1-\frac{1}{2}t^2\parallel g\parallel^2_{L^2(\mathbb{R})}\\ &+ \frac{1}{4}t^4\parallel g\parallel^4_{L^2(\mathbb{R})}+\cdots\end{aligned}$$

Now comparing terms with the same powers of t we have

$$it\int_{\Omega}\langle\omega,g\rangle d\mu(\omega)=0$$

which implies that

$$\mathbb{E}[\langle\omega,f\rangle]=0.$$

This proves Equation (4.3). Comparing terms in t^2 we have

$$-\frac{1}{2}t^2\int_{\Omega}\langle\omega,g\rangle^2d\mu=-\frac{1}{2}t^2\parallel g\parallel^2_{L^2(\mathbb{R})}$$

which implies

$$\mathbb{E}[\langle\omega,f\rangle^2]=\parallel f\parallel^2_{L^2(\mathbb{R})}.$$

This proves Equation (4.4). □

Using Lemma 4.1.1 we can extend the definition of $\langle\omega,f\rangle$ from $f\in S(\mathbb{R})$ to any $f\in L^2(\mathbb{R})$ as follows. We take an approximating sequence $\{f_n\}_{n=1}^{\infty}\in S(\mathbb{R})$ such that

$$\lim_{n\to\infty}\parallel f_n-f\parallel^2_{L^2(\mathbb{R})}=0. \tag{4.5}$$

By using Equation (4.4) we see that the sequence $\{\langle\omega,f_n\rangle\}_{n=1}^{\infty}$ is Cauchy in $L^2(\mu)$, since

$$\begin{aligned}\mathbb{E}[\langle\omega,f_n\rangle-\langle\omega,f_m\rangle]^2 &= \mathbb{E}[\langle\omega,f_n-f_m\rangle^2]\\ &= \parallel f_n-f_m\parallel^2_{L^2(\mathbb{R})},\end{aligned}$$

and hence the sequence converges in $L^2(\mu)$. We denote the limit by $\langle\omega,f\rangle$. This limit is independent of the choice of the approximating sequence.

In particular, this makes

$$\tilde{W}(t):=\tilde{W}(t,\omega)=\langle\omega,\chi_{[0,t]}(\cdot)\rangle \tag{4.6}$$

well-defined since $\chi_{[0,t]}$ is in $L^2(\mathbb{R})$ for all $t\in\mathbb{R}$. By means of Kolmogorov's continuity theorem, the process $\tilde{W}(t)$ can be shown to have a continuous version which we will denote

by $W(t),\ t \in \mathbb{R}$, that is, $W(t,\omega)$ is continuous in t for all t, $P(\tilde{W}_t = W_t) = 1$.
From now on we work with the Brownian motion $W(t),\ t \in \mathbb{R}$ on the white noise probability space $(S'(\mathbb{R}), \mathcal{B}(S'(\mathbb{R})), \mu)$. It can now be shown that $W(t),\ t \in \mathbb{R}$ is a Gaussian process with mean

$$\mathbb{E}[W(t)] = W(0) = 0 \tag{4.7}$$

and covariance

$$\begin{aligned}\mathbb{E}[W(t_1)W(t_2)] &= \mathbb{E}[\int_{\mathbb{R}} \chi_{[0,t_1]} dW_s \int_{\mathbb{R}} \chi_{[0,t_2]}(s) dW_s] \\ &= \min(t_1, t_2).\end{aligned}$$

(See [17] page 49). $W(t)$ is a Brownian motion with respect to the probability law μ. A function of the type $f(t) = \sum_k a_k \chi_{[t_k,t_{k+1})}(t)$ where $t_0 < t_1 < t_2 < \ldots < t_N$ and $a_k \in \mathbb{R}$ is called a simple function and belongs to $L^2(\mathbb{R})$. Using Equation (4.6) and the linearity property of f we obtain

$$\langle \omega, f \rangle = \sum_{k=0}^{N-1} a_k \langle \omega, \chi_{[t_k,t_{k+1})}(\cdot) \rangle = \sum_{k=0}^{N-1} a_k (W(t_{k+1}) - W(t_k)) = \sum_{k=0}^{N-1} a_k \Delta W(t_k). \tag{4.8}$$

Riemann sums suggest that we define

$$\langle \omega, f \rangle = \int_{\mathbb{R}} f(t) dW_t \quad \text{for all} \quad f \in L^2(\mathbb{R}). \tag{4.9}$$

We introduce Hermite polynomials and Hermite functions (see [1]). Hermite polynomials play an important role in probability theory. In what follows we let

$$h_n(x) := (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} \left(e^{-\frac{1}{2}x^2} \right), \quad x \in \mathbb{R},\ n = 0, 1, 2, ... \tag{4.10}$$

denote the Hermite polynomials. This gives, for example,

$$h_0(x) = 1, \quad h_1(x) = x, \quad h_2(x) = x^2 - 1, \quad h_3(x) = x^3 - 3x, \tag{4.11}$$

$$h_4(x) = x^4 - 6x^2 + 3, \quad h_5(x) = x^5 - 10x^3 + 15x, \cdots \tag{4.12}$$

The generating function of Hermite polynomial is

$$\exp(tx - \frac{t^2}{2}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) \quad \text{for all} \quad t, x \in \mathbb{R}. \tag{4.13}$$

We have the following properties of the Hermite polynomials:

$$h_n'(x) = n h_{n-1}(x) \quad \text{and} \quad h_{n+1}(x) = x h_n(x) - n h_{n-1}(x), \qquad n \geq 1. \tag{4.14}$$

Let $\{\xi_n\}_{n=1}^{\infty}$ be the Hermite functions defined by

$$\xi_n(x) := \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}x)e^{-\frac{1}{2}x^2}, \quad n = 1, 2, ... \tag{4.15}$$

The Hermite functions ξ_n form an orthonormal basis for $L^2(\mathbb{R}^n)$ (see [82]) and

$$| \xi_n(x) | \leq \begin{cases} Cn^{-\frac{1}{12}} & \text{if} \quad | x | \leq 2\sqrt{n} \\ Ce^{-\gamma x^2} & \text{if} \quad | x | > \sqrt{n} \end{cases}$$

where C and γ are positive constants independent of n (see [31]). Most of the theory can be developed without the explicit use of a particular basis. However, the choice of an explicit basis is crucial when we want to deduce that the white noise (to be defined later on) belongs to the space $(S)^*$ of Hida distribution.

Let $\mathcal{J}$ denote the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \cdots)$ of finite length $l(\alpha) = \max\{i, \alpha_i \neq 0\}$ with non-negative integers $\alpha_i \in \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ for all i. Then, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{J}$ we put $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$ and $| \alpha | = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. We can construct an orthogonal $L^2(\mu)$ basis $\{H_\alpha(\omega)\}_{\alpha\in\mathcal{J}}$ given by

$$H_\alpha(\omega) := h_{\alpha_1}(\langle\omega, \xi_1\rangle)h_{\alpha_2}(\langle\omega, \xi_2\rangle)\cdots h_{\alpha_n}(\langle\omega, \xi_n\rangle), \quad \omega \in \Omega \tag{4.16}$$

where $\langle\omega, \cdot\rangle$ and ξ_j (resp. h_j, $j = 1, 2, \ldots, n$) are Hermite functions (resp. Hermite polynomials). Thus, for example,

$$\begin{aligned} H_{(4,0,3,2)}(\omega) &= h_4(\langle\omega, \xi_1\rangle)h_0(\langle\omega, \xi_2\rangle)h_3(\langle\omega, \xi_3\rangle)h_2(\langle\omega, \xi_4\rangle) \\ &= (\langle\omega, \xi_1\rangle^4 - 6\langle\omega, \xi_1\rangle^2 + 3)(\langle\omega, \xi_3\rangle^3 - 3\langle\omega, \xi_3\rangle)(\langle\omega, \xi_4\rangle^2 - 1) \end{aligned}$$

where we have applied Equations (4.11) and (4.12). The family $\{H_\alpha\}_{\alpha\in\mathcal{J}}$ is an orthogonal sequence that constitutes a basis for the Hilbert space $L^2(\mu)$. The unit vectors

$$\epsilon^{(k)} = (0, 0, \ldots 0, 1, 0, \ldots, 0) \tag{4.17}$$

with 1 on the k^{th} entry, 0 otherwise, $k = 1, 2, \ldots$ are important special cases of multi-indices (see [27]). We note that

$$H_{\epsilon^{(k)}}(\omega) = h_1(\langle\omega, \xi_k\rangle) = \langle\omega, \xi_k\rangle = \int_{\mathbb{R}} \xi_k(t)dW_t. \tag{4.18}$$

More generally, by a fundamental result of Itô [52], we have

$$I_n(\xi^{\hat{\otimes}\alpha}) = H_\alpha(\omega) \tag{4.19}$$

with $H_0 := 1$. $\otimes$ and $\hat{\otimes}$ denote the tensor product and the symmetrized tensor product respectively. For example, if f and g are real functions on $\mathbb{R}$ then

$$(f \otimes g)(x_1, x_2) = f(x_1)g(x_2)$$

and

$$(f \hat{\otimes} g)(x_1, x_2) = \frac{1}{2}[f(x_1)g(x_2) + f(x_2)g(x_1)].$$

Lemma 4.1.2 *$\{H_\alpha\}_{\alpha \in \mathcal{J}}$ constitutes an orthogonal basis for $L^2(\mu)$. Moreover, if $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{J}$ we have the norm expression*

$$\parallel H_\alpha \parallel^2 = \alpha! := \alpha_1! \alpha_2! \cdots \tag{4.20}$$

Proof

The proof is given in [47] on page 24. We omit the details. □

We can show that $\mathbb{E}[H_\alpha H_\beta] = \alpha! \delta_{\alpha\beta}$. So if $F = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha(\omega)$ we have

$$\mathbb{E}[F \cdot H_\beta] = \mathbb{E}[\left(\sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha(\omega)\right) \cdot H_\beta] = a_\alpha \beta!, \tag{4.21}$$

which gives

$$a_\alpha = \frac{1}{\beta!} \mathbb{E}[F \cdot H_\beta], \qquad \beta! \neq 0. \tag{4.22}$$

We now state the chaos decomposition for the elements of $L^2(\mu)$ (see [27], Theorem 5.2).

Theorem 4.1.3 *Let $F \in L^2(\mu)$, be an $\mathcal{F}_T$-measurable random variable. Then there exists a unique family $\{a_\alpha\}_{\alpha \in \mathcal{J}}$ of constants $a_\alpha \in \mathbb{R}$ such that*

$$F(\omega) = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha. \tag{4.23}$$

Moreover, the Itô isometry is valid:

$$\parallel F \parallel^2_{L^2(\mu)} = \sum_{\alpha \in \mathcal{J}} a_\alpha^2 \parallel H_\alpha \parallel^2_{L^2(\mu)} = \sum_{\alpha \in \mathcal{J}} a_\alpha^2 \alpha! \tag{4.24}$$

Example

For each $t \in \mathbb{R}$, the random variable $W(t) \in L^2(\mu)$ has the expansion

$$\begin{aligned} W(t) &= \langle \omega, \chi_{[0,t]}(\cdot) \rangle = \langle \omega, \sum_{k=1}^{\infty} (\chi_{[0,t]}, \xi_k)_{L^2(\mathbb{R})} \xi_k(\cdot) \rangle \\ &= \int_{\mathbb{R}} \sum_{k=1}^{\infty} (\chi_{[0,t]}, \xi_k)_{L^2(\mathbb{R})} \xi_k(s) dW_s = \sum_{k=1}^{\infty} (\chi_{[0,t]}, \xi_k)_{L^2(\mathbb{R})} \int_{\mathbb{R}} \xi_k(s) dW_s \\ &= \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s) ds \right) \int_{\mathbb{R}} \xi_k(s) dW_s = \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s) ds \right) H_{\epsilon^{(k)}}(\omega) \end{aligned} \tag{4.25}$$

where, in general, $(f, g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(t) g(t) dt$.

Lemma 4.1.4

$$\int_{\mathbb{R}} f(s) dW_s = \sum_{k=1}^{\infty} (\xi_k, f)_{L^2(\mathbb{R})} H_{\epsilon^{(k)}}(\omega) \quad \textit{for} \quad f \in L^2(\mathbb{R}). \tag{4.26}$$

Following the construction of the Wiener-Itô chaos expansion discussed in Chapter 2, we can formulate an equivalent theorem to Theorem 4.1.3 in terms of the multiple Itô integral as follows (see [27], Theorem 5.3).

Theorem 4.1.5 *Let $F \in L^2(\mu)$, be an $\mathcal{F}_T$-measurable random variable. Then there exists a unique sequence $\{f_n\}_{n=1}^{\infty}$ of functions $f_n \in \hat{L}^2(\mathbb{R}^n)$ such that*

$$F(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(t) dW^{\otimes n}(\omega) = \sum_{n=0}^{\infty} I_n(f_n). \tag{4.27}$$

Moreover, the following Itô isometry is valid:

$$\| F \|_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} n! \, \| f_n \|_{L^2(\mathbb{R}^n)}^2 . \tag{4.28}$$

Remark

The connection between the two expansions in Theorem 4.1.3 and Theorem 4.1.5 is given by

$$f_n = \sum_{\alpha \in \mathcal{J} : |\alpha| = n} a_\alpha \xi^{\hat{\otimes} \alpha} \qquad n = 0, 1, 2, \ldots \tag{4.29}$$

where a_α are the coefficients in the expansion in Hermite functions given in Theorem 4.1.3.

4.2 Stochastic test functions and stochastic distribution functions

From Theorem 4.1.3, the condition

$$\sum_{\alpha\in\mathcal{J}} a_\alpha^2 \parallel H_\alpha \parallel_{L^2(\mu)}^2 < \infty \tag{4.30}$$

ensures that

$$F = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha \in L^2(\mu). \tag{4.31}$$

The condition (4.30) can be replaced by various other conditions.

Definition 4.2.1 *For $0 \le \rho \le 1$ the Kondratiev test function space $(S)_\rho$ consists of all $f = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha \in L^2(\mu)$, $a_\alpha \in \mathbb{R}$ such that*

$$\parallel f \parallel_{\rho,k}^2 := \sum_{\alpha\in\mathcal{J}} (\alpha!)^{1+\rho} a_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \quad \textit{for all} \quad k \in \mathbb{N} \tag{4.32}$$

where $(2\mathbb{N})^{k\alpha} := (2\cdot 1)^{k\alpha_1}(2\cdot 2)^{k\alpha_2}\cdots(2\cdot j)^{k\alpha_j}$ if $k\alpha = (k\alpha_1,\ldots,k\alpha_j) \in \mathcal{J}$.

Definition 4.2.2 *For $0 \le \rho \le 1$ the Kondratiev distribution space $(S)_{-\rho}$ consists of all formal series $F = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha \in L^2(\mu)$, $b_\alpha \in \mathbb{R}$ such that*

$$\parallel F \parallel_{-\rho,-q}^2 := \sum_{\alpha\in\mathcal{J}} (\alpha!)^{1-\rho} b_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty \quad \textit{for some} \quad q \in \mathbb{N} \tag{4.33}$$

$(S)_\rho$ is endowed with the projective limit topology and $(S)_{-\rho}$ is endowed with limit topology induced by the above seminorms. We note that for any $f = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha \in (S)_\rho$ and $F = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha \in (S)_{-\rho}$ the action

$$\langle F, f\rangle := \sum_{\alpha\in\mathcal{J}} a_\alpha b_\alpha \alpha! \tag{4.34}$$

is well defined and thus the space $(S)_{-\rho}$ is the dual of $(S)_\rho$
For general $0 \le \rho \le 1$ we have the following inclusions

$$(S)_1 \subset (S)_\rho \subset (S)_0 \subset L^2(\mu) \subset (S)_{-0} \subset (S)_{-\rho} \subset (S)_{-1}. \tag{4.35}$$

Remark
The spaces $(S)_0$ and $(S)_{-0}$ coincide with the Hida spaces (S) and $(S)^*$, respectively, which

we introduce below.

The stochastic (Hida) test functions (S) and stochastic (Hida) distribution space $(S)^*$ relates to $L^2(\mu)$ in a natural way (see [17], [27], [47] and [74]). We use Theorem 4.1.3 and Theorem 4.1.5 to define the Hida test function space (S) which is a subspace of $L^2(\mu)$ and the Hida distribution space $(S)^*$ which is a superset of $L^2(\mu)$ as follows.

Definition 4.2.3 *We define the Hida space (S) of stochastic test functions to be all $f \in L^2(\mu)$ whose expansion*

$$f(\omega) = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in L^2(\mu) \tag{4.36}$$

satisfies

$$\| f \|_k^2 := \sum_{\alpha \in \mathcal{J}} (\alpha!) a_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \quad for \quad all \quad k \in \mathbb{N}. \tag{4.37}$$

Let $(S)_k$ be the completion of (S) in the norm $\| \cdot \|_k$. Then $(S)_k$ is a Hilbert space and the following inclusions are continuous

$$(S) \subset \cdots \subset (S)_{k+1} \subset (S)_k \subset \cdots \subset (S)_1.$$

It can be shown that (S) is complete if and only if

$$(S) = \bigcap_{k=1}^{\infty} (S)_k.$$

$\{((S)_k, \| \cdot \|_k); k \geq 1\}$ is a sequence of Hilbert space with respect to the norm $\| \cdot \|_k$ such that $(S)_{k+1}$ is continuously embedded in $(S)_k$ for each k. Let $(S) = \bigcap_{k=1}^{\infty} (S)_k$ and endow (S) with projective limit topology, that is, the coarsest topology such that for each k the inclusion from (S) into $(S)_k$ is continuous. This topological space (S) is called the projective limit of $\{(S)_k, k \geq 1\}$. A base of neighborhoods of zero in this projective limit topology is given by the choice of $\epsilon > 0$, $k \geq 1$ and the set $\{f \in (S)_k; \| f \|_k < \epsilon\}$. The projective limit topology of (S) is induced by a decreasing sequence of Hilbert spaces and therefore (S) is a countably Hilbert space (see [61]). A sequence $\{f_k, \ k \geq 1\}$ converges to f in (S) with respect to the norm $\| \cdot \|_k$ if and only if it converges to f in every Hilbert space $(S)_k$, that is,

$$\| f_n - f \|_k \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad k.$$

Definition 4.2.4 *We define the Hida space* $(S)^*$ *of stochastic distributions to be the set of formal expansions* $G(\omega) = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha$ *such that*

$$\| G \|_{-q}^2 := \sum_{\alpha \in \mathcal{J}} (\alpha!) b_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty \tag{4.38}$$

for some $q \in \mathbb{N}$.

Let $(S)_{-q}^*$ be the completion of L^2 with respect to the norm $\| \cdot \|_{-q}$. Then $(S)_{-q}^*$ is a Hilbert space. The dual space $(S)^*$ of (S) is given by

$$(S)^* = \bigcup_{q=1}^{\infty} (S)_{-q}^*$$

where $(S)_{-q}^*, q \geq 1$ are Hilbert spaces and we have the inclusions

$$(S)_{-1}^* \subset \cdots \subset (S)_{-q}^* \subset (S)_{-q-1}^* \subset \cdots \subset (S)^*.$$

$\{((S)_{-q}^*, \| \cdot \|_{-q}); q \geq 1\}$ is a sequence of Hilbert spaces with respect to the norm $\| \cdot \|_{-q}$ such that $(S)_{-q}^*$ is continuously imbedded in $(S)_{-q-1}^*$ for each q. Let $(S)^* = \bigcup_{q=1}^{\infty} (S)_{-q}^*$ and endow $(S)^*$ with the finest topology such that for each q the inclusion from $(S)_{-q}^*$ into $(S)^*$ is continuous. This topological space $(S)^*$ is called the inductive limit of the sequence $\{(S)_{-q}^*; q \geq 1\}$ (see [61]). A base of neighborhoods of zero in this inductive limit topology is given by the choice of $\epsilon > 0$, $q \geq 1$ and the set $\{f \in (S)_{-q}^*; \| f \|_{-q} < \epsilon\}$. A sequence $\{f_k,\ k \geq 1\}$ converges to f in $(S)^*$ if and only if there exists some q such that $f_k \in (S)_{-q}^*$ and

$$\| f_k - f \|_{-q} \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for some } q,$$

that is, f_k converges to f in $(S)_{-q}^*$ as $k \to \infty$.

Definition 4.2.5 *If* $F = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in (S)^*$ *we define the generalized expectation* $\mathbb{E}[F]$ *of* F *by*

$$\mathbb{E}[F] := a_0. \tag{4.39}$$

Definition 4.2.6 *The action of* $G(\omega) = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha \in (S)^*$ *on* $f(\omega) = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in (S)$ *is defined by*

$$\langle G, f \rangle = \sum_{\alpha \in \mathcal{J}} \alpha! a_\alpha b_\alpha. \tag{4.40}$$

Equation (4.40) is well-defined since

$$\begin{aligned}\sum_{\alpha\in\mathcal{J}} a_\alpha b_\alpha \alpha! &= \sum_{\alpha\in\mathcal{J}} a_\alpha b_\alpha (\alpha!)^{1/2}(\alpha!)^{1/2}(2\mathbb{N})^{-q\alpha/2}(2\mathbb{N})^{q\alpha/2} \\ &\leq \left(\sum_{\alpha\in\mathcal{J}} a_\alpha^2(\alpha!)(2\mathbb{N})^{-q\alpha}\right)^{\frac{1}{2}} \left(\sum_{\alpha\in\mathcal{J}} b_\alpha^2(\alpha!)(2\mathbb{N})^{q\alpha}\right)^{\frac{1}{2}} \\ &< \infty\end{aligned}$$

for q large enough. We have the inclusions:

$$(S) \subset L^2(\mu) \subset (S)^*. \tag{4.41}$$

The quantity $(2\mathbb{N})^\alpha$ plays an important role in white noise theory. We note that if $\alpha = \epsilon^{(k)}$ we obtain

$$(2\mathbb{N})^{\epsilon^{(k)}} = 2k. \tag{4.42}$$

We can, in a natural way, define $(S)^*$-valued integrals as follows (see [17], [46] and [47]).

Definition 4.2.7 *Suppose $Z : \mathbb{R} \to (S)^*$ has the property that*

$$\langle Z(t), f\rangle \in L^1(\mathbb{R}) \quad for \quad all \quad f \in (S). \tag{4.43}$$

Then $\int_{\mathbb{R}} Z(t)dt$ is defined to be the unique element of $(S)^*$ such that

$$\langle \int_{\mathbb{R}} Z(t)dt, f\rangle = \int_{\mathbb{R}} \langle Z(t), f\rangle dt \quad for \quad all \quad f \in (S). \tag{4.44}$$

We can show that Equation (4.44) defines $\int_{\mathbb{R}} Z(t)dt$ as an element of $(S)^*$ (see [46] Proposition 8.1). If expression (4.43) holds, we say that $Z(t)$ is integrable in $(S)^*$.

One of the important features of the Hida space $(S)^*$ is that it contains the singular white noise $\dot{W}_t$ for all t (see [74]). By formally differentiating (4.25) we arrive at the following definition.

Definition 4.2.8 *The white noise process $\dot{W}(t)$ is defined by the following formal expansion*

$$\dot{W}(t) = \sum_{k=1}^{\infty} \xi_k(t) H_{\epsilon^{(k)}}(\omega), \quad t \in \mathbb{R} \tag{4.45}$$

where $\xi_k(t)$ is the Hermite function and $\epsilon^{(k)}$ is given in Equation (4.17).

The following lemma says that the white noise $\dot{W}(t)$ belongs to $(S)^*$.

Lemma 4.2.9 *For each $t \in \mathbb{R}$, $\dot{W}(t)$ is a generalized function, that is, $\dot{W}(t) \in (S)^*$.*

Proof

We need to show that the expansion (4.45) satisfies the condition (4.38). To do this we recall that

$$| \xi_k(t) | \leq \begin{cases} Ck^{-\frac{1}{12}} & \text{if} \quad | t | \leq 2\sqrt{k} \\ Ce^{-\gamma t^2} & \text{if} \quad | t | > \sqrt{k} \end{cases}$$

for some constants C and γ independent of k and the well-known estimate

$$\sup_{t \in \mathbb{R}} | \xi_k(t) | = O(k^{-\frac{1}{12}}). \tag{4.46}$$

For each t, we have

$$\begin{aligned} \| \dot{W}(t) \|_{-q}^2 &:= \sum_{k=1}^{\infty} \xi_k^2(t) \epsilon^{(k)}!((2\mathbb{N})^{\epsilon^{(k)}})^{-q} \\ &= \sum_{k=1}^{\infty} \xi_k^2(t)(2k)^{-q} \qquad \text{by} \quad (4.42) \\ &\leq C \sum_{k=1} k^{-\frac{1}{6}}(2k)^{-q} \\ &= C \sum_{k=1}^{\infty} k^{-\frac{1}{6}-q} 2^{-q} \end{aligned}$$

for some constant C. Hence, for any $q \geq 2$, we have

$$\| \dot{W}(t) \|_{-q}^2 < \infty \tag{4.47}$$

and so $\dot{W}(t) \in (S)^*$ for all t. Thus, $\dot{W}(t)$ is a generalized function. $\square$

A process $B : \mathbb{R} \to (S)^*$ is differentiable in $(S)^*$ if the limit

$$\lim_{h \to 0} \frac{B(t+h) - B(t)}{h} \tag{4.48}$$

exists in $(S)^*$ for all t. We denote this limit by $\frac{d}{dt}B(t)$.
$\frac{d}{dt}B(t) = B_t'$ if and only if there exists a $q \in \mathbb{N}$ such that

$$\lim_{h \to 0} \| \frac{B(t+h) - B(t)}{h} - B_t' \|_{-q} = 0.$$

Lemma 4.2.10

$$\frac{d}{dt}W(t) \quad \textit{exists in } (S)^* \quad \textit{for} \quad \textit{all} \quad t \in \mathbb{R}. \tag{4.49}$$

Proof

By Equation (4.25) we have

$$W(t) = \sum_{k=1}^{\infty} \int_0^t \xi_k(s) ds H_{\epsilon^{(k)}}(\omega) \quad \text{and} \quad W(t+h) = \sum_{k=1}^{\infty} \int_0^{t+h} \xi_k(s) ds H_{\epsilon^{(k)}}(\omega). \tag{4.50}$$

Then

$$\begin{aligned} W(t+h) - W(t) &= \sum_{k=1}^{\infty} \int_0^{t+h} \xi_k(s) ds H_{\epsilon^{(k)}}(\omega) - \sum_{k=1}^{\infty} \int_0^t \xi_k(s) ds H_{\epsilon^{(k)}}(\omega) \\ &= \sum_{k=1}^{\infty} \int_t^{t+h} \xi_k(s) ds H_{\epsilon^{(k)}}(\omega) \end{aligned}$$

and so, for $h \neq 0$, we have

$$\frac{1}{h}\{W(t+h) - W(t)\} = \sum_{k=1}^{\infty} \frac{1}{h} \int_t^{t+h} \xi_k(s) ds H_{\epsilon^{(k)}}(\omega). \tag{4.51}$$

Set

$$W_h(t) := \sum_{k=1}^{\infty} \frac{1}{h} \int_t^{t+h} \xi_k(s) ds H_{\epsilon^{(k)}}(\omega)$$

and so

$$| W_h(t) | \leq K, \quad \text{for fixed} \quad t$$

for K constant. We need to show that

$$W_h(t) \to \sum_{k=1}^{\infty} \xi_k(t) H_{\epsilon^{(k)}}(\omega) \tag{4.52}$$

as $h \to 0$, that is, for some $q \in \mathbb{N}$,

$$\| \sum_{k=1}^{\infty} \frac{1}{h} \int_t^{t+h} \xi_k(s) ds H_{\epsilon^{(k)}}(\omega) - \sum_{k=1}^{\infty} \xi_k(t) H_{\epsilon^{(k)}}(\omega) \|_{-q}^2 \to 0 \quad \text{as} \quad h \to 0. \tag{4.53}$$

We have

$$\sum_{k=1}^{\infty} \frac{1}{h} \int_t^{t+h} \xi_k(s) ds H_{\epsilon^{(k)}}(\omega) - \sum_{k=1}^{\infty} \xi_k(t) H_{\epsilon^{(k)}}(\omega) = \sum_{k=1}^{\infty} \frac{1}{h} \int_t^{t+h} [\xi_k(s) - \xi_k(t)] ds H_{\epsilon^{(k)}}(\omega). \tag{4.54}$$

Put

$$a_k(h) := \frac{1}{h} \int_t^{t+h} [\xi_k(s) - \xi_k(t)] ds.$$

Therefore we have

$$\begin{aligned}\| \sum_{k=1}^{\infty} \frac{1}{h} \int_t^{t+h} \xi_k(s) ds H_{\epsilon^{(k)}}(\omega) - \sum_{k=1}^{\infty} \xi_k(t) H_{\epsilon^{(k)}}(\omega) \|_{-q}^2 &= \sum_{k=1}^{\infty} \mid a_k(h) \mid^2 \epsilon^{(k)}!(2\mathbb{N})^{-q\epsilon^{(k)}} \text{ by } (4.38) \\ &= \sum_{k=1}^{\infty} \mid a_k(h) \mid^2 (2k)^{-q} \text{ by } (4.42).\end{aligned}$$

Since $\sup_{t\in\mathbb{R}} \mid \varepsilon_k(t) \mid = O(k^{-\frac{1}{2}})$, we see that

$$\sup_h \{a_k(h), \quad h \in [0,1] \quad k = 1, 2, \ldots\} < \infty.$$

So

$$\mid a_k(h) \mid^2 (2k)^{-q} \leq C(2k)^{-q}$$

for some constant C and the right hand side is the term of a convergent series. In addition, since

$$a_k(h) \to 0 \text{ as } h \to 0, \quad k = 1, 2, \ldots$$

we have

$$\| \sum_{k=1}^{\infty} \frac{1}{h} \int_t^{t+h} \xi_k(s) ds H_{\epsilon^{(k)}}(\omega) - \sum_{k=1}^{\infty} \xi_k(t) H_{\epsilon^{(k)}}(\omega) \|_{-q}^2 \to 0 \text{ as } h \to 0$$

for all $q \geq 1$ by dominated convergence and the result then follows by Lemma 4.2.9. □

Remark

$(S)^*$ is too small for the purpose of solving stochastic ordinary and partial differential equations. However, we can find a unique solution in $(S)_{-1}$.

4.3 The Wick product

In this section review the definition of a Wick product $\diamond$ on the space $(S)_{-1}$ (see [27]). The Wick product was first introduced by G. Wick in the early 1950's as a renormalization technique in the theory of quantum physics. New applications have been developed in stochastic analysis (see [27], [47] and [74]).

Definition 4.3.1 *The Wick product $F \diamond G$ of $F = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha \in (S)_{-1}$ and $G = \sum_{\beta\in\mathcal{J}} b_\beta H_\beta \in (S)_{-1}$ with $a_\alpha, b_\beta \in \mathbb{R}$ is defined by*

$$(F \diamond G)(\omega) = \sum_{\alpha,\beta\in\mathcal{J}} a_\alpha b_\beta H_{\alpha+\beta} = \sum_{\gamma\in\mathcal{J}} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) H_\gamma(\omega). \tag{4.55}$$

The Wick product is a commutative, associative and distributive binary operation on each of the spaces $(S)_1, (S), (S)^*$ and $(S)_{-1}$ (see [47] page 47).
As an example, we have

$$(W_t \diamond W_t)(\omega) = W_t^2(\omega) - t \tag{4.56}$$

which follows from the next lemma. We need the following property:

$$H_{\epsilon^{(j)}+\epsilon^{(k)}} = \begin{cases} H_{\epsilon^{(j)}} H_{\epsilon^{(k)}} & \text{if } \ j \neq k \\ H^2_{\epsilon^{(j)}} - 1 & \text{if } \ j = k \end{cases}$$

Lemma 4.3.2 *We have*

$$\left(\int_{\mathbb{R}} f(s)dW_s\right) \diamond \left(\int_{\mathbb{R}} g(s)dW_s\right) = \left(\int_{\mathbb{R}} f(s)dW_s\right) \cdot \left(\int_{\mathbb{R}} g(s)dW_s\right) - \int_{\mathbb{R}} f(s)g(s)ds \tag{4.57}$$

for all $f, g \in L^2(\mathbb{R})$.

Proof
Let $f, g \in L^2(\mathbb{R})$. Then

$$\begin{aligned}
\left(\int_{\mathbb{R}} f(s)dW_s\right) \diamond \left(\int_{\mathbb{R}} g(s)dW_s\right) &= \left(\sum_{j=1}^{\infty}(\xi_j, f)_{L^2(\mathbb{R})} H_{\epsilon^{(j)}}\right) \diamond \left(\sum_{k=1}^{\infty}(\xi_k, g)_{L^2(\mathbb{R})} H_{\epsilon^{(k)}}\right) \text{ by (4.26)} \\
&= \sum_{j,k=1}^{\infty}(\xi_j, f)_{L^2(\mathbb{R})}(\xi_k, g)_{L^2(\mathbb{R})} H_{\epsilon^{(j)}+\epsilon^{(k)}} \text{ by (4.55)} \\
&= \left(\sum_{j=1}^{\infty}(\xi_j, f)_{L^2(\mathbb{R})} H_{\epsilon^{(j)}}\right) \cdot \left(\sum_{k=1}^{\infty}(\xi_k, g)_{L^2(\mathbb{R})} H_{\epsilon^{(k)}}\right) \\
&- \sum_{j=1}^{\infty}(\xi_j, f)_{L^2(\mathbb{R})}(\xi_j, g)_{L^2(\mathbb{R})} \quad \text{(by the property above)} \\
&= \left(\int_{\mathbb{R}} f(s)dW_s\right) \cdot \left(\int_{\mathbb{R}} g(s)dW_s\right) - \sum_{j=1}^{\infty}(\xi_j, f)_{L^2(\mathbb{R})}(\xi_j, g)_{L^2(\mathbb{R})} \\
&= \left(\int_{\mathbb{R}} f(s)dW_s\right) \cdot \left(\int_{\mathbb{R}} g(s)dW_s\right) - (f, g)_{L^2(\mathbb{R})}. \qquad \square
\end{aligned}$$

The Wick powers $X^{\diamond n}$, $n = 0, 1, 2, \ldots$ of $X \in (S)_{-1}$ are defined as follows:

$$X^{\diamond 0} := 1, \quad \text{and} \ \ X^{\diamond n} := X \diamond X \diamond \cdots \diamond X \quad (n \ \ factors). \tag{4.58}$$

The Wick exponential $\exp^{\diamond} X$ of $X \in (S)_{-1}$ is defined by

$$\exp^{\diamond} X := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n} \tag{4.59}$$

provided the series converges in $(S)_{-1}$. We also have the following useful rules.

$$(X+Y)^{\diamond 2} = X^{\diamond 2} + 2X \diamond Y + Y^{\diamond 2} \qquad X, Y \in (S)_{-1} \tag{4.60}$$

and

$$\exp^{\diamond}(X+Y) = \exp^{\diamond}(X) \diamond \exp^{\diamond}(Y) \quad X, Y \in (S)_{-1}. \tag{4.61}$$

We note that

$$\begin{aligned} H_{\epsilon^{(k)}}^{\diamond n} &= H_{\epsilon^{(k)}} \diamond H_{\epsilon^{(k)}} \diamond \cdots \diamond H_{\epsilon^{(k)}} \ \ (n \ \text{ times}) \ \text{ by } \ (4.58) \\ &= H_{\epsilon^{(k)}+\epsilon^{(k)}+\cdots+\epsilon^{(k)}} \ \text{ by } \ (4.55) \\ &= H_{n\epsilon^{(k)}} \end{aligned} \tag{4.62}$$

Lemma 4.3.3 *We have*

$$\exp^{\diamond}(\langle \omega, f\rangle) = \exp\{\langle \omega, f\rangle - \frac{1}{2} \parallel f \parallel^2\} \tag{4.63}$$

for $f \in L^2(\mathbb{R})$.

Proof

By basis independence (see [47] page 73) we assume that $f = c\varepsilon_1$ where c is a constant. We have

$$\begin{aligned} \exp^{\diamond}\langle \omega, f\rangle &= \exp^{\diamond}(\langle \omega, c\varepsilon_1\rangle) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle \omega, c\varepsilon_1\rangle^{\diamond n} \quad \text{by} \ \ (4.59) \\ &= \sum_{n=0}^{\infty} \frac{c^n}{n!} H_{\epsilon^{(1)}}^{\diamond n}(\omega) = \sum_{n=0} \frac{c^n}{n!} H_{n\epsilon^{(1)}}(\omega) \quad \text{by} \ \ (4.62) \\ &= \sum_{n=0}^{\infty} \frac{c^n}{n!} h_n(\langle \omega, \varepsilon_1\rangle) \ \ \text{by} \ \ (4.18) \\ &= \exp\{c\langle \omega, \varepsilon_1\rangle - \frac{1}{2}c^2\} \\ &= \exp\{\langle \omega, f\rangle - \frac{1}{2} \parallel f \parallel^2\} \end{aligned}$$

where the second last equality follows by the generating property of the Hermite polynomials. □

We now use the white noise and the Wick product to define an integration of a general class of processes with respect to $W(t)$ (see [47]).

Definition 4.3.4 *If X is S^*-integrable, then so is $X1_{[a,b]}$ for all $a, b \in \mathbb{R}$ and we put*

$$\int_a^b X(t)dt := \int_{\mathbb{R}} X(t)1_{[a,b]}(t)dt. \tag{4.64}$$

Proposition 4.3.5 *Suppose $X : \mathbb{R} \to (S)^*$ is such that $X(t) \diamond \dot{W}_t$ is integrable in $(S)^*$. Then*

$$\int_{\mathbb{R}} X(t)\delta W_t = \int_{\mathbb{R}} X(t) \diamond \dot{W}_t dt. \tag{4.65}$$

Proof

The proof follows by the arguments in [47] Theorem 2.5.9 on page 57. □

Remark

The integral on the left hand side of Equation (4.65) denotes the Skorohod integral of the stochastic process $X(t) = X(t, \omega)$. The integral on the right hand side of Equation (4.65) is interpreted as an $(S)^*$-valued integral. We note that the integral on the right may exist even if X is not Skorohod integrable. Thus, the right hand side of Equation (4.65) may be regarded as an extension of the Skorohod integral.

Definition 4.3.6 *Suppose X is an $(S)^*$-valued process such that*

$$\int_{\mathbb{R}} X_t \diamond \dot{W}_t dt \in (S)^*$$

then we call this integral the generalized Skorohod integral of X.

Lemma 4.3.7 *Let $f \in (S)$ and $G_t \in (S)_{-q}$ for all $t \in \mathbb{R}$, for some $q \in \mathbb{N}$. Put $\hat{q} = q + \frac{1}{\log 2}$. Then*

$$\int_{\mathbb{R}} | \langle G_t \diamond \dot{W}_t, f \rangle | \, dt \leq \| f \|_{\hat{q}} \left(\int_{\mathbb{R}} \| G_t \|_{-q}^2 \, dt \right)^{\frac{1}{2}}. \tag{4.66}$$

The following theorem gives conditions for the generalized Skorohod integral to exist (see [27] page 77).

Theorem 4.3.8

1. *Suppose $G : \mathbb{R} \to (S)_{-q}$ satisfies $\int_{\mathbb{R}} \| G_t \|_{-q}^2 \, dt < \infty$ for some $q \in \mathbb{N}$. Then*

   $$\int_{\mathbb{R}} G_t \diamond \dot{W}_t dt \quad \text{exists in} \quad (S)^*.$$

2. *Suppose $F(t)$ and $F_n(t)$, $n = 1, 2, \ldots.$ are elements of $(S)^*$ for all $t \in \mathbb{R}$ and $\int_{\mathbb{R}} \| F_n(t) - F(t) \|_{-q}^2 \, dt \to 0, \quad n \to \infty$. Then*

   $$\int_{\mathbb{R}} F_n(t) \diamond \dot{W}_t dt \to \int_{\mathbb{R}} F(t) \diamond \dot{W}_t dt, \quad n \to \infty$$

   in the weak-topology on $(S)^*$*

Proof

1. The proof follows from Lemma 4.3.7 and Definition 4.2.7 .

2. We have

$$\begin{aligned} \mid \langle \int_{\mathbb{R}} (F_n(t) - F(t)) \diamond \dot{W}_t dt, f \rangle \mid &\leq \int_{\mathbb{R}} \mid \langle (F_n(t) - F(t)) \diamond \dot{W}_t, f \rangle \mid dt \quad \text{by} \quad (4.44) \\ &\leq \parallel f \parallel_{\hat{q}} \int_{\mathbb{R}} \parallel F_n(t) - F(t) \parallel_{-q}^2 dt \quad \text{by} \quad (4.66) \\ &\to 0, \quad n \to \infty. \end{aligned}$$

□

4.4 The Hermite Transform

The Wick product faces challenges when limit operations are involved in computations. To handle such situations we use a transformation called Hermite transform or $\mathcal{H}$-transform (see [47] Section 2.6) which transforms an element $F \in (S)_{-1}$ into deterministic functions $\mathcal{H}F(z_1, z_2, \ldots)$ of complex variables $z_j \in \mathbb{C}, \quad j = 1, 2, \ldots$ with values in $\mathbb{C}$.

Definition 4.4.1 *Let $F(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(\omega) \in (S)_{-1}$. Then the Hermite transform of F, denoted by $\mathcal{H}F$ or $\tilde{F}$, is defined by*

$$\mathcal{H}F(z) = \tilde{F}(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha \in \mathbb{C} \tag{4.67}$$

where $z = (z_1, z_2, \ldots,) \in \mathbb{C}^{\mathbb{N}}$ (the set of all sequences of complex numbers) and
$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \cdots$ if $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{J}$ where $z_j^0 = 1$.

One can show that the sum in Equation (4.67) converges on the infinite dimensional neighborhood

$$\mathbb{K}_q(R) = \{(z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}; \sum_{\alpha \in \mathcal{J}} (2\mathbb{N})^{q\alpha} \mid z^\alpha \mid^2 < R^2\} \tag{4.68}$$

for some $0 < q, R < \infty$ (see [47] Proposition 2.6.5).

Example

Let $W(x, \omega) = \sum_{j=0}^{\infty} \int_0^x \xi_j(s) ds H_{\epsilon^{(j)}}(\omega)$. Then we have

$$\mathcal{H}W(z) = \tilde{W}(x)(z) = \sum_{j=0}^{\infty} \int_0^x \xi_j(s) ds z_j, \qquad z = (z_1, z_2, \ldots) \in \mathbb{C}_c^{\mathbb{N}}.$$

The following proposition is an immediate consequence of Definitions 4.3.1 and 4.4.1.

Proposition 4.4.2 *If $F, G \in (S)_{-1}$ then*

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z). \tag{4.69}$$

for all z such that $\mathcal{H}F(z)$ and $\mathcal{H}G(z)$ exist. In general

$$\mathcal{H}(f^{\diamond}(F))(z) = f(\mathcal{H}F(z)) \quad (\textit{when convergent}) \tag{4.70}$$

if $f : \mathbb{C} \to \mathbb{C}$ is entire, $f(\mathbb{R}) \subset \mathbb{R}$ and $f^{\diamond}(F) := \sum_{\alpha \in \mathcal{J}} c_\alpha F^{\diamond n} \in (S)_{-1}$.

One can use the Hermite transform to characterize distributions in $(S)_{-1}$. This is given in the next theorem. Its proof is found in [47] on page 68.

Theorem 4.4.3 *1. If $F = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in (S)_{-1}$ then there exists $q, M_q < \infty$ such that*

$$| \mathcal{H}F(z) | \leq \sum_{\alpha \in \mathcal{J}} | a_\alpha || z^\alpha | \leq M_q \left(\sum_{\alpha \in \mathcal{J}} (2\mathbb{N})^{q\alpha} | z^\alpha |^2 \right)^{\frac{1}{2}}$$

for all $z \in (\mathbb{C}^{\mathbb{N}})_c$. In particular, $\mathcal{H}F$ is a bounded analytic function on $\mathbb{K}_q(R)$ for all $R < \infty$.

2. Conversely, assume that $g(z) := \sum_{\alpha \in \mathcal{J}} b_\alpha z^\alpha$ is a power series of $z \in (\mathbb{C}^{\mathbb{N}})_c$ such that there exist $q < \infty$ and $\delta > 0$ with $g(z)$ absolutely convergent and bounded on $\mathbb{K}_q(\delta)$. Then there exists a unique $G \in (S)^*$ such that $\mathcal{H}G = g$, namely

$$G = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha.$$

The Hermite transform also serves as a useful tool to describe the topology of $(S)_{-1}$. In particular, the convergence of sequences of Hida distributions can be characterized as follows.

Theorem 4.4.4 *A sequence $X_n, n \geq 1$ converges to X in $(S)_{-1}$ if and only if there exists $q, M < \infty,\ R > 0$ such that*

$$\sup_{z \in \mathbb{K}_q(R)} | \mathcal{H}X_n(z) | \leq M \tag{4.71}$$

for all $n \geq 1$ and

$$\mathcal{H}X_n(z) \to \mathcal{H}X(z) \tag{4.72}$$

as $n \to \infty$ for all $z \in \mathbb{K}_q(R)$.

Proof

Details of the proof is given in [47] on page 81. □

We have the following important lemma (see [47] pages 85.

Lemma 4.4.5 *Suppose* $X(t,\omega)$ *and* $F(t,\omega)$ *are* $(S)_{-1}$ *processes such that*

$$\frac{d\tilde{X}(t,z)}{dt} = \tilde{F}(t,z) \quad \text{for each} \quad t \in (a,b), \quad z \in \mathbb{K}_q(\delta) \tag{4.73}$$

and that $\tilde{F}(t,z)$ *is a bounded function of* $(t,z) \in (a,b) \times \mathbb{K}_q(\delta)$, *continuous in* $t \in (a,b)$ *for each* $z \in \mathbb{K}_q(\delta)$. *Then* $X(t,\omega)$ *is a differentiable* $(S)_{-1}$ *process and*

$$\frac{dX(t,\omega)}{dt} = F(t,\omega) \tag{4.74}$$

for all $t \in (a,b)$.

Definition 4.4.6 *An* $(S)_{-1}$*-process* X *is strongly integrable over an interval* $[a,b]$ *if*

$$\int_a^b X(t,\omega)dt := \lim_{\triangle t_k \to 0} \sum_{k=0}^{n-1} X(t_k^*,\omega)\triangle t_k \tag{4.75}$$

exists in $(S)_{-1}$ *for all partitions* $a = t_0 < t_1 < \ldots < t_n = b$ *of* $[a,b]$, $\triangle t_k = t_{k+1-t_k}$ *and* $t_k^* \in [t_k, t_{k+1}]$ *for* $k = 1,\ldots,n-1$.

Taking the Hermite transform in Equation (4.75) and using Lemma 4.4.5, we have the following result (see [47] page 86).

Lemma 4.4.7 *Let* $X(t)$ *be an* $(S)_{-1}$ *process. Suppose there exists* $q < \infty$, $\delta > 0$ *such that*

$$\sup\{\tilde{X}(t,z) : t \in [a,b], \; z \in \mathbb{K}_q(\delta)\} < \infty \tag{4.76}$$

and $\tilde{X}(t,z)$ *is a continuous function of* $t \in [a,b]$ *for each* $z \in \mathbb{K}_q(\delta)$. *Then* $X(t)$ *is strongly integrable and*

$$\mathcal{H}\left(\int_a^b X(t)dt\right) = \int_a^b \tilde{X}(t)dt \tag{4.77}$$

where the integral to the right is the Lebesgue integral.

We also have the following useful lemma.

Lemma 4.4.8 *Let G be a bounded open subset of $\mathbb{R}_+ \times \mathbb{R}$. Assume a process $F : G \to (S)_{-1}$ with $\mathcal{H}(F) = \tilde{F}$ such that $\tilde{F}$ and its derivatives are bounded on $G \times \mathbb{K}_q(R)$, continuous with respect to t for all $z \in \mathbb{K}_q(R)$ and analytic in $z \in \mathbb{K}_q(R)$ for all t, $q < \infty$, $R > 0$. Then*

$$\mathcal{H}\left(\frac{d}{dt}F\right) = \frac{d}{dt}\left(\mathcal{H}(F)\right) = \frac{d}{dt}\tilde{F}. \tag{4.78}$$

on $\mathbb{K}_q(R)$.

Proof

The proof is based on Lemma 4.4.5. The mean value theorem implies that

$$\frac{\tilde{F}(t+h)(z) - \tilde{F}(t)(z)}{h} = \frac{d}{dt}\tilde{F}(t + \varepsilon h)(z)$$

for some $\varepsilon \in [0, 1]$, for all $z \in \mathbb{K}_q(\mathbb{R})$. By the assumptions on $\tilde{F}$ we conclude that

$$\frac{\tilde{F}(t+h)(z) - \tilde{F}(t)(z)}{h} \to \frac{d}{dt}\tilde{F}(t)(z) \quad \text{as} \quad h \to 0 \tag{4.79}$$

pointwise boundedly for $z \in \mathbb{K}_q(\mathbb{R})$. Since $(S)_1$ is a nuclear space (see Lemma 2.8.2 in [47]) we can show using Theorem 4.4.4 that statement (4.79) is equivalent to convergence in $(S)_{-1}$. That is

$$\frac{F(t+h) - F(t)}{h} \to \frac{d}{dt}F(t) \quad \text{in} \quad (S)_{-1}$$

for all t. The result then follows since the Hermite transform is a continuous linear functional in $(S)_{-1}$ □

We have the following chain rule in $(S)_{-1}$.

Proposition 4.4.9 *Suppose that $t \to X_t : \mathbb{R} \to (S)_{-1}$ is continuously differentiable and let $f : \mathbb{C} \to \mathbb{C}$ be entire (analytic on $\mathbb{C}$) such that $f(\mathbb{R}) \subset \mathbb{R}$ and $f^{\diamond}(X_t) \in (S)_{-1}$ for all t, then*

$$\frac{d}{dt}f^{\diamond}(X_t) = (f')^{\diamond}(X_t) \diamond \frac{d}{dt}X_t \quad \textit{in} \quad (S)_{-1}. \tag{4.80}$$

Proof

$$\begin{aligned}
\mathcal{H}\left(f'^{\diamond}(X_t)\diamond\frac{d}{dt}X_t\right)(z) &= \mathcal{H}(f'^{\diamond}(X_t))(z)\cdot\mathcal{H}\left(\frac{d}{dt}X_t\right)(z) \quad \text{by} \quad (4.69)\\
&= f'(\mathcal{H}(X_t)(z))\cdot\frac{d}{dt}\mathcal{H}(X_t)(z) \quad \text{by} \quad (4.70) \quad \text{and} \quad (4.78)\\
&= \frac{d}{dt}f(\mathcal{H}(X_t)(z))\\
&= \frac{d}{dt}\mathcal{H}(f^{\diamond}(X_t))(z) \quad \text{by} \quad (4.70)\\
&= \mathcal{H}\left(\frac{d}{dt}f^{\diamond}(X_t)\right)(z) \quad \text{by} \quad (4.78)
\end{aligned}$$

The result follows by the uniqueness of the Hermite transform (see Theorem 4.4.3). □

Example

$$\frac{d}{dt}\exp^{\diamond}(W_t) = \exp^{\diamond}(W_t)\diamond\frac{dW_t}{dt} = \exp^{\diamond}(W_t)\diamond\dot{W}_t.$$

Note

The Hermite transform is closely related to the so called $\mathcal{S}$-transform (see [46] and [47] page 80). The $\mathcal{S}$-transform maps random variables into non-random functionals. The use of Hermite transform has some advantages, for instance it enables the application of methods of complex analysis.

4.5 Hida-Malliavin derivative

Now that the basic white noise theory have been given, we can proceed to define the Hida-Malliavin derivative. We follow the construction in [27].

Definition 4.5.1 *Let $F : S'(\mathbb{R}) \to \mathbb{R}$ be a given function and let $\gamma(t) = \int_0^t g(s)ds$ be deterministic, $g \in L^2([0,T])$. We say that F has a directional derivative in $(S)^*$ in the direction γ if*

$$D_\gamma F(\omega) := \lim_{\varepsilon\to 0}\frac{F(\omega+\varepsilon\gamma)-F(\omega)}{\varepsilon} \tag{4.81}$$

exists in $(S)^*$.

Example

Let $F(\omega) = \langle \omega, f \rangle = \int_{\mathbb{R}} f(t) dW_t(\omega)$ for some $f \in S(\mathbb{R})$. Then

$$\begin{aligned} D_\gamma F(\omega) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\langle \omega + \varepsilon\gamma, f\rangle - \langle \omega, f \rangle] \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle \varepsilon\gamma, f \rangle = \langle \gamma, f \rangle \\ &= \int_{\mathbb{R}} f(t)\gamma(t) dt \end{aligned}$$

for all $\gamma \in L^2([0, T])$.

Definition 4.5.2 *We say that $F : S'(\mathbb{R}) \to \mathbb{R}$ is differentiable if there exists a map $\psi : \mathbb{R} \to (S)^*$ such that $\psi(t)g(t) = \psi(t, \omega)g(t)$ is $(S)^*$-integrable and*

$$D_\gamma F = \int_{\mathbb{R}} \psi(t, \omega) g(t) dt \quad for\ all \quad g \in L^2([0, T]). \tag{4.82}$$

Then $D_t F(\omega)$ is defined to be $\psi(t, \omega)$. This is the Hida-Malliavin derivative of F at t in $(S)^*$.

Example

Let $F(\omega) = \langle \omega, f \rangle = \int_{\mathbb{R}} f(t) dW_t(\omega)$ for some $f \in S'(\mathbb{R})$. Then by the above example, F is differentiable and its Hida-Malliavin derivative is

$$D_t F(\omega) = f(t) \text{ for all } (t, \omega).$$

Assume that the Hida-Malliavin derivative of $F \in L^2(\mu)$ exists and suppose that φ is continuously differentiable, $D_t F \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$ and $\varphi'(F) D_t F \in L^2(\lambda \times \mu)$ then the Hida-Malliavin derivative of $\varphi(F)$ exists and

$$D_t \varphi(F) = \varphi'(F) D_t F. \tag{4.83}$$

More generally, we have the following extension (see [27] page 82).

Theorem 4.5.3 *Assume that the Hida-Malliavin derivative of $F_1, \ldots, F_n \in L^2(\mu)$ exists and that φ is continuously differentiable in $\mathbb{R}^n$, $D_t F_i \in L^2(\mu)$ for all $t \in \mathbb{R}$ and $\sum_{i=1}^n \frac{\partial \varphi(F)}{\partial x_i} DF_i \in L^2(\mu \times \lambda)$ for $i = 1, \ldots, n$ where $F = (F_1, \ldots, F_n)$. Then the Hida-Malliavin derivative of $\varphi(F)$ exists and*

$$D_t \varphi(F) = \sum_{i=1}^n \frac{\partial \varphi(F)}{\partial x_i} D_t F_i. \tag{4.84}$$

By using Equation (4.83) and the example above we have

$$D_t(h_n(\langle\omega, f\rangle)) = h_n'(\langle\omega, f\rangle)f(t) = nh_{n-1}(\langle\omega, f\rangle)f(t) \tag{4.85}$$

where $\{h_n(t)\}_{n\geq 0},\ t \in \mathbb{R}$ are Hermite polynomials of order n and $f \in L^2(\mathbb{R})$. In particular, choosing $n = 1$ we have

$$D_t(h_1(\langle\omega, f\rangle) = f(t).$$

The following theorem gives some versions of the Wick chain rule for Hida-Malliavin derivatives (see [27] page 87).

Theorem 4.5.4 *1. Let $F, G \in \mathbb{D}_{1,2}$. Then $F \diamond G \in \mathbb{D}_{1,2}$ and*

$$D_t(F \diamond G) = F \diamond D_t G + D_t F \diamond G, \qquad t \in \mathbb{R}.$$

2. Let $F \in \mathbb{D}_{1,2}$. Then $F^{\diamond n} \in \mathbb{D}_{1,2}$ and

$$D_t(F^{\diamond n}) = nF^{\diamond(n-1)} \diamond D_t F, \qquad n = 1, 2, \ldots$$

3. Let $F \in \mathbb{D}_{1,2}$ be Hida-Malliavin differentiable and assume that

$$\exp^{\diamond} F = \sum_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n} \in \mathbb{D}_{1,2}.$$

Then

$$D_t \exp^{\diamond} F = \exp^{\diamond} F \diamond D_t F.$$

Proof

The proofs for parts 1 and 3 are given in [27] page 87. Here we give the proof to part 2 which is not given. By the closability of the Hida-Malliavin derivative (see Theorem 6.12 in [27] page 86) it suffice to prove 2 in the case when $F = \exp^{\diamond}\langle\omega, f\rangle = \exp(\langle\omega, f\rangle - \frac{1}{2} \parallel f \parallel^2_{L^2(\mathbb{R})})$ where $f \in L^2(\mathbb{R})$ is a deterministic function and $\langle\omega, f\rangle = \int_{\mathbb{R}} f(s)dW_s$. Then

$$\begin{aligned}
D_t F^{\diamond n} &= D_t(F \diamond F \diamond \cdots \diamond F) \\
&= D_t(\exp^{\diamond}\langle\omega, f\rangle \diamond \exp^{\diamond}\langle\omega, f\rangle \diamond \cdots \diamond \exp^{\diamond}\langle\omega, f\rangle) \\
&= D_t(\exp^{\diamond}\langle\omega, f + f + \cdots + f\rangle) \text{ by } (4.61) \\
&= D_t(\exp(\langle\omega, f + f + \cdots + f\rangle - \frac{1}{2} \parallel f + f + \cdots + f \parallel^2_{L^2(\mathbb{R})})) \text{ by } (4.63) \\
&= \exp(\langle\omega, f + f + \cdots + f\rangle - \frac{1}{2} \parallel f + f + \cdots + f \parallel^2_{L^2(\mathbb{R})}) \\
&\quad (f(t) + f(t) + \cdots + f(t)) \text{ by } (4.84) \\
&= \exp^{\diamond}(\langle\omega, nf\rangle)nf(t) \text{ by } (4.63).
\end{aligned} \tag{4.86}$$

On the other hand,

$$\begin{aligned}
nF^{\diamond(n-1)} \diamond D_t F &= n(F \diamond F \diamond \cdots \diamond F) \diamond D_t(\exp^\diamond \langle \omega, f\rangle) \\
&= n(\exp^\diamond\langle \omega, f\rangle \diamond \exp^\diamond\langle \omega, f\rangle \diamond \cdots \diamond \exp^\diamond\langle \omega, f\rangle) \\
&\diamond \quad D_t(\exp(\langle \omega, f\rangle - \frac{1}{2} \parallel f \parallel^2_{L^2(\mathbb{R})})) \quad \text{by} \quad (4.63) \\
&= n \exp^\diamond(\langle \omega, f + f + \cdots + f\rangle) \diamond \exp(\langle \omega, f\rangle - \frac{1}{2} \parallel f \parallel^2_{L^2(\mathbb{R})}) f(t) \quad \text{by} \quad (4.84) \\
&= n \exp^\diamond(\langle \omega, (n-1)f\rangle) \diamond \exp^\diamond(\langle \omega, f\rangle) f(t) \\
&= n \exp^\diamond(\langle \omega, nf\rangle) f(t) \quad \text{by} \quad (4.63).
\end{aligned} \tag{4.87}$$

The result then follows by comparing (4.86) and (4.87). □

Example
For $f \in L^2(\mathbb{R})$ we have

$$D_t(\langle \omega, f\rangle^n) = n\langle \omega, f\rangle^{n-1} f(t) \quad \text{and} \quad D_t(\langle \omega, f\rangle^{\diamond n}) = n\langle \omega, f\rangle^{\diamond(n-1)} \diamond f(t) \quad \text{for a.a } t. \tag{4.88}$$

These examples illustrate that the Hida-Malliavin derivative satisfies the chain rule.

We recall that, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{J}$, we have

$$H_\alpha(\omega) = h_{\alpha_1}(\langle \omega, \xi_1\rangle) \cdots h_{\alpha_n}(\langle \omega, \xi_n\rangle) = \langle \omega, \xi_1\rangle^{\diamond \alpha_1} \cdots \langle \omega, \xi_n\rangle^{\diamond \alpha_n}. \tag{4.89}$$

Motivated by the examples above, we now extend Definition 4.5.2 to elements in $(S)^*$ (see [27] page 83).

Definition 4.5.5 *Let $F(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \in (S)^*$. Then we define the Hida-Malliavin derivative $D_t F$ of F at t by*

$$D_t F(\omega) := \sum_{\alpha \in \mathcal{J}} c_\alpha \sum_{k=1}^{\infty} \alpha_k H_{\alpha - \epsilon^{(k)}}(\omega) \xi_k(t) = \sum_{\beta \in \mathcal{J}} \left(\sum_{k=1}^{\infty} c_{\beta + \epsilon^{(k)}} (\beta_k + 1) \xi_k(t) \right) H_\beta(\omega) \tag{4.90}$$

whenever the sum converges in $(S)^*$.

The following result shows the Hida-Malliavin derivative operator is closable.

Lemma 4.5.6 *Let $F \in Dom(D_t) \subset (S)^*$ for a.a t and $q \in \mathbb{N}$. Then*

1. $\int_{\mathbb{R}} \parallel D_t F \parallel^2_{-\hat{q}} dt \leq \parallel F \parallel^2_{-q}, \quad \hat{q} \geq 2q + \frac{1}{\log 2}$.

2. $D_tF \in (S)^*$ *for a.a.* $t \in \mathbb{R}$.

3. *Suppose* $F_n \in Dom(D_t) \subset (S)^*$ *for a.a* t *and for all* $n = 1, 2, \ldots$ *and* $F_n \to F, \quad n \to \infty$ *in* $(S)^*$. *Then there exists a subsequence* $F_{n_k}, \ k = 1, 2, \ldots$ *such that*

$$D_tF_{n_k} \to D_tF, \quad k \to \infty, \quad in \ \ (S)^* \tag{4.91}$$

for a.a. $t \in \mathbb{R}$.

Proof

The proof is similar to the one given in [27] on page 96. We give it here for completeness.

1. Suppose $F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha$. Then, by Equation (4.90), we have

$$D_tF \;=\; \sum_{\alpha \in \mathcal{J}} \sum_{k=1}^{\infty} c_\alpha \alpha_k \xi_k(t) H_{\alpha - \epsilon^{(k)}} = \sum_{\beta \in \mathcal{J}} \left(\sum_{k=1}^{\infty} c_{\beta + \epsilon^{(k)}} (\beta_k + 1) \xi_k(t) \right) H_\beta = \sum_{\beta \in \mathcal{J}} g_\beta(t) H_\beta.$$

where $g_\beta(t) = \sum_{k=1}^{\infty} c_{\beta+\epsilon^{(k)}} (\beta_k + 1)\xi_k(t)$. Since $F \in (S)^*$ there exists a $q \in \mathbb{N}$ such that

$$\| \ F \ \|_{-q}^2 := \sum_{\alpha \in \mathcal{J}} c_\alpha^2 \alpha! (2\mathbb{N})^{-q\alpha} < \infty.$$

We note that

$$\int_{\mathbb{R}} g_\beta^2(t) dt = \int_{\mathbb{R}} \left(\sum_{k=1}^{\infty} c_{\beta+\epsilon^{(k)}} (\beta_k + 1) \xi_k(t) \right)^2 dt = \sum_{k=1}^{\infty} c_{\beta+\epsilon^{(k)}}^2 (\beta_k + 1)^2.$$

Therefore, using $(x+1)e^{-x} \leq 1$ for all $x \geq 0$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \| \ D_tF \ \|_{-\hat{q}}^2 \ dt \;&=\; \int_{\mathbb{R}} \sum_{\beta \in \mathcal{J}} g_\beta^2(t) \beta! (2\mathbb{N})^{-\hat{q}\beta} d = \sum_{\beta \in \mathcal{J}} \sum_{k=1}^{\infty} c_{\beta+\epsilon^{(k)}}^2 (\beta_k + 1)^2 \beta! (2\mathbb{N})^{-\hat{q}\beta} \\ &=\; \sum_{\beta \in \mathcal{J}} \sum_{k=1}^{\infty} c_{\beta+\epsilon^{(k)}}^2 (\beta_k + \epsilon^{(k)})! (2k)^{-\frac{\beta_k}{\log 2}} (2\mathbb{N})^{-2q\beta_k} \\ &\leq\; \sum_{\beta \in \mathcal{J}} \sum_{k=1}^{\infty} c_{\beta+\epsilon^{(k)}}^2 (\beta_k + \epsilon^{(k)})! (2\mathbb{N})^{-q(\beta+\epsilon^{(k)})} \\ &\leq\; \sum_{\alpha \in \mathcal{J}} c_\alpha^2 \alpha! (2\mathbb{N})^{-q\alpha} = \| \ F \ \|_{-q}^2 \end{aligned}$$

which proves 1 and hence 2.

3. For a given F and $F_n, \quad n = 1, 2, \ldots$ we have

$$\int_{\mathbb{R}} \| \ D_t(F_n - F) \ \|_{-\hat{q}}^2 \ dt \leq \| \ F_n - F \ \|_{-q}^2 \to 0, \quad n \to \infty$$

by 1, if $q \in \mathbb{N}$ is large enough. Hence, there exists a subsequence $F_{n_k}, k = 1, 2, \ldots$ such that

$\| \ D_t(F_{n_k} - F) \ \|_{-q}^2 \to 0, \quad k \to \infty$ for a.a $t \in \mathbb{R}$. □

4.6 Conditional expectation on $(S)^*$

Definition 4.6.1 *Let* $F = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n dW^{\otimes n} \in (S)^*$. *Then the conditional expectation of* F *with respect to* $\mathcal{F}_t$ *is defined by*

$$\mathbb{E}[F \mid \mathcal{F}_t] = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n \cdot \chi_{[0,t]^n} dW^{\otimes n} \tag{4.92}$$

when convergent in $(S)^*$

Lemma 4.6.2 *Let* $F, G, \mathbb{E}[F \mid \mathcal{F}_t], \mathbb{E}[G \mid \mathcal{F}_t] \in (S)^*$. *Then*

$$\mathbb{E}[F \diamond G \mid \mathcal{F}_t] = \mathbb{E}[F \mid \mathcal{F}_t] \diamond \mathbb{E}[G \mid \mathcal{F}_t]. \tag{4.93}$$

Proof
Assume without loss of generality that $F = I_n(f_n) = \int_{\mathbb{R}^n} f_n dW^{\otimes n}$ and
$G = I_m(g_m) = \int_{\mathbb{R}^m} g_m dW^{\otimes m}$ for some $f_n \in \hat{L}^2(\mathbb{R}^n)$ and $g_m \in \hat{L}^2(\mathbb{R}^m)$. Then we have

$$\begin{aligned}
\mathbb{E}[F \diamond G \mid \mathcal{F}_t] &= \mathbb{E}[I_n(f_n) \diamond I_m(g_m) \mid \mathcal{F}_t] = \mathbb{E}[I_{n+m}(f_n \hat{\otimes} g_m) \mid \mathcal{F}_t] \\
&= \mathbb{E}[\int_{\mathbb{R}^{n+m}} f_n \hat{\otimes} g_m \cdot \chi_{[0,t]^{n+m}} dW^{\otimes(n+m)} \mid \mathcal{F}_t] \quad \text{by} \quad (4.27) \\
&= \int_{\mathbb{R}^{n+m}} f_n \cdot \chi_{[o,t]^n} \hat{\otimes} g_m \cdot \chi_{[0,t]^m} dW^{\otimes(n+m)} \quad \text{by} \quad (4.92) \\
&= \mathbb{E}[F \mid \mathcal{F}_t] \diamond \mathbb{E}[G \mid \mathcal{F}_t].
\end{aligned}$$

□

Lemma 4.6.3 *Suppose* $F \in (S)^*$ *and* $\exp^\diamond F \in (S)^*$. *Then*

$$\mathbb{E}[\exp^\diamond F \mid \mathcal{F}_t] = \exp^\diamond \mathbb{E}[F \mid \mathcal{F}_t]. \tag{4.94}$$

In particular, if $F \in L^1(\mu)$, *we have*

$$\mathbb{E}[\exp^\diamond F] = \exp\{\mathbb{E}[F]\}. \tag{4.95}$$

Proof
By Lemma 4.6.2 we have

$$\mathbb{E}[\exp^\diamond F \mid \mathcal{F}_t] \;=\; \mathbb{E}[\sum_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n} \mid \mathcal{F}_t] = \sum_{n=0}^{n} \frac{1}{n!} \mathbb{E}[F \mid \mathcal{F}_t]^{\diamond n} = \exp^\diamond\{\mathbb{E}[F \mid \mathcal{F}_t]\}$$

where we have used Equations (4.59) and (4.92). □

Let $P(x) = \sum_{\alpha \in \mathcal{J}} c_\alpha x^\alpha$ be a polynomial where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}, \ldots\ x \in \mathbb{R}^N,\ \ c_\alpha \in \mathbb{R}$ and $x_j^0 = 1$. Then we can define its Wick version at $X = (X_1, \ldots, X_n) \in ((S)^*)^n$ by

$$P^\diamond(X) = \sum_{\alpha \in \mathcal{J}} c_\alpha X^{\diamond\alpha}.$$

Let $X^{(t)} = (X_1^{(t)}, \ldots, X_n^{(t)})$ be of the form

$$X_i^{(t)}(\omega) = \int_0^t \xi_i(s) dW_s = \int_{\mathbb{R}} \xi_i(s) 1_{[0,t]}(s) dW_s, \qquad i = 1, 2, \ldots$$

Then $t \to X_i^{(t)}$ is differentiable in $(S)^*$ and

$$\frac{d}{dt}\left(\int_0^t \xi_i(s) dW_s\right) = \frac{d}{dt}\left(\int_0^t \xi_i(s) \dot{W}_s ds\right) = \xi_i(t)\dot{W}_t \in (S)^*. \tag{4.96}$$

The following Wick chain rule for polynomials follows by induction (see [27] page 94).

Lemma 4.6.4 *Let $P(x) = \sum_{\alpha \in \mathcal{J}} c_\alpha x^\alpha$ be a polynomial in $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $X_j^{(t)} = \int_0^t \xi_j(s) dW_s,\ \ j = 1, \ldots, n$. Then*

$$\frac{d}{dt} P^\diamond(X^{(t)}) = \sum_{j=1}^n \left(\frac{\partial P}{\partial x_j}\right)^\diamond (X^{(t)}) \diamond \xi_j(t)\dot{W}_t. \tag{4.97}$$

Proof
Since $X_j^{(t)} = \int_{\mathbb{R}} \xi_j(s) dW_s,\ \ j = 1, \ldots, n$, we can write this, using Equation (4.65), as

$$X_j^{(t)} = \int_0^t \xi_j(s) \diamond \dot{W}_s ds.$$

Now

$$\begin{aligned} \frac{d}{dt} P^\diamond(X^{(t)}) &= \sum_{j=1}^n \left(\frac{\partial P}{\partial x_j}\right)^\diamond (X^{(t)}) \frac{d}{dt}\left(\int_0^t \xi_j(s) \diamond \dot{W}_s ds\right) \\ &= \sum_{j=1}^n \left(\frac{\partial P}{\partial x_j}\right)^\diamond (X^{(t)}) \diamond \xi_j(t)\dot{W}_t. \end{aligned}$$

□

4.7 The Donsker delta function

The Donsker delta function is a generalized white noise functional which has been studied in several monographs within white noise analysis (see [27], [46], [59], [61] and the references therein). Here we give its definition within the white noise framework as discussed in the previous sections. We follow the presentation in [27].

Definition 4.7.1 *Let $X : \Omega \to \mathbb{R}$ be a random variable which also belongs to $(S)_{-1}$. Then a continuous function*

$$\delta_X(\cdot) : \mathbb{R} \to (S)_{-1}$$

is called a Donsker delta function of X if it satisfies

$$\int_{\mathbb{R}} \varphi(x)\delta_X(x)dx = \varphi(X) \qquad a.e \tag{4.98}$$

for all (measurable) $\varphi : \mathbb{R} \to \mathbb{R}$ such that the integral on the left hand side converges in $(S)_{-1}$.

Proposition 4.7.2 *Suppose X is a normally distributed random variable with variance $v > 0$. Then δ_X is unique and is given by the expression*

$$\delta_X(x) = \frac{1}{\sqrt{2\pi v}} \exp^{\diamond}[-\frac{(x-X)^{\diamond 2}}{2v}] \in (S)_{-1} \tag{4.99}$$

Proof

We follow the proof in [27] Proposition 7.2. Let

$$f(x) = \frac{1}{\sqrt{2\pi v}} \exp^{\diamond}[-\frac{(x-X)^{\diamond 2}}{2v}].$$

The characterization theorem for $(S)_{-1}$ (see Theorem 4.4.3) ensures that $f(x) \in (S)_{-1}$ for all x and that $x \to f(x)$ is continuous for $x \in \mathbb{R}$. We show that $f(x)$ satisfies Equation (4.98), that is,

$$\int_{\mathbb{R}} \varphi(x)f(x)dx = \varphi(X) \qquad a.s. \tag{4.100}$$

We first assume that φ has the form

$$\varphi(x) = e^{\lambda x} \quad \text{for} \quad \text{some} \quad \lambda \in \mathbb{C}. \tag{4.101}$$

Taking the Hermite transform of the left hand side of Equation (4.100) gives

$$\mathcal{H}\left(\int_{\mathbb{R}} \varphi(x)f(x)dx\right) = \int_{\mathbb{R}} e^{\lambda x}\mathcal{H}(f(x))dx = \int_{\mathbb{R}} e^{\lambda x}\frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{(x-\tilde{X})^2}{2v}\right)dx \tag{4.102}$$

where $\tilde{X} = \tilde{X}(z)$ is the Hermite transform of X at $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$.
Put

$$\mathbb{E}[e^{\lambda Z}] = \int_{\mathbb{R}} e^{\lambda x}\frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{(x-\tilde{X})^2}{2v}\right)dx$$

where Z is a normally distributed random variable with mean $\tilde{X}$ and variance v. Denote the mean of X by m. Now $Z := X - m + \tilde{X}$ is such a random variable. Hence,

$$\mathbb{E}[e^{\lambda(X-m+\tilde{X})}] = \int_{\mathbb{R}} e^{\lambda x} \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(x-\tilde{X})^2}{2v}\right) dx = \exp(\lambda \tilde{X} + \frac{1}{2}\lambda^2 v)$$

where we have used a well known formula for the characteristic function of a normal random variable in the last equality. We, therefore, have

$$\begin{aligned} \mathcal{H}\left(\int_{\mathbb{R}} \varphi(x) f(x) dx\right) &= \exp\left(\lambda\tilde{X} + \frac{1}{2}\lambda^2 v\right) \\ &= \mathcal{H}\left(\exp^{\diamond}\left(\lambda X + \frac{1}{2}\lambda^2 v\right)\right) \quad \text{by} \quad (4.70) \\ &= \mathcal{H}(\exp(\lambda X)) \quad \text{by} \quad (4.63) \\ &= \mathcal{H}(\varphi(X)) \quad \text{by} \quad (4.101). \end{aligned}$$

This proves that Equation (4.100) holds for functions φ given by Equation (4.101). Therefore Equation (4.100) also holds for linear combinations of such functions. By a density argument, Equation (4.100) holds for all φ such that the integral on the left hand side of Equation (4.98) converges. We then prove uniqueness in the following way: if $\varrho_1 : \mathbb{R} \to (S)_{-1}$ and $\varrho_2 : \mathbb{R} \to (S)_{-1}$ are two continuous functions such that

$$\int_{\mathbb{R}} \varphi(x)\varrho_i(x) dx = \varphi(X), \quad i = 1, 2 \tag{4.103}$$

for all φ such that the integral on the left hand side converges, then in particular Equation (4.103) must hold for all continuous functions with compact support. But then clearly we must have

$$\varrho_1(x) = \varrho_2(x) \quad \text{for a.a} \quad x \in \mathbb{R}$$

and hence for all x by continuity. □

Lemma 4.7.3 *Let $\psi : [0,T] \to \mathbb{R}$, $\phi : [0,T] \to \mathbb{R}$ be deterministic functions such that $\int_0^T \mid \psi_s \mid ds < \infty$ and $\parallel \phi \parallel^2_{L^2([0,T])} := \int_0^T \phi_s^2 ds < \infty$. Define*

$$X_t = \int_0^t \psi_s ds + \int_0^t \phi_s dW_s, \qquad 0 \le t \le T. \tag{4.104}$$

Then

$$\begin{aligned} \exp^{\diamond}\left(-\frac{(x-X_T)^{\diamond 2}}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) &= \exp^{\diamond}\left(-\frac{x^2}{2 \parallel \phi \parallel^2_{L^2([0,T])]}}\right) + \int_0^T \exp^{\diamond}\left(-\frac{(x-X_t)^{\diamond 2}}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) \\ &\diamond \frac{x - X_t}{\parallel \phi \parallel^2_{L^2([0,T])}} \diamond (\psi_t + \phi_t \dot{W}_t) dt \end{aligned} \tag{4.105}$$

where $\dot{W}$ denotes the white noise of W.

Proof

The proof is a result of an application of the fundamental theorem of calculus and Proposition 4.4.9. Define $\varrho : [0,T] \to (S)_{-1}$ by

$$\varrho_t = \exp^{\diamond}\left(-\frac{(x-X_t)^{\diamond 2}}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right), \quad 0 \le t \le T. \tag{4.106}$$

Then

$$\begin{aligned}
\varrho_T &= \varrho_0 + \int_0^T \frac{d\varrho}{dt} dt \\
&= \exp^{\diamond}\left(-\frac{x^2}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) + \int_0^T \exp^{\diamond}\left(-\frac{(x-X_t)^{\diamond 2}}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) \diamond \frac{d}{dt}\left(-\frac{(x-X_t)^{\diamond 2}}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) dt \\
&= \exp^{\diamond}\left(-\frac{x^2}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) + \int_0^T \exp^{\diamond}\left(-\frac{(x-X_t)^{\diamond 2}}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) \diamond -2\frac{x-X_t}{2 \parallel \phi \parallel^2_{L^2([0,T])}} \diamond \frac{d}{dt}(x-X_t) \\
&= \exp^{\diamond}\left(-\frac{x^2}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) + \int_0^T \exp^{\diamond}\left(-\frac{(x-X_t)^{\diamond 2}}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) \diamond \frac{x-X_t}{\parallel \phi \parallel^2_{L^2([0,T])}} \\
&\diamond \ (\psi_t + \phi_t \dot{W}_t) dt
\end{aligned}$$

□

Next we state the main theorem for the Donsker delta function as presented in [27] (see Theorem 7.4).

Theorem 4.7.4 *Let $\phi : [0,T] \to \mathbb{R}$, $\alpha : [0,T] \to \mathbb{R}$, $\psi = \phi\alpha$ be deterministic functions such that $0 < \parallel \phi \parallel^2_{L^2([0,T])} = \int_0^T \phi_s^2 ds < \infty$ and $0 \le \int_0^T \psi_s^2 ds < \infty$. Define*

$$X_t = \int_0^t \phi_s \alpha_s ds + \int_0^t \phi_s dW_s, \qquad 0 \le t \le T.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be bounded. Then

$$f(X_T) = V_0 + \int_0^T u_t \diamond (\psi + \dot{W}_t) dt \tag{4.107}$$

where

$$V_0 = \int_{\mathbb{R}} \frac{f(x)}{\sqrt{2\pi} \parallel \phi \parallel_{L^2([0,T])}} \exp\left(-\frac{x^2}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) dx$$

and

$$u(t) = \phi(t) \int_{\mathbb{R}} \frac{f(x)}{\sqrt{2\pi} \parallel \phi \parallel_{L^2([0,T])}} \exp^{\diamond}\left(-\frac{(x-X_T)^{\diamond 2}}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) \diamond \frac{x-X_T}{\parallel \phi \parallel^2_{L^2([0,T])}} dx. \tag{4.108}$$

Proof

The proof is a consequence of Proposition 4.7.2 and Lemma 4.7.3 for $\psi_s = \phi_s \alpha_s$. □

Remark

If ϕ is continuous at $t = T$ it can be shown that Equation (4.108) implies

$$\lim_{t \to T} u_t = \phi_T \int_{\mathbb{R}} \frac{g(x)}{\sqrt{2\pi} \parallel \phi \parallel_{L^2([0,T])}} \exp^{\diamond}\left(-\frac{(x - X_T)^{\diamond 2}}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) \diamond \frac{x - X_T}{\parallel \phi \parallel^2_{L^2([0,T])}} dx. \tag{4.109}$$

This limit clearly exists in $(S)_{-1}$. The following corollary provides a more explicit representation than the one in Theorem 4.7.4.

Corollary 4.7.5 *Let ϕ and X_t be as in Theorem 4.7.4. In addition, assume $\parallel \phi \parallel^2_{L^2([0,T])} > 0$ for all $t < T$. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded. Then*

$$f(X_T) = V_0 + \int_0^T u_t(\psi dt + dW_t) \tag{4.110}$$

where

$$V_0 = \int_{\mathbb{R}} \frac{f(x)}{\sqrt{2\pi} \parallel \phi \parallel_{L^2([0,T])}} \exp\left(-\frac{x^2}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) dx$$

and

$$u(t) = \phi(t) \int_{\mathbb{R}} \frac{f(x)}{\sqrt{2\pi} \parallel \phi \parallel_{L^2([0,T])}} \exp\left(-\frac{(x - X_T)^2}{2 \parallel \phi \parallel^2_{L^2([0,T])}}\right) \frac{x - X_T}{\parallel \phi \parallel^2_{L^2([0,T])}} dx. \tag{4.111}$$

Proof

The proof follows the same arguments in Theorem 4.7.4 and using the fact that $\int_0^T u_t \diamond \dot{W}_t dt = \int_0^T u_t dW_t$ in the L^2-case. We omit the details. □

4.8 Financial Application: Calculating Greeks

We consider the following model with two securities (see [1] and [74]):

1. A risk-free asset (for example a bank account) where the price A_t at time t is given by

$$dA_t = r_t A_t dt, \quad A_0 = 1 \tag{4.112}$$

2. A risky asset (for example a stock) where the price S_t at time t is given by

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad S_0 = x > 0. \tag{4.113}$$

where r_t, μ_t and σ_t are deterministic functions satisfying the property

$$\int_0^T \left(| r_t | + | \mu_t | + \sigma_t^2\right) ds < \infty.$$

We assume that σ is bounded away from zero. The exact solutions of the differential Equations (4.113) is given by

$$S_t = x \exp\left(\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s\right). \tag{4.114}$$

Then, we consider Δ of a digital option. The digital option payoff then takes the form

$$\chi_{[K,\infty)}(S_T) \tag{4.115}$$

with strike price K and S_T is the value of a stock price at final time T. Following the constructions in the preceding sections, we apply the concept of white noise analysis together with the Donsker delta function to compute Δ for the digital option. Here we only illustrate the computation of Δ.

We define ν by:

$$d(\log S) = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW := \nu_t dt + \sigma_t dW. \tag{4.116}$$

Let

$$v_T = \int_0^T \sigma_u^2 du. \tag{4.117}$$

Then

$$\log S_T \sim N\left(\int_0^T \nu_u du, v_T\right).$$

So we may apply (4.98) to get a.s.

$$f(S_T) = f(e^{\log S_T}) = \frac{1}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \exp^{\diamond}\left(-\frac{(y - \log S_T)^{\diamond 2}}{2v_T}\right) dy. \tag{4.118}$$

We note that for $f \in L^1(\mathbb{R})$ and with compact support the integral belongs to the distribution space $(S)_{-1}$ (by Lemma 4.4.7). The option price with the payoff function of the form (4.115) is given by

$$u(x) \;=\; \mathbb{E}[e^{-rT} f(S_T)] = \mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \exp^{\diamond}\left(-\frac{(y - \log S_T)^{\diamond 2}}{2v_T}\right) dy].$$

Theorem 4.8.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth. Then*

$$\begin{aligned} &\frac{\partial}{\partial x}\mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \exp^{\diamond}\left(-\frac{(y - \log S_T)^{\diamond 2}}{2v_T}\right) dy] \\ = \;& \mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \exp^{\diamond}\left(-\frac{(y - \log S_T)^{\diamond 2}}{2v_T}\right) \diamond \frac{(y - \log S_T)}{v_T}\frac{1}{x} dy]. \end{aligned} \tag{4.119}$$

Proof

Let

$$u(x) \;=\; \mathbb{E}[e^{-rT}f(S_T)] = \mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\exp^{\diamond}\left(-\frac{(y-\log S_T)^{\diamond 2}}{2v_T}\right)dy].$$

First we assume that $f \in L^2(\mathbb{R})$ and has compact support. Then by Lemma 4.4.7 the strong integral exists in $(S)_{-1}$. Taking the Hermite transform on both sides and using Lemma 4.4.7 since condition (4.76) holds we obtain the following deterministic equation

$$\tilde{u}(x)(z) \;=\; \mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\mathcal{H}\left(\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)\right)dy]$$

where $\tilde{u}$ denote the Hermite transforms of u and the expectation is taken in the generalized sense.

We note that

$$\mathcal{H}\left(\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)\right) = \exp\left(-\frac{(y-\log \tilde{S}_T(x)(z))^2}{2v_T}\right)$$

where we have used Proposition 4.4.2. Therefore we have

$$\tilde{u}(x)(z) \;=\; \mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\exp\left(-\frac{(y-\log \tilde{S}_T(x)(z))^2}{2v_T}\right)dy].$$

where $\tilde{S}_T$ is the Hermite transform of S_T.

Then

$$\begin{aligned}
&\tilde{u}(x+\varepsilon)(z) - \tilde{u}(x)(z)\\
=\;& \mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)[\exp\left(-\frac{(y-\log \tilde{S}_T(x+\varepsilon)(z))^2}{2v_T}\right) - \exp\left(-\frac{(y-\log \tilde{S}_T(x)(z))^2}{2v_T}\right)]dy]
\end{aligned}$$

and so, for $\varepsilon \neq 0$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\tilde{u}(x+\varepsilon)(z) - \tilde{u}(x)(z)}{\varepsilon}\\
=\;& \lim_{\varepsilon\to 0}\mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\frac{1}{\varepsilon}[\exp\left(-\frac{(y-\log \tilde{S}_T(x+\varepsilon)(z))^2}{2v_T}\right) - \exp\left(-\frac{(y-\log \tilde{S}_T(x)(z))^2}{2v_T}\right)]dy].
\end{aligned}$$

Put

$$\begin{aligned}
\tilde{Z}_\varepsilon(y)(z) \;:=\;& \frac{1}{\varepsilon}[\exp\left(-\frac{(y-\log \tilde{S}_T(x+\varepsilon)(z))^2}{2v_T}\right) - \exp\left(-\frac{(y-\log \tilde{S}_T(x)(z))^2}{2v_T}\right)]\\
=\;& \exp\left(-\frac{(y-\log \tilde{S}_T(x)(z))^2}{2v_T}\right)\\
\cdot\;& \frac{1}{\varepsilon}\{\exp\left(\frac{\varepsilon(y\log \tilde{S}_T(x)(z) - \log \tilde{S}_T^2(x)(z))}{v_T} - \frac{\log \varepsilon^2\tilde{S}_T^2(x)(z)}{2v_T}\right) - 1\}.
\end{aligned}$$

Using Taylor expansions and letting $\varepsilon \to 0$, we have

$$\tilde{Z}_\varepsilon(y)(z) \to \exp\left(-\frac{(y-\log \tilde{S}_T(x)(z))^2}{2v_T}\right)\left(\frac{y\log \tilde{S}_T(x)(z) - \log \tilde{S}_T^2(x)(z)}{v_T}\right) \text{ in } L^2(\mu) \text{ as } \varepsilon \to 0$$

since

$$\begin{aligned} \mid S_T(x+\varepsilon) \mid &= \mid (x+\varepsilon) \mid \exp\left(\int_0^T \left(\mu - \frac{1}{2}\sigma^2\right) dt + \int_0^T \sigma dW_t\right) \\ &\leq (x+1)\exp\left(\int_0^T \left(\mu - \frac{1}{2}\sigma^2\right) ds + \int_0^T \sigma dW_t\right) \in L^1(du). \end{aligned}$$

Thus, we have the following estimate

$$\begin{aligned} \mid \tilde{K}_\varepsilon(y)(z) \mid &:= \mid \frac{e^{-rT}}{\sqrt{2\pi v_T}} f(e^y)\frac{1}{\varepsilon}[\exp\left(-\frac{(y-\log \tilde{S}_T(x+\varepsilon)(z))^2}{2v_T}\right) - \exp\left(-\frac{(y-\log \tilde{S}_T(x)(z))^2}{2v_T}\right)] \mid \\ &= \mid \frac{e^{-rT}}{\sqrt{2\pi v_T}} f(e^y)\exp\left(-\frac{(y-\log \tilde{S}_T(x)(z))^2}{2v_T}\right)\left(\frac{y\log \tilde{S}_T(x)(z) - \log \tilde{S}_T^2(x)(z)}{v_T}\right) \mid \\ &\leq A_1 \cdot \mid f(e^y) \mid e^{-A_2(y^2 - |y|)}(A_3 \mid y \mid + A_4) \in L^1(\mathbb{R}) \end{aligned} \tag{4.120}$$

for some positive constants A_1, A_2, A_3, A_4 for fixed z. Using a similar estimate as in (4.120) we obtain

$$\int_{\mathbb{R}} \mid \tilde{K}_\varepsilon(y)(z) \mid dy \leq A_1 \int_{\mathbb{R}} \mid f(e^y) \mid e^{-A_2(y^2-|y|)}(A_3 \mid y \mid + A_4) dy < \infty$$

for fixed z with constants independent of ε. Since $f(e^y)$ grows polynomially, we can use the dominated convergence theorem to interchange the order of taking the limit and expectation and obtain

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{\tilde{u}(x+\varepsilon)(z) - \tilde{u}(x)(z)}{\varepsilon} \\ &= \mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(e^y)\frac{1}{\varepsilon}[\exp\left(-\frac{(y-\log \tilde{S}_T(x+\varepsilon)(z))^2}{2v_T}\right) - \exp\left(-\frac{(y-\log \tilde{S}_T(x)(z))^2}{2v_T}\right)]dy]. \end{aligned}$$

Using the estimate (4.120) for some positive constants A_1, A_2, A_3, A_4 which are independent of ε and for fixed z we can use the dominated convergence theorem to interchange the order of taking the limit and the integral and obtain

$$\lim_{\varepsilon\to 0}\frac{\tilde{u}(x+\varepsilon)(z)-\tilde{u}(x)(z)}{\varepsilon}$$

$$= \mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\exp\left(-\frac{(y-\log\tilde{S}_T(x+\varepsilon)(z))^2}{2v_T}\right)-\exp\left(-\frac{(y-\log\tilde{S}_T(x)(z))^2}{2v_T}\right)\}dy]$$

$$= \mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\frac{d}{dx}\left(\exp\left(-\frac{(y-\log\tilde{S}_T(x)(z))^2}{2v_T}\right)\right)dy]$$

$$= \mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\exp\left(-\frac{(y-\log\tilde{S}_T(x)(z))^2}{2v_T}\right)\frac{(y-\log\tilde{S}_T(x)(z))}{v_T}\frac{1}{x}dy]$$

where we have used the chain rule in the last equality. Thus we have

$$\frac{d}{dx}\tilde{u}(x)(z)=\mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\exp\left(-\frac{(y-\log\tilde{S}_T(x)(z))^2}{2v_T}\right)\frac{(y-\log\tilde{S}_T(x)(z))}{v_T}\frac{1}{x}dy].$$

We can write the above equation as

$$\frac{d}{dx}\tilde{u}(x)(z)=\mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\mathcal{H}\left(\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)\right)\cdot\mathcal{H}\left(\frac{(y-\log S_T(x))}{v_T}\frac{1}{x}\right)dy].$$

Since $S_T(x)\in(S)_{-1}$ the application of Proposition 4.4.2 yields

$$\frac{d}{dx}\tilde{u}(x)(z)=\mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\mathcal{H}\left(\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)\diamond\frac{(y-\log S_T(x))}{v_T}\frac{1}{x}\right)dy].$$

By using Lemma 4.4.7 twice (recall that f has compact support) on the right hand side and Lemma 4.78 on the left hand side of the above equation we obtain

$$\mathcal{H}\left(\frac{d}{dx}u(x)\right)(z)=\mathcal{H}\left(\mathbb{E}[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)\diamond\frac{(y-\log S_T(x))}{v_T}\frac{1}{x}dy]\right).$$

The result then follows by the uniqueness of the Hermite transform.

For the general case we consider, for f of polynomial growth, the sequence

$$f_n=f\chi_{[-n,n]}$$

whose functions have compact support so that $|f_n|\leq|f|$. Then $f_n\to f$ in L^2 as $n\to\infty$. Let

$$u_n(x)=\mathbb{E}[e^{-rT}\int_{\mathbb{R}} f_n(e^y)\frac{1}{\sqrt{2\pi v_T}}\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)dy].$$

Then it is clear that

$$u_n(x)\to u(x)\ \text{ in }\ (S)_{-1}. \tag{4.121}$$

By Theorem 4.4.4 this is equivalent to

$$\mathcal{H}(u_n(x))(z) \to \mathcal{H}(u(x))(z).$$

Put

$$g(x) = \mathbb{E}[e^{-rT} \int_{\mathbb{R}} f(e^y) \frac{1}{\sqrt{2\pi v_T}} \exp^\diamond \left(-\frac{(y - \log S_T)^{\diamond 2}}{2v_T} \right) \diamond \frac{(y - \log S_T)}{v_T} \diamond \frac{1}{x} dy].$$

Using the Cauchy-Schwartz inequality we have

$$\begin{aligned}
& \mid \frac{d}{dx} u_n(x) - g(x) \mid \\
= & \mid \mathbb{E}[e^{-rT} \int_{\mathbb{R}} (f_n(e^y) - f(e^y)) \frac{1}{\sqrt{2\pi v_T}} \exp^\diamond \left(-\frac{(y - \log S_T(x))^{\diamond 2}}{2v_T} \right) \diamond \frac{(y - \log S_T(x))}{v_T} \diamond \frac{1}{x} dy] \mid \\
\leq & \ \mathbb{E}[e^{-rT} \int_{\mathbb{R}} \mid f_n(e^y) - f(e^y) \mid\mid \frac{1}{\sqrt{2\pi v_T}} \exp^\diamond \left(-\frac{(y - \log S_T(x))^{\diamond 2}}{2v_T} \right) \diamond \frac{(y - \log S_T(x))}{v_T} \diamond \frac{1}{x} \mid dy] \\
\leq & \ A_1 \mathbb{E}[\int_{\mathbb{R}} \mid f_n(e^y) - f(e^y) \mid e^{-A_2(y^2 - |y|)} (A_3 \mid y \mid + A_4) dy].
\end{aligned}$$

Since $f(e^y)$ grows polynomially the dominated convergence theorem implies that $\mid f_n(e^y) - f(e^y) \mid$ converges to 0 as $n \to \infty$. Therefore using (4.120) the above inequality proves that

$$\frac{d}{dx} u_n(x) \to g(x) \ \ \text{pointwise.} \tag{4.122}$$

From (4.121) and (4.122) we can deduce that $u(x)$ is continuously differentiable and that

$$\frac{d}{dx} u(x) = g(x).$$

□

Chapter 5

Malliavin calculus for Pure Jump Lévy SDEs

In this chapter we aim at deriving a Malliavin derivative representation for a pure jump Lévy stochastic differential equation X_t in terms of its first variation process. We are inspired by the ideas discussed Chapter 2 Section 2.4 where we have reviewed a representation for the Malliavin derivative of X_t in the Brownian motion case. The result we obtain can be considered as a generalization of the ideas developed in Chapter 2 Section 2.4 to pure jump Lévy case.

We begin this chapter by introducing the Lévy process with both continuous part and jump part. We then restrict ourselves to a pure jump case. As in the second half of Chapter 2, we define a chaos expansion valid in a pure jump setting. We review the definition of the Skorohod integral and the stochastic derivative for the pure jump case (see [27]). We then give some known formulae for the Skorohod integral and the stochastic derivative in the case of pure jump Lévy process. We mention that these formulae generalize the known results for the Malliavin calculus in the case of Brownian motion. We define the first variation process of the pure jump Lévy stochastic differential equation X_t and then derive the Malliavin derivative representation (see Section 5.5).

In Section 5.6 we derive the necessary and sufficient conditions for a function to serve as a weighting function in the pure jump Lévy case. Here we are motivated by the ideas in [9] and [10] where the author gives the necessary and sufficient conditions for a function to serve as a weighting function in the Brownian motion case.

5.1 Basic definitions and results for Lévy processes

This section presents the basic concepts and results for Lévy processes. For a detailed account of Lévy processes we refer to [3], [16], [79] and [80].

Definition 5.1.1 *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space. An $\mathcal{F}_t$-adapted process $(X_t)_{t\geq 0}$ with $X_0 = 0$ a.s is called a Lévy process if X_t is continuous in probability and has stationary and independent increments.*

A Lévy process has a càdlàg modification (see [16]) and we will always assume that we are using the càdlàg version. By càdlàg we mean right continuous and having left limits. Let $\{\mathcal{F}_t, t \geq 0\}$ be the natural filtration of X_t completed with the P-null sets of $\mathcal{F}$. For the Lévy process X_t we denote by

$$X_{t^-} = \lim_{s\to t,\ s<t} X_s, \quad t \geq 0 \tag{5.1}$$

the left limit process and by

$$\triangle X_t = X_t - X_{t^-} \tag{5.2}$$

the jump size at time t. Put $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the σ-algebra generated by the family of all Borel subsets $\Lambda \subset \mathbb{R}$ whose closure $\bar{\Lambda}$ does not contain 0. The jump measure of X_t is defined by

$$N(t,\Lambda) := N(t,\Lambda,\omega) = \sum_{0\leq s\leq t} \chi_\Lambda(\triangle X_s), \quad \Lambda \in \mathcal{B}(\mathbb{R}_0). \tag{5.3}$$

$N(t,\Lambda)$ describes the number of jumps whose size $\triangle X_s$ belongs to Λ and which occur before or at time t. The derivative form is denoted by $N(dt,dz),\ t \geq 0,\ z \in \mathbb{R}_0$. Moreover, $N(t,\Lambda)$ defines, in a natural way, a Poisson random measure N on $\mathcal{B}(0,\infty) \times \mathcal{B}(\mathbb{R}_0)$ given by

$$(a,b] \times \Lambda \to N(b,\Lambda) - N(a,\Lambda), \quad \Lambda \in \mathcal{B}(\mathbb{R}_0),\ 0 < a \leq b \tag{5.4}$$

and its standard extension. The Lévy measure ν of X_t is defined by

$$\nu(\Lambda) := \mathbb{E}[N(1,\Lambda)], \quad \Lambda \in \mathcal{B}(\mathbb{R}_0). \tag{5.5}$$

The Lévy measure ν always satisfies

$$\int_{\mathbb{R}_0} \min(1, z^2)\nu(dz) < \infty, \tag{5.6}$$

but it is possible that

$$\int_{\mathbb{R}_0} \min(1, |\, z\,|)\nu(dz) = \infty. \tag{5.7}$$

This places a bound on the sizes of small jumps. The situation is of interest in financial modelling (see [71]).
The Law of X_t is infinitely divisible (see [80]) with characteristic function of the form

$$\mathbb{E}[\exp(iuX_t)] = (\phi(u))^t = e^{t\psi(u)} \tag{5.8}$$

where $\psi(u) = \log \phi(u)$ and $\phi(u)$ is the characteristic function of X_1. The function $\psi(u)$ is called the characteristic exponent and it satisfies the following Lévy-Khintchine formula (see [16]):

$$\psi(u) = iau - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz1_{\{|z|<1\}})\nu(dz) \tag{5.9}$$

where $a \in \mathbb{R}_0,\ \sigma^2 > 0$ and ν is a measure on $\mathbb{R}_0$.
We assume further that the Lévy measure ν satisfies the following condition: for each $\varepsilon > 0$ there exists $\lambda > 0$ such that

$$\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda \mid z \mid)\nu(dz) < \infty \tag{5.10}$$

where $(-\varepsilon, \varepsilon)^c$ denotes the complement of the interval $(-\varepsilon, \varepsilon)$. This condition implies that

$$\int_{-\infty}^{\infty} \mid z \mid^i \nu(dz) < \infty, \quad i \geq 2 \tag{5.11}$$

and that the characteristic function $\mathbb{E}[\exp(iuX_t)]$ is analytic in a neighborhood of 0. As a consequence, X_t has moments of all orders and the polynomials are dense in $L^2(\mathbb{R}_0, P \circ X_t^{-1})$ for all $t \geq 0$ (see [71]). The Lévy process X_t admits a decomposition

$$X_t = a_1 t + \sigma W_t + \int_0^t \int_{|z|<1} z\widetilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(ds, dz) \tag{5.12}$$

for some constants $a_1,\ \sigma \in \mathbb{R}_0$ where $W = \{W_t,\ t \geq 0\}$ is a standard Brownian motion, $\widetilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ is a compensated Poisson random measure of X_t and $N(dt, dz)$ is a Poisson random measure. The Brownian motion W_t is independent of the compensated Poisson random measure $\widetilde{N}(dt, dz)$.

We can write the representation in Equation (5.12) as

$$X_t = at + \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(dt, dz) \tag{5.13}$$

where $a = a_1 + \int_{|z|\geq 1} z\nu(dz)$. Lévy processes may be regarded as natural generalizations of the Brownian motion to discontinuous processes. Following the representation in Equation

(5.13) we consider the more general stochastic differential equation of the form

$$dX_t = \alpha(t)dt + \sigma(t)dW_t + \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(dt,dz) \tag{5.14}$$

where α, σ and γ are predictable processes satisfying

$$\int_0^t \{\mid \alpha(s) \mid + \sigma^2(s) + \int_{\mathbb{R}_0} \gamma^2(s,z)\nu(dz)\}ds < \infty \text{ P. a.s. for all } t \geq 0, z \in \mathbb{R}_0. \tag{5.15}$$

This is called an Itô-Lévy process.

Theorem 5.1.2 *(1-dimensional Itô formula [75])*
Let $X = X_t,\ t \geq 0$ be the Itô-Lévy process given by Equation (5.14) and let $f \in C^2(\mathbb{R}^2)$ and define

$$Y_t := f(t, X_t).$$

Then $Y = Y_t, t \geq 0$ is also an Itô-Lévy process and

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)[\alpha(t)dt + \sigma(t)dW_t] + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\sigma^2(t)dt \\ &+ \int_{\mathbb{R}_0} \{(f(t,X_t) + \gamma(t,z)) - f(t,X_t) - \frac{\partial f}{\partial x}(t,X_t)\gamma(t,z)\}\nu(dz)dt \\ &+ \int_{\mathbb{R}_0} \{f(t,X_t + \gamma(t,z)) - f(t,X_t)\}\widetilde{N}(dt,dz). \end{aligned} \tag{5.16}$$

Assumption
From now onwards we will assume that

$$X_t = \int_0^t \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(ds,dz), \quad 0 \leq t \leq T, \tag{5.17}$$

that is, $\alpha = \sigma = 0$ in Equation (5.14).

Lemma 5.1.3 *Fix $T > 0$. The set of random variables*

$$\{f(X_{t_1}, \cdots, X_{t_n});\ t_i \in [0,T] \quad i = 1, 2, \ldots, n\}, \tag{5.18}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is an infinitely many times continuously differentiable function with bounded derivatives, is dense in $L^2(\mathcal{F}_T, P)$.

Proof
The proof follows the same arguments as in [73]. We omit the details. □

Let $\psi = \psi(t,z,\omega)$ be predictable. Then, the following Itô isometry is valid:

$$\mathbb{E}\Big[\Big(\int_0^T\int_{\mathbb{R}_0}\psi(t,z)\widetilde{N}(dt,dz)\Big)^2\Big] = \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}_0}\psi(t,z)^2\nu(dz)dt\Big]. \tag{5.19}$$

We introduce a continuous function $\gamma : \mathbb{R}_0 \to (-1,0)\cup(0,1)$ given by

$$\gamma(z) = \begin{cases} e^z - 1 & \text{if } z < 0 \\ 1 - e^{-z} & \text{if } z > 0 \end{cases}$$

(see [64]) which is totally bounded and has an inverse $\gamma^{-1} : (-1,0)\cup(0,1) \to \mathbb{R}_0$. In addition, $\lim_{z\to 0}\frac{\gamma(z)}{z} = 1$, so γ approaches zero just as fast as z. $\rho((-\infty,1]\cup[1,\infty)) < \infty$ and since γ is totally bounded it follows that $\gamma \in L^2(\rho)$ and $e^{\lambda\gamma} - 1 \in L^2(\rho)$ for all $\lambda \in \mathbb{R}$. Hence, if $h \in C([0,T])$ then

$$e^{h\gamma} - 1 \in L^2(\nu), \quad h\gamma \in L^2(\nu) \;\text{ and }\; e^{h\lambda} - 1 - h\gamma \in L^1(\nu). \tag{5.20}$$

The function $\gamma(z)$ ensures that the exponential function $e^{\lambda X_t}$ belongs to $L^2(\mu)$.

Lemma 5.1.4 *The linear span of random variables of the type*

$$\exp\{\int_0^T\int_{\mathbb{R}_0}h(t)\gamma(z)\widetilde{N}(ds,dz) - \int_0^T\int_{\mathbb{R}_0}(e^{h(t)\gamma(z)} - 1 - h(t)\gamma(z))\nu(dz)dt\}, \tag{5.21}$$

where $h \in C([0,T])$*, is dense in* $L^2(\mu)$*.*

Proof

The proof follows the same arguments as in [63]. We omit the details. □

5.2 Chaos expansion

Here, we extend the Wiener-Itô chaos expansion for Brownian motion developed in the second half of Chapter 2 to pure jump Lévy processes (see [27]). However, in this case the corresponding iterated integrals are with respect to the compensated Poisson measure rather than the Lévy process itself. Let $\lambda(dt) = dt$ denotes the Lebesgue measure on $[0,T]$ and let $L^2((\lambda\times\nu)^n) = L^2([0,T]\times\mathbb{R}_0)^n$ be the space of all deterministic functions $f : ([0,T]\times\mathbb{R}_0)^n \to \mathbb{R}_0$ such that

$$\| f \|^2_{L^2((\lambda\times\nu)^n)} := \int_{([0,T]\times\mathbb{R}_0)^n} f^2(t_1,z_1,\ldots,t_n,z_n)dt_1\nu(dz_1)\cdots dt_n\nu(dz_n) < \infty. \tag{5.22}$$

If f is a real function on $([0,T]\times\mathbb{R}_0)^n$ we define its symmetrization $\hat{f}$ with respect to the variables $(t_1,z_1,\dots,t_n,z_n)$ by

$$\hat{f}(t_1,z_1,\dots,t_n,z_n)=\frac{1}{n!}\sum_{\sigma} f(t_{\sigma(1)},z_{\sigma(1)},\dots,t_{\sigma(n)},z_{\sigma(n)}) \tag{5.23}$$

where the sum is taken over all permutations σ of $(1,\dots,n)$. A function $f\in L^2((\lambda\times\nu)^n)$ is called symmetric if $f=\hat{f}$ and we denote by $\hat{L}^2((\lambda\times\nu)^n)$ the space of symmetric functions in $L^2((\lambda\times\nu)^n)$. We assume that the symmetric function $f\in\hat{L}^2(\lambda\times\nu)^n$ vanishes on the diagonal, that is,

$$f(t_1,z_1,\dots,t_n,z_n)=0 \ \text{ if } \ t_i=t_j \ \text{ and } \ z_i=z_j \ \text{ for some } \ i\neq j. \tag{5.24}$$

Define

$$G_n:=\{(t_1,z_1,\dots,t_n,z_n):0\le t_1\le\ldots\le T,\quad z_i\in\mathbb{R}_0 \ \text{ and } \ i=1,n\} \tag{5.25}$$

and let $L^2(G_n)$ be the set of functions $g:G_n\to\mathbb{R}_0$ such that

$$\| g \|^2_{L^2(G_n)}:=\int_{G_n} g^2(t_1,z_1,\dots,t_n,z_n)dt_1\nu(dz_1)\cdots dt_n\nu(dz_n)<\infty. \tag{5.26}$$

Definition 5.2.1 *For any $g\in L^2(G_n)$ the n-fold iterated Itô integral $J_n(g)$ is the random variable with respect to $\widetilde{N}(\cdot,\cdot)$ in $L^2(\mu)$ defined as*

$$J_n(g):=\int_0^T\int_{\mathbb{R}_0}\cdots\int_0^{t_2^-}\int_{\mathbb{R}_0} g(t_1,z_1,\dots,t_n,z_n)\widetilde{N}(dt_1,dz_1)\cdots\widetilde{N}(dt_n,dz_n). \tag{5.27}$$

We set $J_0(g)=g$ for any $g\in\mathbb{R}$.

Note that in each step the corresponding integrand is adapted because of the limits of the preceding integrals. For $f\in\hat{L}^2((\lambda\times\nu)^n)$ we have

$$\| f \|^2_{L^2((\lambda\times\nu)^n)}=n!\int_0^T\int_{\mathbb{R}_0}\cdots\int_0^{t_2^-}\int_{\mathbb{R}_0} f^2(t_1,z_1,\dots,t_n,z_n)dt_1\nu(dz_1)\cdots dt_n\nu(dz_n). \tag{5.28}$$

If $f\in\hat{L}^2((\lambda\times\nu)^n)$ we also define

$$I_n(f):=n!J_n(f). \tag{5.29}$$

For $f\in\hat{L}^2((\lambda\times\nu)^n)$ and $g\in\hat{L}^2((\lambda\times\nu)^m)$ we obtain the following orthogonality relation

$$\mathbb{E}[I_n(f)I_m(g)]=\begin{cases} 0 \quad \text{if} \quad n\neq m \\ n!(f,g)_{L^2((\lambda\times\nu)^n)} \ \text{ if } \ n=m \ \ m,n=1,2,\dots\end{cases}$$

where $(\cdot,\cdot)_{L^2((\lambda\times\nu)^n)}$ is the inner product in $L^2((\lambda\times\nu)^n)$. The relation follows by applying Itô isometry iteratively and due to the fact that the expected value of an Itô integral is zero. With the notation above the following chaos expansion in terms of iterated integrals with respect to $\widetilde{N}(dt,dz)$ holds.

Theorem 5.2.2 *Let $F \in L^2(\mu)$ be an $\mathcal{F}_T$-measurable random variable. Then there exists a unique sequence of functions $\{f_n\}_{n=0}^{\infty}$ where $f_n \in \hat{L}^2((\lambda\times\nu)^n),\ n \geq 1$ such that*

$$F = \sum_{n=0}^{\infty} I_n(f_n). \tag{5.30}$$

Moreover, the following Itô isometry is valid:

$$\| F \|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n! \| f_n \|^2_{L^2((\lambda\times\nu)^n)} . \tag{5.31}$$

Proof

The proof is similar the corresponding Brownian motion case given in [27] on page 11. We omit the details. □

Example

Choose $h \in \hat{L}^2(\lambda\times\nu)$ and let $F(\omega) = \int_0^\infty \int_{\mathbb{R}} h(s,z)\widetilde{N}(ds,dz)$. Then we have the following chaos expansion

$$F(\omega) = I_1(h).$$

Corollary 5.2.3 *Let $F = Y_T$ where*

$$Y_t = \exp\left(\int_0^t \int_{\mathbb{R}_0} h(s)\gamma(z)\widetilde{N}(ds,dz) - \int_0^t \int_{\mathbb{R}_0} (e^{h(s)\gamma(z)} - 1 - h(s)\gamma(z))\nu(dz)ds\right), \quad t \in [0,T] \tag{5.32}$$

with $h(s) \in L^2([0,T])$. Then

$$F = 1 + \sum_{n=1}^{\infty} I_n(\frac{1}{n!}(e^{h\gamma} - 1)^{\otimes n}). \tag{5.33}$$

Proof

Put

$$X_t = \int_0^t \int_{\mathbb{R}_0} h(s)\gamma(z)\widetilde{N}(ds,dz) - \int_0^t \int_{\mathbb{R}_0} (e^{h(s)\gamma(z)} - 1 - h(s)\gamma(z))\nu(dz)ds$$

so that

$$Y_t = e^{X_t}. \tag{5.34}$$

Applying the Itô formula (5.16) with

$$\alpha(t) = -\int_{\mathbb{R}_0} (e^{h(t)\gamma(z)} - 1 - h(t)\gamma(z))\nu(dz)dt, \quad \beta(t) = 0 \text{ and } \gamma(t, X_t, z) = h(t)\gamma(z)$$

we obtain

$$\begin{aligned} dY_t &= e^{X_t}[-\int_{\mathbb{R}_0} (e^{h(t)\gamma(z)} - 1 - h(t)\gamma(z))\nu(dz)dt] + \int_{\mathbb{R}_0} (e^{X_t+h(t)\gamma(z)} - e^{X_t} - e^{X_t}h(t)\gamma(z))\nu(dz)dt \\ &+ \int_{\mathbb{R}_0} (e^{X_t+h(t)\gamma(z)} - e^{X_t})\widetilde{N}(dt, dz) \\ &= \int_{\mathbb{R}_0} e^{X_t}(e^{h(t)\gamma(z)} - 1)\widetilde{N}(dt, dz) \\ &= \int_{\mathbb{R}_0} Y_t^-(e^{h(t)\gamma(z)} - 1)\widetilde{N}(dt, dz). \end{aligned}$$

Hence

$$Y_T = 1 + \int_0^T \int_{\mathbb{R}_0} Y_s^-(e^{h(s)\gamma(z)} - 1)\widetilde{N}(ds, dz).$$

Applying the Itô formula to Equation (5.34) for a second time and then integrate from 0 to T we obtain

$$\begin{aligned} Y_T &= 1 + \int_0^T \int_{\mathbb{R}_0} (1 + \int_0^{t_1^-} \int_{\mathbb{R}_0} Y_{t_2}(e^{h(t_2)\gamma(z_2)} - 1)\widetilde{N}(dt_2, dz_2))(e^{h(t_1)\gamma(z_1)} - 1)\widetilde{N}(dt_1, dz_1) \\ &= 1 + \int_0^T \int_{\mathbb{R}_0} (e^{h(t_1)\gamma(z_1)} - 1)\widetilde{N}(dt_1, dz_1) \\ &+ \int_0^T \int_{\mathbb{R}_0} \int_0^{t_1^-} \int_{\mathbb{R}_0} Y_{t_2^-}(e^{h(t_2)\gamma(z_2)} - 1)(e^{h(t_1)\gamma(z_1)} - 1)\widetilde{N}(dt_2, dz_2)\widetilde{N}(dt_1, dz_1). \end{aligned}$$

Applying the Itô formula to Equation (5.34) repeatedly and then integrate from 0 to T we obtain

$$Y_T = \sum_{n=0}^{N-1} I_n(f_n) + \int_0^T \int_{\mathbb{R}_0} \cdots \int_0^{t_2^-} \int_{\mathbb{R}_0} Y_{t_1^-} \prod_{i=1}^{N}(e^{h(t_i)\gamma(z_i)} - 1)\widetilde{N}(dt_1, dz_1) \cdots \widetilde{N}(dt_N, dz_N)$$

where

$$f_n(t_1, z_1, \ldots, t_n, z_n) = \frac{1}{n!}\prod_{i=1}^{n}(e^{h(t_i)\gamma(z_i)} - 1) = \frac{1}{n!}(e^{h(t)\gamma(z)} - 1)^{\otimes n}. \tag{5.35}$$

This leads to a chaos expansion

$$Y_T = \sum_{n=0}^{\infty} I_n(f_n).$$

Next we show that Y_T converges in $L^2(\mu)$. By the Itô isometry (5.19) we have

$$\mathbb{E}[Y_T^2] = 1 + \int_0^T \int_{\mathbb{R}_0} \cdots \int_0^{t_2^-} \int_{\mathbb{R}_0} \mathbb{E}[Y_{t_1^-}^2] \mid (e^{h(t)z} - 1)^{\otimes N} \mid^2 dt_1\nu(dz_1) \cdots dt_N\nu(dz_N)$$

which has a unique solution given by

$$\mathbb{E}[Y_T^2] = \exp\left(\int_0^T \int_{\mathbb{R}_0} (e^{h(t_1)z_1} - 1)^2 dt_1 \nu(dz_1)\right)^2.$$

Therefore

$$\| Y_T \|^2_{L^2(\mu)} = \exp(\| e^{h\gamma} - 1 \|^2_{L^2(\nu)}).$$

Since $e^{h\gamma} - 1 \in L^2(\nu)$ by (5.20) we conclude that $Y_T \in L^2(\mu)$ and the proof is complete. □

5.3 Skorohod integral

We recall the definition of the Skorohod integral in terms of Wiener-Itô chaos expansion (see [27]). Let $X = X(t,z),\ t \in [0,T],\ z \in \mathbb{R}_0$ be a stochastic process such that $X(t,z)$ is a $\mathcal{F}_T$-measurable random variable for all $(t,z) \in [0,T] \times \mathbb{R}_0$ and

$$\mathbb{E}[X(t,z)^2] < \infty, \quad (t,z) \in [0,T] \times \mathbb{R}_0. \tag{5.36}$$

Then, by Theorem 5.2.2, the random variable $X(t,z)$ has a chaos expansion of the form

$$X(t,z) = \sum_{n=0}^{\infty} I_n(f_n(t_1,z_1,\ldots,t_n,z_n;t,z)) \text{ for each } (t,z) \tag{5.37}$$

where $f_n(\ldots,t,z) \in \hat{L}^2((\lambda\times\nu)^n),\ n \geq 1$ and where $I_0(f_0) := \mathbb{E}[X(t,z)]$. Let $\hat{f}_n(t_1,z_1,\ldots,t_{n+1},z_{n+1})$ be the symmetrization of $f_n(t_1,z_1,\ldots,t_n,z_n;t,z)$ with respect to the $n+1$ pairs of variables $(t_1,z_1),\ldots,(t_{n+1},z_{n+1})$ with $t_{n+1} = t$ and $z_{n+1} = z$.

Definition 5.3.1 *Assume that*

$$\sum_{n=0}^{\infty}(n+1)! \| \hat{f}_n \|^2_{L^2((\lambda\times\nu)^{n+1})} < \infty. \tag{5.38}$$

Then the Skorohod integral of X with respect to $\widetilde{N}$, denoted by

$$\delta(X) = \int_0^T \int_{\mathbb{R}_0} X(t,z)\widetilde{N}(\delta t, dz),$$

is defined by

$$\delta(X) := \sum_{n=0}^{\infty} I_{n+1}(\hat{f}_n(t_1,z_1,\ldots,t_{n+1},z_{n+1})). \tag{5.39}$$

Condition (5.38) and Equation (5.39) imply that the Skorohod integral belongs to $L^2(\mu)$ and

$$\| \int_0^T \int_{\mathbb{R}_0} X(t,z)\widetilde{N}(\delta t, dz) \|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} (n+1)! \| \hat{f}_n \|^2_{L^2((\lambda\times\nu)^{n+1})} < \infty. \tag{5.40}$$

The following proposition says that if $X(t,z)$ is adapted, then the Skorohod integral coincides with the Itô integral.

Proposition 5.3.2 *Suppose $X(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}_0$ is a stochastic process such that*

$$\mathbb{E}[\int_0^T \int_{\mathbb{R}_0} X^2(t,z)\nu(dz)dt] < \infty. \tag{5.41}$$

Then the Skorohod integral and the Itô integral coincide in $L^2(\mu)$, that is,

$$\int_o^T \int_{\mathbb{R}_0} X(t,z)\widetilde{N}(\delta t, dz) = \int_0^T \int_{\mathbb{R}_0} X(t,z)\widetilde{N}(dt,dz). \tag{5.42}$$

Proof

The proof follows the same arguments as in [27] on page 23. We omit the details. □

Corollary 5.3.3 *Assume the conditions in Proposition 5.3.2 hold. Then*

$$\mathbb{E}[\int_0^T \int_{\mathbb{R}_0} X(t,z)\widetilde{N}(\delta t, dz)] = 0. \tag{5.43}$$

Proof

$$\begin{aligned}
\mathbb{E}[\int_0^T \int_{\mathbb{R}_0} X(t,z)\widetilde{N}(\delta t, dz)] &= \mathbb{E}[\int_0^T \int_{\mathbb{R}_0} X(t,z)\widetilde{N}(dt, dz)] \quad \text{by} \quad (5.42) \\
&= \mathbb{E}[\int_0^T \int_{\mathbb{R}_0} X(t,z)(N(dt,dz) - \nu(dz)dt)] \\
&= \mathbb{E}[\int_0^T \int_{\mathbb{R}_0} X(t,z)N(dt,dz)] - \mathbb{E}[\int_0^T \int_{\mathbb{R}_0} X(t,z)\nu(dz)dt] \\
&= T\int_{\mathbb{R}_0} X(t,z)\nu(dz) - T\int_{\mathbb{R}_0} X(t,z)\nu(dz) = 0. \qquad \square
\end{aligned}$$

5.4 Stochastic derivative

The stochastic derivative $D_{t,z}$ can be defined in several ways. Here we recall the definition of the stochastic derivative by means of a chaos expansion (see [27] page 176).
We introduce the set $\mathbb{D}_{1,2} \subset L^2(\mu)$ defined by

$$\mathbb{D}_{1,2} := \{F = \sum_{n=0}^{\infty} I_n(f_n) : \sum_{n=1}^{\infty} nn! \parallel f_n \parallel^2_{L^2((\lambda\times\nu)^n)} < \infty\}. \tag{5.44}$$

Definition 5.4.1 *The stochastic derivative* $D_{t,z} : \mathbb{D}_{1,2} \to L^2(\lambda \times \nu \times \mu)$ *is defined by*

$$D_{t,z}F := \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t, z)) \tag{5.45}$$

where F *is an* $\mathcal{F}_T$*-measurable random variable of the form* $F = \sum_{n=0}^{\infty} I_n(f_n)$ *with* $f_n(\cdot, t, z) = f_n(t_1, z_1, \ldots, t_{n-1}, z_{n-1}, t, z)$ *and* $I_{n-1}(f_n(\cdot, t, z))$ *means that the* $(n-1)$*-fold iterated integral of* f_n *is regarded as a function of its* $(n-1)$ *first pairs of variables* $(t_1, z_1), \ldots, (t_{n-1}, z_{n-1})$ *while the final pair* (t, z) *is kept as a parameter.*

$D_{t,z}F$ is well-defined since

$$\begin{aligned}
\parallel D_{t,z}F \parallel^2_{L^2(\lambda\times\nu\times\mu)} &= \int_0^T \int_{\mathbb{R}_0} \parallel \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t, z)) \parallel^2_{L^2(\mu)} \nu(dz)dt \\
&= \int_0^T \int_{\mathbb{R}_0} \sum_{n=1}^{\infty} n^2(n-1)! \parallel f_n(\cdot, t, z) \parallel^2_{L^2((\lambda\times\nu)^{n-1})} \nu(dz)dt \\
&= \sum_{n=1}^{\infty} nn! \parallel f_n \parallel^2_{L^2((\lambda\times\nu)^n)} \\
&< \infty.
\end{aligned} \tag{5.46}$$

This implies that $D_{t,z}F \in L^2(\lambda \times \nu \times \mu)$ if $F \in \mathbb{D}_{1,2}$ which shows that $\mathbb{D}_{1,2}$ is the domain of $D_{t,z}$. We can write Equation (5.45) as

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot))f(t, z), \qquad F \in \mathbb{D}_{1,2}. \tag{5.47}$$

This suggests that the operator $D_{t,z}$ satisfy a chain rule. The form is appealing because it has some resemblance to the derivative of a monomial:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Thus, it is natural to call $D_{t,z}F$ the stochastic derivative of F at (t, z).

Example

For

$$F = \int_0^T \int_{\mathbb{R}_0} f(t,z)\widetilde{N}(dt,dz), \quad f \in L^2(\lambda \times \nu)$$

we have seen that $F = I_1(f)$ from the previous example. Then

$$D_{t,z}F = I_0(f(\cdot,t,z)) = f(t,z).$$

In particular, if $F = \int_0^T \int_{\mathbb{R}_0} z\widetilde{N}(dt,dz)$ we have

$$D_{t,z}F = z.$$

Example

For $\eta(t) = \int_0^T \int_{\mathbb{R}_0} z\widetilde{N}(ds,dz)$, let $F = \eta^2(T)$.

Define

$$Y_t = \eta^2(t) = \left(\int_0^T \int_{\mathbb{R}_0} z\widetilde{N}(ds,dz)\right)^2.$$

An application of the Itô formula (5.16) with $\alpha(,t,X_t) = \beta(t,X_t) = 0, \quad \gamma(t,X_t,z) = z$ and $f = \eta^2(t)$ gives

$$\begin{aligned} dY_t &= d\eta^2(t) = \int_{\mathbb{R}_0} [(\eta(t)+z)^2 - \eta^2(t) - 2\eta(t)z]\nu(dz)dt + \int_{\mathbb{R}_0} [(\eta(t)+z)^2 - \eta^2(t)]\widetilde{N}(dt,dz) \\ &= \int_{\mathbb{R}_0} z^2\nu(dz)dt + \int_{\mathbb{R}_0} [2\eta(t)z + z^2]\widetilde{N}(dt,dz). \end{aligned}$$

Hence

$$\eta^2(T) = T\int_{\mathbb{R}_0} z^2\nu(dz) + \int_0^T \int_{\mathbb{R}_0} [2\eta(t)z + z^2]\widetilde{N}(dt,dz).$$

Therefore

$$D_{t,z}F = D_{t,z}\eta^2(T) = 2\eta(t)z + z^2.$$

Example

By Corollary 5.2.3 we have

$$F = Y_T = 1 + \sum_{n=1}^{\infty} I_n\left(\frac{1}{n!}(e^{h\gamma}-1)^{\otimes n}\right).$$

Then

$$\begin{aligned} D_{t,z}F &= \sum_{n=1}^{\infty} nI_{n-1}\left(\frac{1}{n!}(e^{h\gamma}-1)^{\otimes(n-1)}(e^{h\gamma}-1)\right) = \sum_{n=1}^{\infty} \frac{n}{n!}(e^{h(t)\gamma(z)}-1)I_{n-1}\left((e^{h\gamma}-1)^{\otimes(n-1)}\right) \\ &= (e^{h(t)\gamma(z)}-1)\sum_{n=1}^{\infty} \frac{1}{(n-1)!}I_{n-1}((e^{h\gamma}-1)^{\otimes(n-1)}) = \left(e^{h(t)\gamma(z)}-1\right)F. \end{aligned}$$

The following theorem is taken from [27] on page 177. We include the proof here for easy reading.

Theorem 5.4.2 *Assume* $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\mu)$. *Let* $\{F_k\}_{k=1}^{\infty}$ *be a sequence with* $F_k = \sum_{n=0}^{\infty} I_n(f_n^{(k)}) \in \mathbb{D}_{1,2}, \quad k = 1, 2, \ldots$ *such that* $F_k \to F, \quad k \to \infty$ *in* $L^2(\mu)$ *and that* $D_{t,z}F_k,\ k = 1, 2, \ldots$*converges in* $L^2(\lambda \times \nu \times \mu)$. *Then* $F \in \mathbb{D}_{1,2}$ *and*

$$D_{t,z}F = \lim_{k\to\infty} D_{t,z}F_k$$

in $L^2(\lambda \times \nu \times \mu)$.

Proof

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} \| f_n^{(k)} - f_n \|_{L^2((\lambda\times\nu)^n)} = 0 \ \text{ in } \ L^2((\lambda \times \nu)^n) \ \text{ for all } \ n = 0, 1, \ldots$$

Since $F_k \in \mathbb{D}_{1,2}$ and $D_{t,k}F_k$ converges, then $D_{t,k}F_k$ is a Cauchy sequence in L^2 and therefore from calculations in (5.46) it follows that

$$\mathbb{E}[\int_0^T \int_{\mathbb{R}_0} (D_{t,z}F_k - D_{t,z}F_j)^2 \nu(dz)dt] = \sum_{n=0}^{\infty} nn! \| f_n^{(k)} - f_n^{(j)} \|^2_{L^2(\lambda\times\nu\times\mu)} \to 0 \quad k, j \to \infty.$$

Thus, by the Fatou lemma, we have

$$\begin{aligned}
\lim_{k\to\infty} \sum_{n=0}^{\infty} nn! \| f_n^{(k)} - f_n \|^2_{L^2((\lambda\times\nu)^n)} &= \lim_{k\to\infty} \left(\sum_{n=0}^{\infty} \lim_{j\to\infty} nn! \| f_n^{(k)} - f_n^{(j)} \|^2_{L^2((\lambda\times\nu)^n)} \right) \\
&\leq \lim_{k\to\infty} \left(\liminf_{j\to\infty} \sum_{n=0}^{\infty} nn! \| f_n^{(k)} - f_n^{(j)} \|^2_{L^2((\lambda\times\nu)^n)} \right) \\
&= 0
\end{aligned}$$

which implies that $F \in \mathbb{D}_{1,2}$ and

$$D_{t,z}F_k \to D_{t,z}F, \quad k \to \infty \ \text{ in } \ L^2(\lambda \times \nu \times \mu).$$ □

The stochastic derivative $D_{t,z}$ satisfies the following "product rule" (see [29]).

Lemma 5.4.3 *Let* $F, G \in \mathbb{D}_{1,2}$ *with* G *bounded. Then* $FG \in \mathbb{D}_{1,2}$ *and*

$$D_{t,z}(FG) = FD_{t,z}G + GD_{t,z}F + D_{t,z}FD_{t,z}G \quad \lambda \times \nu \ \ a.e. \tag{5.48}$$

Proof

With the help of Theorem 5.4.2 the result is satisfied for F and G of the form $f(X_{t_1}, \cdots, X_{t_k})$ and $g(X_{t_1}, \cdots, X_{t_k})$ respectively where f and g are differentiable functions with compact support. Then, by using a limit argument the proof follows from the closedness of $D_{t,z}$. □

Let $G = F$ in Equation (5.48) so that we have

$$D_{t,z}(F^2) = 2FD_{t,z}F + D_{t,z}FD_{t,z}F = (F + D_{t,z}F)^2 - F^2.$$

By induction it follows that if $F \in \mathbb{D}_{1,2}$ then we have

$$D_{t,z}(F^n) = (F + D_{t,z}F)^n - F^n. \tag{5.49}$$

The following result is the "chain rule" which is useful in the evaluation of stochastic derivatives.

Theorem 5.4.4 *Let $F \in \mathbb{D}_{1,2}$ and let φ be a real continuous function on $\mathbb{R}_0$. Suppose $\varphi(F) \in L^2(\mu)$ and $\varphi(F + D_{t,z}F) \in L^2(\lambda \times \nu \times \mu)$. Then $\varphi(F) \in \mathbb{D}_{1,2}$ and*

$$D_{t,z}\varphi(F) = \varphi(F + D_{t,z}F) - \varphi(F) \tag{5.50}$$

Proof

We follow the proof in [27] on page 178. We first assume that φ has compact support and $F \in \mathbb{D}_{1,2}$. Then φ has the inverse Fourier transform of its Fourier transform $\hat{\varphi}$:

$$\varphi(F) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_0} e^{iyF} \hat{\varphi}(y) dy$$

where

$$\hat{\varphi}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_0} e^{-ixy} \varphi(x) dx.$$

By Equation (5.49) and Theorem 5.4.2 we have

$$\begin{aligned}
D_{t,z}\varphi(F) &= D_{t,z}\left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_0} \sum_{n=0}^{\infty} \frac{1}{n!}(iy)^n (F^n) \hat{\varphi}(y) dy\right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_0} \sum_{n=0}^{\infty} \frac{1}{n!}(iy)^n ((F + D_{t,z}F)^n - F^n) \hat{\varphi}(y) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_0} (e^{iy(F+D_{t,z}F)} - e^{iyF}) \hat{\varphi}(y) dy \\
&= \varphi(F + D_{t,z}F) - \varphi(F).
\end{aligned}$$

Hence the result holds in this case. To prove the general case we proceed by approximation. We choose $F_n \in \mathbb{D}_{1,2},\ n = 1, 2\ldots$ such that $F_n \to F,\ n \to \infty$ in $\mathbb{D}_{1,2}$. Then $\varphi(F_n) \to \varphi(F)$ in $L^2(\mu)$ by the dominated convergence theorem and

$$\varphi(F_n + D_{t,z}F_n) - \varphi(F_n) \to \varphi(F + D_{t,z}F) - \varphi(F)$$

in $\mathbb{D}_{1,2}$. Hence the result holds for all $F \in \mathbb{D}_{1,2}$ in the case of φ with compact support. The extension to the case when $\varphi(F) \in L^2(\mu)$ and $\varphi(F + D_{t,z}F) \in L^2(\lambda \times \nu \times \mu)$ follows by a limit argument using the closedness of $D_{t,z}$. □

Examples

1. For $\varphi(F) = \ln(F)$ where $F = 1 + \theta(t, z)$ with $\theta(t, z) \in \mathbb{D}_{1,2}$ we have

$$\begin{aligned} D_{t,z}\varphi(F) &= D_{t,z}\ln(1 + \theta(s, z)) \\ &= \ln(1 + \theta(t, z) + D_{t,z}(1 + \theta(s, z))) - \ln(1 + \theta(t, z)) \\ &= \ln(1 + \theta(t, z) + D_{t,z}\theta(s, z)) - \ln(1 + \theta(t, z)) \\ &= \ln\left(1 + \frac{D_{t,z}\theta(s, z)}{1 + \theta(t, z)}\right). \end{aligned}$$

2. Let $\eta(T) = \int_0^T \int_{\mathbb{R}} z\widetilde{N}(ds, dz)$ and $F = (\eta(T) - K)^+$. Then, we have

$$D_{t,z}F = (\eta(T) + D_{t,z}\eta(T) - K)^+ - (\eta(T) - K)^+ = (\eta(T) + z - K)^+ - (\eta(T) - K)^+.$$

The next theorem gives the relationship between the stochastic derivative and the Skorohod integral (see [27] page 180).

Theorem 5.4.5 *Let $X(t, z),\ t \in [0, T],\ \ z \in \mathbb{R}_0$ be Skorohod integrable and $F \in \mathbb{D}_{1,2}$. Then*

$$\mathbb{E}[\int_0^T \int_{\mathbb{R}_0} X(t, z)D_{t,z}F\nu(dz)dt] = \mathbb{E}[F \int_0^T \int_{\mathbb{R}_0} X(t, z)\widetilde{N}(\delta t, dz)]. \tag{5.51}$$

Proof

The proof is similar to the proof of the corresponding Brownian motion given in [27] on page 34. We omit the details. □

The following corollary gives the closability of the Skorohod integral (see [27] page 181).

Corollary 5.4.6 *Suppose that $X_n(t, z), t \in [0, T], z \in \mathbb{R}_0$ is a sequence of Skorohod integrable stochastic processes and that*

$$\delta(X_n) := \int_0^T \int_{\mathbb{R}_0} X_n(t, z)\widetilde{N}(\delta t, dz), \quad n = 1, 2, .. \tag{5.52}$$

converges in $L^2(\mu)$. *In addition if*

$$\lim_{n\to\infty} X_n(t,z) = 0 \quad in \quad L^2(\lambda\times\nu\times\mu) \tag{5.53}$$

then we have

$$\lim_{n\to\infty} \delta(X_n) = 0 \qquad in \quad L^2(\mu). \tag{5.54}$$

Proof

By Theorem 5.4.5 and assumption in Equation (5.53) we have that

$$\mathbb{E}[\int_0^T\int_{\mathbb{R}_0} X_n(t,z)D_{t,z}F\nu(dz)dt] = \mathbb{E}[F\int_0^T\int_{\mathbb{R}_0} X_n(t,z)\widetilde{N}(dt,dz)] \to 0 \quad \text{as} \quad n\to\infty$$

for all $F\in\mathbb{D}_{1,2}$. Then we have $\lim_{n\to\infty}\delta(X_n)=0$ weakly in $L^2(\mu)$. The result follows because the sequence $\delta(X_n),\ n=1,2,\dots$ is convergent in $L^2(\mu)$. □

If $X_n,\ n=1,2,\dots$ is a sequence of Skorohod integrable random variable such that

$$X(t,z) = \lim_{n\to\infty} X_n(t,z) \ \text{ in } \ L^2(\lambda\times\nu\times\mu)$$

then the Skorohod integrable of $X(t,z)$ can be defined as

$$\delta(X) := \int_0^T\int_{\mathbb{R}_0} X(t,z)\widetilde{N}(\delta t,dz) = \lim_{n\to\infty}\int_{\mathbb{R}_0} X_n(t,z)\widetilde{N}(\delta t,dz) := \lim_{n\to\infty}\delta(X_n) \tag{5.55}$$

provided the limit exists in $L^2(\mu)$. The following theorem is the fundamental theorem of calculus (see [27] page 181).

Theorem 5.4.7 *Let* $X = X(s,y),\quad (s,y)\in[0,T]\times\mathbb{R}_0$ *be a stochastic process such that*

$$\mathbb{E}[\int_0^T\int_{\mathbb{R}_0} X^2(s,y)\nu(dy)ds] < \infty. \tag{5.56}$$

Assume that $X(s,y)\in\mathbb{D}_{1,2}$ *for all* $(s,y)\in[0,T]\times\mathbb{R}_0$ *and that* $D_{t,z}X(\cdot,\cdot)$ *is Skorohod integrable with*

$$\mathbb{E}[\int_0^T\int_{\mathbb{R}_0}\left(\int_0^T\int_{\mathbb{R}_0} D_{t,z}X(s,y)\widetilde{N}(\delta s,dy)\right)^2\nu(dz)dt] < \infty. \tag{5.57}$$

Then $\int_0^T\int_{\mathbb{R}_0} X(s,y)\widetilde{N}(\delta s,dy)\in\mathbb{D}_{1,2}$ *and*

$$D_{t,z}\int_0^T\int_{\mathbb{R}_0} X(s,y)\widetilde{N}(\delta s,dy) = \int_0^T\int_{\mathbb{R}_0} D_{t,z}X(s,y)\widetilde{N}(\delta s,dy) + X(t,z). \tag{5.58}$$

Proof

The proof can be found in [27] on page 182. We omit the details. □

Example

Let

$$F = \int_0^T \int_{\mathbb{R}_0} \ln(1 + \theta(s, z))\widetilde{N}(\delta s, dz).$$

Then

$$\begin{aligned}
D_{t,z}F &= D_{t,z}\left(\int_0^T \int_{\mathbb{R}_0} \ln(1 + \theta(s, z))\widetilde{N}(\delta s, dz)\right) \\
&= \int_0^T \int_{\mathbb{R}_0} D_{t,z}(\ln(1 + \theta(s, z)))\widetilde{N}(\delta s, dz) + \ln(1 + \theta(t, z)) \\
&= \int_0^T \int_{\mathbb{R}_0} (\ln(1 + \theta(s, z) + D_{t,z}(1 + \theta(s, z))) - \ln(1 + \theta(s, z)))\widetilde{N}(\delta s, dz) + \ln(1 + \theta(t, z)) \\
&= \int_0^T \int_{\mathbb{R}_0} \ln\left(1 + \frac{D_{t,z}\theta(s, z)}{1 + \theta(s, z)}\right)\widetilde{N}(\delta s, dz) + \ln(1 + \theta(t, z)).
\end{aligned}$$

Theorem 5.4.8 *Let $X \in L^2(\lambda \times \nu \times \mu)$ and $D_{t,z}X \in L^2(\lambda \times \nu \times \mu)$. Then the Itô isometry holds*

$$\begin{aligned}
\mathbb{E}[\left(\int_0^T \int_{\mathbb{R}_0} X(t, z)\widetilde{N}(\delta t, dz)\right)^2] &= \mathbb{E}[\int_0^T \int_{\mathbb{R}_0} X^2(t, z)\nu(dz)dt] \\
&+ \mathbb{E}[\int_0^T \int_{\mathbb{R}_0} \int_0^T \int_{\mathbb{R}_0} D_{t,z}X(s, y)D_{s,y}X(t, z)\nu(dy)ds\nu(dz)dt].
\end{aligned}$$

Proof

The proof is given in [29]. We omit the details. □

5.5 Differentiability of pure jump Lévy stochastic differential equation

We consider the following pure jump Lévy stochastic differential equation

$$X_t = x + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X_{s^-}, z)\widetilde{N}(ds, dz), \quad X_0 = x \in \mathbb{R} \tag{5.59}$$

where $\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is a compensated Poisson random measure. The function $\gamma : [0, T] \times \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuously differentiable with bounded derivatives and satisfies the linear growth condition:

$$\int_{\mathbb{R}_0} \mid \gamma(t,x,z) \mid^2 \nu(dz) \leq C(1+ \mid x \mid^2), \quad 0 \leq t \leq T, \quad x \in \mathbb{R} \tag{5.60}$$

for a constant $C < \infty$. In addition, we assume that the function γ satisfies the Lipschitz condition:

$$\int_{\mathbb{R}_0} \mid \gamma(t,x,z) - \gamma(t,y,z) \mid^2 \nu(dz) \leq K \mid x-y \mid^2, \quad 0 \leq t \leq T, \quad x,y \in \mathbb{R}, \tag{5.61}$$

for a constant $K < \infty$. The conditions (5.60) and (5.61) ensure the existence of a unique solution $\{X_t,\ 0 \leq t \leq T\}$ to the stochastic differential equation (5.59). That is, there exists a unique cádlág adapted solution $\{X_t,\ 0 \leq t \leq T\}$, such that

$$\mathbb{E}[\sup_{0 \leq t \leq T} \mid X_t \mid^2] < \infty.$$

The following theorem says that the solution to the stochastic differential equation (5.59) is Malliavin differentiable, that is, the random variable X_t belongs to $\mathbb{D}_{1,2}$ (see [27] page 310).

Theorem 5.5.1 *Suppose the conditions (5.60) and (5.61) hold. Then there exists a unique strong solution $\{X_t,\ 0 \leq t \leq T\}$ to the stochastic differential equation (5.59) such that $X_t \in \mathbb{D}_{1,2}$ for all $0 \leq t \leq T$,*

$$\sup_{0 \leq r \leq t} \mathbb{E}[\sup_{r \leq s \leq T} \mid D_{r,z} X_s \mid^2] < \infty \tag{5.62}$$

and the stochastic derivative $D_{r,z}X_t$ follows a linear equation

$$D_{r,z}X_t = \int_r^t \int_{\mathbb{R}_0} (\gamma(s, X_s + D_{r,z}X_s, z) - \gamma(s, X_s, z)) \widetilde{N}(ds,dz) + \gamma(r, X_r, z) \tag{5.63}$$

for $r \leq t$ a.e and $D_{r,z}X_t = 0$ for $r > t$ a.e.

Proof

We consider the Picard approximation $X_n(t),\ n \geq 0$ to X_t given by

$$X_{n+1}(t) = x + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X_n(s^-), z) \widetilde{N}(ds,dz), \qquad X_0(t) = x. \tag{5.64}$$

We first prove, by induction on n, that

$$X_n(t) \in \mathbb{D}_{1,2} \quad \text{for all} \quad 0 \leq t \leq T \tag{5.65}$$

and that

$$\psi_{n+1}(t) \leq k_1 + k_2 \int_0^t \psi_n(s) ds \tag{5.66}$$

for all $0 \le t \le T,\ n \ge 0$ where k_1 and k_2 are constants and

$$\psi_n(t) := \sup_{0\le r\le t} \mathbb{E}[\int_{\mathbb{R}_0} \sup_{r\le s\le t} \mid D_{r,z}X_n(s) \mid^2 \nu(dz)] < \infty.$$

It can be shown that conditions (5.65) and (5.66) hold for $n = 0$ since

$$D_{t,z} \int_0^T \int_{\mathbb{R}_0} \gamma(s,x,\zeta)\widetilde{N}(ds,d\zeta) = \gamma(t,x,z).$$

Suppose that it holds for all n. Then, the closability of the stochastic derivative $D_{t,z}$ and Theorem 5.4.4 imply that

$$D_{r,z}\gamma(t,X_n(t^-),z) = \gamma(t,X_n(t^-) + D_{r,z}X_n(t^-),z) - \gamma(t,X_n(t^-),z)$$

for $r \le t$ a.e and ν- a.e. Using Theorem 5.4.7 we deduce that the Itô integral $\int_0^t \int_{\mathbb{R}_0} \gamma(s,X_n(s^-),z)\widetilde{N}(ds,dz)$ belongs to $\mathbb{D}_{1,2}$ and that for $r \le t$ we have

$$\begin{aligned} D_{r,z}X_{n+1}(t) &= \int_0^t \int_{\mathbb{R}_0} D_{r,z}\gamma(s,X_n(s^-),\zeta)\widetilde{N}(ds,d\zeta) + \gamma(r,X_n(r^-),z) \\ &= \int_0^t \int_{\mathbb{R}_0} (\gamma(s,X_n(s^-)+D_{r,z}X_n(s^-),\zeta) - \gamma(s,X_n(s^-),\zeta))\widetilde{N}(ds,d\zeta) + \gamma(r,X_n(r^-),z) \\ &= \int_r^t \int_{\mathbb{R}_0} (\gamma(s,X_n(s^-)+D_{r,z}X_n(s^-),\zeta) - \gamma(s,X_n(s^-),\zeta))\widetilde{N}(ds,d\zeta) + \gamma(r,X_n(r^-),z) \end{aligned}$$

for $r \le t$ a.e and ν- a.e. Then, by Doob maximal inequality, Fubini Theorem, Itô isometry, (5.60) and (5.61) we get

$$\begin{aligned} \mathbb{E}[\int_{\mathbb{R}_0} \sup_{r\le s\le t} (D_{r,z}X_{n+1}(s))^2\nu(dz)] &\le 8K\int_r^t \mathbb{E}[\int_{\mathbb{R}_0} \mid D_{r,z}X_n(u^-) \mid^2 \nu(dz)]du \\ &+ 2C(1+\mathbb{E}[\mid X_n(r^-) \mid^2]) \\ &\le 8K\int_r^t \mathbb{E}[\int_{\mathbb{R}_0} \mid D_{r,z}X_n(u^-) \mid^2 \nu(dz)]du \\ &+ 2C(1+\lambda) \end{aligned} \tag{5.67}$$

for all $0 \le r \le t$ where C is a constant and

$$\lambda := \sup_{n\ge 0} \mathbb{E}[\sup_{0\le s\le T} \mid X_n(s) \mid^2] < \infty.$$

Applying a discrete version of Gronwall's inequality to (5.67) we get

$$\sup_{n\ge 0} \mathbb{E}[\int_0^T \int_{\mathbb{R}_0} \mid D_{s,z}X_n(t) \mid^2 \nu(dz)ds] < \infty$$

for all $0 \leq t \leq T$. Thus, the inequality (5.67) shows that the conditions (5.65) and (5.66) hold for $n+1$. Moreover, we note that

$$\mathbb{E}[\sup_{0\leq s\leq T} \mid X_n(s) - X(s) \mid^2] \to 0$$

as n goes to infinity by Picard approximation. Hence, by Theorem 5.4.2, we conclude that X_t belongs to $\mathbb{D}_{1,2}$.
Finally, applying the operator $D_{t,z}$ to Equation (5.59) and using Theorem (5.4.4) and Theorem (5.4.7) we obtain Equation (5.63). □

We will use the following assumption (see [48]).

Assumption 2 *Let the function $\gamma(t, X_t, z)$ be of the form*

$$\gamma(t, X_t, z) = g(t, z)X_t \tag{5.68}$$

for some deterministic function g.

Assumption 2 implies, for any adapted random variable Y, the following

1. $\gamma(t, X + Y, z) = \gamma(t, X, z) + \gamma(t, Y, z)$.
2. $\gamma(t, XY, z) = Y\gamma(t, X, z)$.

Under Assumption 2 we have

$$\begin{aligned} D_{r,z}X_t &= \int_r^t \int_{\mathbb{R}_0} (g(s,z)(X_s + D_{r,z}X_s) - g(s,z)X_s)\widetilde{N}(ds,dz) + g(r,z)X(r) \\ &= \int_r^t \int_{\mathbb{R}_0} g(s,z)D_{r,z}X_s\widetilde{N}(ds,dz) + g(r,z)X(r) \end{aligned}$$

The following proposition gives a representation for $D_{t,z}X_t$ under Assumption 2.

Proposition 5.5.2 *Let γ be as in Assumption 2 and let the first variation process Y_t of X_t satisfies the following equation*

$$Y_t = Y_0 + \int_0^t \int_{\mathbb{R}_0} g(s,z)Y_s\widetilde{N}(ds,dz) \tag{5.69}$$

with $Y_0 = 1$. Then the stochastic derivative of the process X_t in Equation (5.59) is given by

$$D_{r,z}X_t = Y_tY_r^{-1}\gamma(r, X_r, z)1_{\{r\leq t\}}. \tag{5.70}$$

Proof

Using Assumption 2 we have

$$X_t = x + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X_s, z)\widetilde{N}(ds, dz) = x + \int_0^t \int_{\mathbb{R}_0} g(s, z) X_s \widetilde{N}(ds, dz).$$

Taking the partial derivative with respect to x on both sides we have

$$\frac{\partial}{\partial x} X_t = I + \int_0^t \int_{\mathbb{R}_0} g(s, z) \frac{\partial X_s}{\partial x} \widetilde{N}(ds, dz). \tag{5.71}$$

Setting $Y_t = \frac{\partial}{\partial x} X_t$ we get

$$Y_t = I + \int_0^t \int_{\mathbb{R}_0} g(s, z) Y_s \widetilde{N}(ds, dz).$$

That is,

$$dY_t = \int_{\mathbb{R}_0} g(t, z) Y_t \widetilde{N}(dt, dz) \quad Y_0 = I. \tag{5.72}$$

Using the Itô formula the solution to Equation (5.72) is given by

$$Y_t = \exp\{\int_0^t \int_{\mathbb{R}_0} [\log(1+g(s, z)) - g(s, z)]\nu(dz)ds + \int_0^t \int_{\mathbb{R}_0} \log(1+g(s, z))\widetilde{N}(ds, dz)\} \tag{5.73}$$

On the other hand, an application of Theorem 5.4.7 to the stochastic differential equation (5.59) yields

$$D_{r,z} X_t = \int_r^t \int_{\mathbb{R}_0} (\gamma(s, X_s + D_{r,z} X_s, z) - \gamma(s, X_s, z))\widetilde{N}(ds, dz) + \gamma(r, X_r, z)$$

Under Assumption 2 we have

$$\begin{aligned} D_{r,z} X_t &= \int_r^t \int_{\mathbb{R}_0} (g(s, z)(X_s + D_{r,z} X_s) - g(s, z) X_s)\widetilde{N}(ds, dz) + \gamma(r, X_r, z) \\ &= \int_r^t \int_{\mathbb{R}_0} g(s, z) D_{r,z} X_s \widetilde{N}(ds, dz) + \gamma(r, X_r, z). \end{aligned}$$

Fix r, z and set $Z_t := D_{r,z} X_t$. We have

$$Z_t = \int_r^t \int_{\mathbb{R}_0} g(s, z) Z_s \widetilde{N}(ds, dz) + \gamma(r, X_r, z).$$

That is,

$$dZ_t = \int_{\mathbb{R}_0} g(t, z) Z_t \widetilde{N}(dt, dz) \tag{5.74}$$

with initial condition $Z_r = \gamma(r, X_r, z)$. Using the Itô formula the solution to Equation (5.74) is given by

$$\begin{aligned} Z_t &= \gamma(r, X_r, z) \exp\{\int_r^t \int_{\mathbb{R}_0} [\log(1 + g(s,z)) - g(s,z)]\nu(dz)ds \\ &+ \int_r^t \int_{\mathbb{R}_0} \log(1 + g(s,z))\widetilde{N}(ds,dz)\} \end{aligned} \tag{5.75}$$

Matching Equation (5.73) with Equation (5.75) yields

$$D_{r,z}X_t = Y_t Y_r^{-1} \gamma(r, X_r, z) 1_{\{r \leq t\}}.$$

□

Proposition 5.5.3 *The first variation process Y_t is invertible a.s. Further the inverse Y_t^{-1} satisfies a.s*

$$\begin{aligned} Y_t^{-1} &= I - \int_0^t \int_{\mathbb{R}_0} Y_s^{-1}(I + \gamma'(s, X_{s^-}, z))^{-1} \gamma'(s, X_{s^-}, z)^2 \nu(dz)ds \\ &- \int_0^t \int_{\mathbb{R}_0} Y_s^{-1}(I + \gamma'(s, X_{s^-}, z))^{-1} \gamma'(s, X_{s^-}, z) \widetilde{N}(ds, dz) \end{aligned} \tag{5.76}$$

where $\gamma'(\cdot,\cdot,\cdot)$ denotes the derivative of $\gamma(\cdot,\cdot,\cdot)$ with respect to the second argument.

Proof

Since X_t satisfies the stochastic differential equation (5.59) its first variation process satisfies

$$Y_t = I + \int_0^t \int_{\mathbb{R}_0} \gamma'(s, X_{s^-}, z) Y_s \widetilde{N}(ds, dz).$$

Now we consider the linear stochastic differential equation for unknown matrix valued Z_t:

$$\begin{aligned} Z_t &= I - \int_0^t \int_{\mathbb{R}_0} Z_s(I + \gamma'(s, X_{s^-}, z))^{-1} \gamma'(s, X_{s^-}, z)^2 \nu(dz)ds \\ &- \int_0^t \int_{\mathbb{R}_0} Z_s(I + \gamma'(s, X_{s^-}, z))^{-1} \gamma'(s, X_{s^-}, z) \widetilde{N}(ds, dz). \end{aligned}$$

It has a unique solution Z_t. It can be shown that the product $Z_t Y_t$ satisfies

$$d(Z_t Y_t) = 0.$$

Therefore

$$Z_t Y_t = I$$

holds a.s, proving that Y_t is invertible, that is, $Z_t = Y_t^{-1}$ and the inverse Y_t^{-1} satisfies (5.76).

□

5.6 The necessary and sufficient condition for a function to serve as a weighting function

In this section we use the results discussed in the preceding sections. Greeks can be expressed as (see [35])

$$\text{Greeks} = \mathbb{E}[e^{-rT}\Phi(X_T)\pi \mid X_0 = x]$$

where π is a weight function and $\Phi(X_T)$ is the payoff function which is square integrable. The weight function could be expressed in the form of a Skorohod integral. Here we examine the set of functions expressed as a Skorohod integral and determine which conditions these functions should satisfy to serve as a weighting function. We mention that similar conditions were given in [7] in the Brownian motion case. We consider the market model whose dynamics is given by Equation (5.59).

The following result shows the necessary and sufficient conditions to be satisfied by a function to serve as a weight function of Δ in the pure jump case. This is an extension of the work of [7] to pure jump cases.

Theorem 5.6.1 *Let X_t be a stochastic process of the form (5.59). Let $\Phi : \mathbb{R} \to \mathbb{R}$ be of polynomial growth and let $u \in L^2([0,T]\times\Omega)$. Then a necessary and sufficient conditions for a function $u(\cdot,\cdot)$ to serve as a weight function of Δ is that it satisfy the following*

1. *$u(\cdot,\cdot)$ is Skorohod integrable.*

2.

$$\begin{aligned} & \mathbb{E}[e^{-rT}\Phi'(X_T)Y_T \mid X_T] \\ = \; & \mathbb{E}[e^{-rT}\int_0^T\int_{\mathbb{R}_0} u(t,z)(\Phi(X_T + D_{t,z}X_T) - \Phi(X_T))\nu(dz)dt \mid X_T]. \end{aligned} \quad (5.77)$$

where $D_{t,z}X_T = Y_T Y_t^{-1}\gamma(t, X_t, z)1_{\{t<T\}}(t)$ and $Y_T = \frac{\partial}{\partial x}X_T$.

Proof

Necessary condition: First, assume that the payoff function Φ is continuously differentiable with bounded derivatives. Then

$$\begin{aligned}\Delta &= \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X_T) \mid X_0 = x] \\ &= \mathbb{E}[e^{-rT}\Phi'(X_T)\frac{\partial}{\partial x}X_T \mid X_0 = x] \\ &= \mathbb{E}[e^{-rT}\Phi'(X_T)Y_T \mid X_0 = x] \qquad (5.78)\end{aligned}$$

where the interchange of the derivative and the expectation operator is justified by the dominated convergence theorem.

We want to write $\mathbb{E}[e^{-rT}\Phi'(X_T)Y_T \mid X_0 = x]$ as $\mathbb{E}[e^{-rT}\Phi(X_T)\delta(u) \mid X_0 = x]$ where $\delta(u)$ is the Skorohod integral of a certain $u \in L^2([0,T] \times \Omega)$, that is,

$$\Delta = \mathbb{E}[e^{-rT}\Phi(X_T)\delta(u) \mid X_0 = x].$$

Using the notation in Definition 5.3.1 we have

$$\begin{aligned}\Delta &= \mathbb{E}[e^{-rT}\Phi(X_T)\int_0^T\int_{\mathbb{R}_0} u(t,z)\widetilde{N}(\delta s, dz) \mid X_0 = x] \\ &= \mathbb{E}[e^{-rT}\int_0^T\int_{\mathbb{R}_0} u(t,z)D_{t,z}(\Phi(X_T))\nu(dz)dt \mid X_0 = x] \quad \text{by} \quad (5.51) \\ &= \mathbb{E}[e^{-rT}\int_0^T\int_{\mathbb{R}_0} u(t,z)[\Phi(X_T + D_{t,z}X_T) - \Phi(X_T)]\nu(dz)dt \mid X_0 = x] \qquad (5.79)\end{aligned}$$

where we have used Equation (5.50) in the last equality.

So $u(t,z)$ should satisfy the following equation

$$\begin{aligned}&\mathbb{E}[e^{-rT}\Phi'(X_T)Y_T \mid X_0 = x] \\ = &\ \mathbb{E}[e^{-rT}\int_0^T\int_{\mathbb{R}_0} u(t,z)[\Phi(X_T + D_{t,z}X_T) - \Phi(X_T)]\nu(dz)dt \mid X_0 = x]. \qquad (5.80)\end{aligned}$$

Using the fact that Equation (5.80) should hold for any continuously differentiable function Φ, we get that the following equality holds on any function measurable, leading to conditions expressed with conditional expectations:

$$\begin{aligned}&\mathbb{E}[e^{-rT}\Phi'(X_T)Y_T \mid X_T] \\ = &\ \mathbb{E}[e^{-rT}\int_0^T\int_{\mathbb{R}_0} u(t,z)(\Phi(X_T + D_{t,z}X_T) - \Phi(X_T))\nu(dz)dt \mid X_T] \qquad (5.81)\end{aligned}$$

and this is Equation (5.77).

Sufficient condition: We assume that the function $u \in L^2([0,T]\times\Omega)$ satisfies Equation (5.81) and its Skorohod integral. Then the proof can be conducted backwards. □

Next we give the necessary and sufficient conditions for a function to serve as a weight function for $\mathcal{V}$. We interpret $\mathcal{V}$ as the Vega for pure jump Lévy stochastic differential equation (5.59). We first define a jump-perturbed process $\{X_t^\varepsilon, 0 \le t \le T\}$, for small $\epsilon > 0$, as the solution of a perturbed stochastic differential equation in the direction of $\bar{\gamma}$:

$$dX_t^\varepsilon = \int_{\mathbb{R}_0} (\gamma(s, X_s^\varepsilon, z) + \varepsilon\bar{\gamma}(s, X_s^\varepsilon, z))\widetilde{N}(ds, dz) \tag{5.82}$$

where $\bar{\gamma}$ is a continuously differentiable function with bounded derivatives. Writing Equation (5.82) in integral form we have

$$X_t^\varepsilon = x + \int_0^t \int_{\mathbb{R}_0} (\gamma(s, X_s^\varepsilon, z) + \varepsilon\bar{\gamma}(s, X_s^\varepsilon, z))\widetilde{N}(ds, dz).$$

The corresponding option price (under the risk-neutral probability measure) is given by

$$u^\varepsilon(x) = \mathbb{E}[e^{-rT}\Phi(X_T^\varepsilon) \mid X_0^\varepsilon = x]. \tag{5.83}$$

We define $\mathcal{V}$ as follows.

Definition 5.6.2

$$\mathcal{V} := \frac{\partial}{\partial\varepsilon} u^\varepsilon(x) \mid_{\varepsilon=0} . \tag{5.84}$$

The variation process Z_t^ε of X_t^ε satisfies the following equation

$$Z_t^\varepsilon = \int_0^t \int_{\mathbb{R}_0} (\gamma'(s, X_s^\varepsilon, z) + \varepsilon\bar{\gamma}'(s, X_s^\varepsilon, z))Z_s^\varepsilon \widetilde{N}(ds, dz). \tag{5.85}$$

Note that $Z_t^\varepsilon := \frac{\partial X_t^\varepsilon}{\partial\varepsilon}$.

The following theorem gives the necessary and sufficient conditions for a function to serve as a weight function for $\mathcal{V}$.

Theorem 5.6.3 *Let X_t be a stochastic process of the form (5.59). Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be of polynomial growth and let $u \in L^2([0,T]\times\Omega)$. Then a necessary and sufficient conditions for a function $u(\cdot,\cdot)$ to serve as a weight function of $\mathcal{V}$ is that it satisfy the following*

1. *$u(\cdot,\cdot)$ is Skorohod integrable.*

2.

$$\begin{aligned}&\mathbb{E}[e^{-rT}\Phi'(X_T^\epsilon)Z_T^\epsilon \mid X_T^\epsilon]\\ =\;& \mathbb{E}[e^{-rT}\int_0^T\int_{\mathbb{R}_0} u(t,z)(\Phi(X_T^\epsilon + D_{t,z}X_T^\epsilon) - \Phi(X_T^\epsilon))\nu(dz)dt \mid X_T^\epsilon] \qquad (5.86)\end{aligned}$$

where $Z_t^\epsilon = \frac{\partial}{\partial\epsilon}X_t^\epsilon$ *and* $D_{t,z}X_T^\varepsilon = Y_T^\varepsilon(Y_t^\varepsilon)^{-1}\gamma(t, X_t^\varepsilon, z)1_{\{t<T\}}(t)$.

Proof

Necessary condition: As in the proof of Theorem 5.6.1, we first assume that the payoff function Φ is continuously differentiable with bounded derivatives. Then we have

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}u^\varepsilon(x) &= \frac{\partial}{\partial\varepsilon}\mathbb{E}[e^{-rT}\Phi(X_T^\varepsilon) \mid X_0^\varepsilon = x]\\ &= \mathbb{E}[e^{-rT}\Phi'(X_T^\varepsilon)\frac{\partial}{\partial\varepsilon}X_T^\varepsilon \mid X_0^\varepsilon = x]\\ &= \mathbb{E}[e^{-rT}\Phi'(X_T^\varepsilon)Z_T^\varepsilon \mid X_0^\varepsilon = x] \qquad (5.87)\end{aligned}$$

where the interchange of the derivative operator and the expectation operator is justified by the dominated convergence theorem.

We want to write $\mathbb{E}[e^{-rT}\Phi'(X_T^\varepsilon)Z_T^\varepsilon \mid X_0^\varepsilon = x]$ as $\mathbb{E}[e^{-rT}\Phi(X_T^\varepsilon)\delta(u) \mid X_0^\varepsilon = x]$ where $\delta(u)$ is the Skorohod integral of a certain $u \in L^2([0,T]\times\Omega)$, that is,

$$\frac{\partial}{\partial\varepsilon}u^\varepsilon(x) = \mathbb{E}[e^{-rT}\Phi(X_T^\varepsilon)\delta(u) \mid X_0^\varepsilon = x].$$

Using the notation in Definition 5.3.1 we have

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}u^\varepsilon(x) &= \mathbb{E}[e^{-rT}\Phi(X_T^\varepsilon)\int_0^T\int_{\mathbb{R}_0} u(t,z)\widetilde{N}(\delta s, dz) \mid X_0^\varepsilon = x]\\ &= \mathbb{E}[e^{-rT}\Phi(X_T^\varepsilon)\int_0^T\int_{\mathbb{R}_0} u(t,z)\widetilde{N}(ds, dz) \mid X_0^\varepsilon = x] \text{ by (5.42)}\\ &= \mathbb{E}[e^{-rT}\int_0^T\int_{\mathbb{R}_0} u(t,z)D_{t,z}(\Phi(X_T^\varepsilon))\nu(dz)dt \mid X_0^\varepsilon = x] \text{ by (5.51)}\\ &= \mathbb{E}[e^{-rT}\int_0^T\int_{\mathbb{R}_0} u(t,z)[\Phi(X_T^\varepsilon + D_{t,z}X_T^\varepsilon) - \Phi(X_T^\varepsilon)]\nu(dz)dt \mid X_0^\varepsilon = x] \quad (5.88)\end{aligned}$$

where we have used Equation (5.50) in the last equality.

So $u(t,z)$ should satisfy the following equation

$$\begin{aligned}&\mathbb{E}[e^{-rT}\Phi'(X_T^\varepsilon)Z_T^\varepsilon \mid X_0^\varepsilon = x]\\ =\;& \mathbb{E}[e^{-rT}\int_0^T\int_{\mathbb{R}_0} u(t,z)[\Phi(X_T^\varepsilon + D_{t,z}X_T^\varepsilon) - \Phi(X_T^\varepsilon)]\nu(dz)dt \mid X_0^\varepsilon = x]. \qquad (5.89)\end{aligned}$$

Using the fact that Equation (5.89) should hold for any continuously differentiable function Φ, we get that the following equality holds on any function measurable, leading to conditions expressed with conditional expectations:

$$\begin{aligned} & \mathbb{E}[e^{-rT}\Phi'(X_T^\varepsilon)Z_T^\varepsilon \mid X_T^\varepsilon] \\ = & \mathbb{E}[e^{-rT}\int_0^T\int_{\mathbb{R}_0} u(t,z)(\Phi(X_T^\varepsilon + D_{t,z}X_T^\varepsilon) - \Phi(X_T^\varepsilon))\nu(dz)dt \mid X_T^\varepsilon] \end{aligned} \tag{5.90}$$

and this is Equation (5.86).

Sufficient condition: We assume that the function $u \in L^2([0,T]\times\Omega)$ satisfies Equation (5.90) and its Skorohod integral. Then the proof can be conducted backwards. □

Remark

As in the pure Brownian motion case (see [7]) the necessary and sufficient conditions for a function to serve as a weight function are different for each Greek.

Chapter 6

Calculations of Greeks for Jump Diffusion Processes

In this chapter we compute Greeks of processes driven by both continuous process and jump process. We mention that there are several papers that have considered the computation of Greeks of processes driven by both continuous process and jump process (see [25],[26], [32], [34], [76] and the references therein). In particular, Davis and Johansson [25] calculate Greeks of models driven by a Brownian motion and a Poisson process with deterministic jump sizes. El-Khatib and Privault [32] consider a market driven by jumps alone. The authors defined a Malliavin calculus on the Poisson space and were able to calculate Greeks of processes having Poisson jump times with random jump sizes but imposing a regularity condition on the payoff function. We note that the papers mentioned above have advantages for specific applications. We extend the results in [25] to more general cases.

The main difficulty is in establishing a chain rule that is valid for both the continuous part and the jump part since the stochastic derivative for the jump part is a difference operator. We, however, are able to circumvent the difficulty by working on the Wiener-Poisson space on which a chain rule has been defined (see [62]). Using the chain rule, Davis and Johansson [25] obtain Greeks for jump diffusion models which satisfy a separability condition. We review the generalized chain rule following the work in [76] and use it to compute Greeks for different models. In particular, we compute Δ for the Heston model with jumps. This is new.

In the last section we use a slightly different approach. We are inspired by ideas developed in [4] and [13]. In [4] the authors give approximation of a Lévy process by a Brownian motion while in [13] the authors use the likelihood approach to compute Greeks for approximation of Lévy processes. We use the Malliavin calculus approach. We first approximate the small jumps for the Lévy process by a Brownian motion (see [4]). Then we calculate the corresponding Greeks by using the chain rule and the integration by parts formula and then use limit arguments to obtain the Greeks for the original Lévy process. The approach is applicable to more cases as it can be applied to random variables whose density function is not explicitly known.

6.1 Basic elements of a Lévy chaotic calculus

Let $X = \{X_t, 0 \le t \le T\}$ be a real-valued Lévy process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We assume, as before, that we are using a càdlàg version. As in Chapter 5, we assume that the Lévy measure ν of X_t satisfies the condition (5.10), that is,

$$\int_{(-\varepsilon,\varepsilon)} e^{\lambda|z|}\nu(dz) < \infty. \tag{6.1}$$

We will consider a Lévy process X_t of the form

$$X_t = \sigma W(t) + \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz), \quad 0 \le t \le T \tag{6.2}$$

where $\{W_t, 0 \le t \le T\}$ is a standard Brownian motion, $\widetilde{N}(ds, dz)$ is the compensated Poisson random measure associated with X_t and

$$\int_{\mathbb{R}_0} z^2\nu(dz)dt < \infty. \tag{6.3}$$

We perform the following transformation to the Lévy process X_t and this will play an important role in the analysis (see [30] and [71]). We set

$$X_t^{(i)} = \sum_{0<s\le t} (\Delta X_s)^{(i)}, \quad i = 2, 3, \ldots \tag{6.4}$$

For convenience we put

$$X_t^{(1)} = X_t, \quad 0 \le t \le T. \tag{6.5}$$

The processes $X^{(i)} = \{X_t^{(i)}, 0 \le t \le T\}$, $i = 1, 2, \ldots$ are again Lévy processes and are called power jump processes which jump at the same points as the original Lévy process X_t.
We have

$$\mathbb{E}[X_t] = \mathbb{E}[X_t^{(1)}] = tm_1 < \infty \tag{6.6}$$

where $m_1 = \mathbb{E}[X_1]$ and

$$E[X_t^{(i)}] = E[\sum_{0<s\leq t} (\Delta X_s)^{(i)}] = t \int_{\mathbb{R}_0} z^i \nu(dz) := m_i t < \infty, \qquad i \geq 2 \tag{6.7}$$

where $m_i = \int_{\mathbb{R}_0} z^i \nu(dz)$ (see [79] page 29). We define the compensated power jump processes $\{Y_t^{(i)}, 0 \leq t \leq T\}$ of order i as follows

$$Y_t^{(i)} := X_t^{(i)} - E[X_t^{(i)}] = X_t^{(i)} - m_i t, \quad i = 1, 2, \ldots \tag{6.8}$$

Example

$$Y_t^{(1)} = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z \widetilde{N}(ds, dz) \quad \text{and} \quad Y_t^{(i)} = \int_0^t \int_{\mathbb{R}_0} z^i \widetilde{N}(ds, dz), \quad i \geq 2.$$

It turns out that the processes $Y^{(i)} = \{Y_t^{(i)}, 0 \leq t \leq T\}$ are martingales. An important question is the orthogonalization of the set $\{Y^{(i)}, i = 1, 2, \ldots\}$ of martingales. We denote by $\mathcal{M}^2$ the space of square integrable martingales M such that

$$\sup_t \mathbb{E}[M_t^2] < \infty \quad \text{and} \quad M_0 = 0 \ \text{ a.s.} \tag{6.9}$$

We note that if $M \in \mathcal{M}^2$ then

$$\lim_{t\to\infty} \mathbb{E}[M_t^2] = \mathbb{E}[M_\infty^2] < \infty \quad \text{and} \quad M_t = \mathbb{E}[M_\infty \mid \mathcal{F}_t]. \tag{6.10}$$

Thus, each $M \in \mathcal{M}^2$ can be identified with the terminal value M_∞. Let $N \in \mathcal{M}^2$ be another martingale. Two martingales are said to be orthogonal if $\mathbb{E}[N_\infty M_\infty] = 0$. We give a stronger notion of orthogonality for martingales in $\mathcal{M}^2$ (see [79] page 179).

Definition 6.1.1 *Two martingales M and N are said to be strongly orthogonal if the product $L = MN$ is a (uniformly integrable) martingale.*

We note that if N and M are strongly orthogonal then $\mathbb{E}[N_\infty M_\infty] = \mathbb{E}[L_\infty] = \mathbb{E}[L_0] = 0$, so strong orthogonality implies orthogonality. The converse is not true (see [79] page 179).

Definition 6.1.2 *The predictable quadratic covariation process of $Y^{(i)}$ and $Y^{(j)}$ denoted by $\langle Y^{(i)}, Y^{(j)} \rangle_t$ is defined by*

$$\begin{aligned} \langle Y^{(i)}, Y^{(j)} \rangle_t &= E\Big[\Big(\int_0^t \int_{\mathbb{R}_0} z^i \widetilde{N}(ds, dz)\Big) \Big(\int_0^t \int_{\mathbb{R}_0} z^j \widetilde{N}(ds, dz)\Big)\Big] \\ &= \int_0^t \int_{\mathbb{R}_0} z^{i+j} \nu(dz) ds = m_{i+j} t, \quad i, j \geq 2 \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\nu)$.

Definition 6.1.3 *Let $Y^1, Y^2 \in \mathbb{R}^n$ be two real-valued Lévy type stochastic integrals. The quadratic covariation of Y^1 and Y^2 denoted by $[Y^1, Y^2]_t$ is defined as*

$$Y^1(t) \cdot Y^2(t) = Y^1(0) \cdot Y^2(0) + \int_0^t Y^1(s^-) dY^2(s) + \int_0^t Y^2(s^-) dY^1(s) + [Y^1, Y^2]_t. \quad (6.11)$$

Example

For an Itô-Lévy process of the form

$$Y_t^{(i)} = \sigma_i W_t + \int_0^t \int_{\mathbb{R}_0} z^i \widetilde{N}(ds, dz), \quad i \geq 1, \quad (6.12)$$

the quadratic covariation process of $Y^{(i)}$ and $Y^{(j)}$ using Definition 6.1.3 is given by

$$\begin{aligned} [Y^{(i)}, Y^{(j)}]_t &= \int_0^t \sigma_i \sigma_j ds + \int_0^t \int_{\mathbb{R}_0} z^i z^j \nu(dz) ds + \int_0^t \int_{\mathbb{R}_0} z^i z^j \widetilde{N}(ds, dz) \\ &= \sigma_i \sigma_j t + t \int_{\mathbb{R}_0} z^{i+j} \nu(dz) + \int_0^t \int_{\mathbb{R}_0} z^{i+j} \widetilde{N}(dz) ds \\ &= 1_{i=j=1} \sigma^2 t + m_{i+j} t + X_t^{(i+j)}, \quad i, j \geq 1 \end{aligned}$$

where σ is a parameter corresponding to the Gaussian part of the Lévy process X_t.

We seek for a set of pairwise strongly orthogonal martingales $\{H^{(i)},\ i = 1, 2, \ldots,\}$ such that each $H^{(i)}$ is a linear combination of the $Y^{(j)},\ j = 1, 2, \ldots, i$ with leading coefficient equal to 1. We set

$$H^{(i)} = Y^{(i)} + a_{i,j-1} Y^{(i-1)} + \cdots + a_{i,1} Y^{(1)} = \sum_{j=1}^{i} a_{i,j} Y_t^{(j)}, \quad i \geq 1 \quad (6.13)$$

where the constants $a_{i,j}$ are chosen in such a way that $a_{i,i} = 1$ and $H^{(i)}$ are martingales. Moreover, $H^{(i)},\ i = 1, 2, \ldots$ are strongly orthogonal martingales. It is shown in [71] that the coefficients $a_{i,j}$ correspond to the coefficients of the orthonormalization of the polynomials $\{z^n,\ n \geq 0\}$ with respect to the measure $\mu(dz) = z^2 \nu(dz) + \sigma^2 \delta_0(dz)$ where δ_0 is the Dirac measure at point 0. In particular, the polynomials $p_n(z)$ defined by

$$p_n(z) = \sum_{j=1}^{n} a_{n,j} z^{j-1} \quad (6.14)$$

are orthogonal with respect to the measure $d\mu$, that is,

$$\int_{\mathbb{R}_0} p_n(z) p_m(z) d\mu(z) = 0, \quad n \neq m. \quad (6.15)$$

Furthermore, the processes $\{H_t^{(i)},\ 0 \le t \le T\}$ are martingales with predictable quadratic variation process given by

$$\langle H^{(i)}, H^{(i)} \rangle_t = q_i t \tag{6.16}$$

where

$$q_i = a_{i,1}^2 \sigma^2 + \sum_{j,j'=1,\dots,i} a_{i,j} a_{i,j'} m_{j+j'}. \tag{6.17}$$

Using the Definition 6.1.3 we deduce that the quadratic covariation of $H^{(i)}$ and $H^{(j)}$ is given by

$$[H^{(i)}, H^{(j)}]_t = a_{i,1} a_{j,1} \sigma^2 t + a_{i,k} a_{j,k'} m_{k+k'} t + \sum_{k=1}^{i} \sum_{k'=1}^{j} a_{i,k} a_{j,k'} X^{(k+k')}(t). \tag{6.18}$$

The $H_t^{(i)}$ are called the *orthogonal power jump processes* of a Lévy process X_t. We can now state the result in [71] on the chaos expansion of a random variable in terms of iterated integral with respect to $H^{(i)}$.

Theorem 6.1.4 *Every random variable $F \in L^2(P)$ can be represented in the form*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{0 \le i_1,\dots,i_n \le m} \int_0^{\infty} \int_0^{t_m} \cdots \int_0^{t_2} f_{i_1,\dots,i_n}(t_1,\dots,t_n) dH_{t_1}^{(i_1)} \cdots dH_{t_n}^{(i_n)} \tag{6.19}$$

for any deterministic function $f_{i_1,\dots,i_n} \in L^2([0,T]^n)$.

An immediate consequence of Theorem 6.1.4 is the following corollary (see [71]).

Corollary 6.1.5 *Every $F \in L^2(P)$ can be written as*

$$F = \mathbb{E}[F] + \sum_{n \ge 1} \int_0^{\infty} \varphi^{(n)}(t) dH_t^{(n)} \tag{6.20}$$

where $\varphi^{(n)}, n = 1, 2, \dots$ are predictable processes.

6.2 Chaos expansion

In order to define a stochastic integral with respect to Lévy processes we use iterated integrals (instead of multiple integrals that is used in the case of Gaussian processes or Poisson processes) because of the chaotic representation given in [71] which consists of $H^{(i)}, i = 1, 2, \dots$ as integrators. We will adopt the following notation

$$U_0 = [0, T] \quad \text{and} \quad U_1 = [0, T] \times \mathbb{R}_0. \tag{6.21}$$

$$u_k^l = \begin{cases} t \ \text{ if } \ l=0 \\ (t,z) \ \text{ if } \ l=1. \end{cases}$$

$$H_0(du) = dW_t \quad \text{and} \quad H_1(du) = \widetilde{N}(dt,dz).$$

$$\langle H_0\rangle(du) = dt \quad \text{and} \quad \langle H_1\rangle(du) = \nu(dz)dt.$$

For $i_1, \ldots, i_n = 0, 1$, we let

$$G_{i_1,\ldots,i_n} = \{(u_1^{i_1}, \ldots, u_n^{i_n}) \in \prod_{k=1}^{n} U_{i_k} : 0 < t_1 < t_2 < \ldots < t_n < T, \ k = 1, \ldots, n\} \tag{6.22}$$

be a positive simplex of $\mathbb{R}^n$.

Definition 6.2.1 *We define the (n-fold) iterated integrals $L_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n})$ of $f_{i_1,\ldots,i_n} \in L^2(G_{i_1,\ldots,i_n})$ with respect to $H_{i_1}, \ldots, H_{i_n}$ by*

$$L_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}) := \int_{G_{i_1,\ldots,i_n}} f_{i_1,\ldots,i_n}(u_1^{i_1}, \ldots, u_n^{i_n}) H_{i_1}(du_1^{i_1}) \cdots H_{i_n} d(u_n^{i_n}). \tag{6.23}$$

The integrals are well-defined since all the processes $H_j, \ j = 1, \ldots, n$ are martingales with respect to the filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$. Using the Itô isometry it can be shown that $L_n^{i_1,\ldots,i_n}$ and $L_m^{j_1,\ldots,j_m}$ are orthogonal whenever $n = m$ and $(i_1, \ldots, i_n) \neq (j_1, \ldots, j_m)$. We now give a chaotic representation property of the Lévy process which can be considered as a reformulation of the result in [71].

Proposition 6.2.2 *Let $F \in L^2(\Omega)$. Then there exists a unique sequence $\{f_{i_1,\ldots,i_n}\}_{n=0}^{\infty}$, $i_1, \ldots, i_n = 0, 1$, where $f_{i_1,\ldots,i_n} \in L^2(G_{i_1,\ldots,i_n})$ such that*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=0,1} L_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}) \tag{6.24}$$

where $f_{i_1,\ldots,i_n}$ are deterministic functions in $L^2(G_{i_1,\ldots,i_n})$. We also have the Itô isometry:

$$\| F \|_{L^2(P)}^2 = [\ | \ F \ |]^2 + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=0,1} \| L_n^{i_1,\ldots,i_n}(f_{i_1,\ldots,i_n}) \|_{L^2(G_{i_1,\ldots,i_n})}^2 . \tag{6.25}$$

6.2.1 Directional derivative

In this subsection we present the properties of the directional derivative in the context of calculus of variations (see [76]).

Definition 6.2.3 *Let $f_{i_1,\ldots,i_n} \in L^2(G_{i_1,\ldots,i_n})$. Then we define the derivative of $L_n^{i_1,\ldots,i_n}(f_{i_1,\ldots,i_n})$ in the l^{th}-direction by*

$$D_{u^l}^{(l)} L_n^{i_1,\ldots,i_n}(f_{i_1,\ldots,i_n}) := \sum_{k=1}^{n} 1_{\{i_k=l\}} L_{n-1}^{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n}(f_{i_1,\ldots,i_n}(\ldots,u^l,\ldots) 1_{G^{(k)}_{i_1,\ldots,i_n}(t)}) \tag{6.26}$$

where u^l appears in the k^{th} position, $l = 0, 1$ and

$$\begin{aligned} G^{(k)}_{i_1,\ldots,i_n}(t) &= \{(u_1^{i_1},\ldots,u_{k-1}^{i_{k-1}},u_{k+1}^{i_{k+1}},\ldots,u_n^{i_n}) \in G_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n} : \\ &\quad 0 < t_1 < \ldots < t_{k-1} < t < t_{k+1} < \ldots < t_n < T\}. \end{aligned} \tag{6.27}$$

We note that, for $k \neq m$, we have $G^{(k)}_{i_1,\ldots,i_n}(t) \cap G^{(m)}_{i_1,\ldots,i_n}(t) = \varnothing$. We define the space of random variables that are differentiable in the l^{th}-direction and its respective derivative in the following definitions (see [62]).

Definition 6.2.4 *We say that F is differentiable in the l^{th}-direction ($l = 0, 1$) if $F \in \mathbb{D}^{(l)}_{1,2}$ where*

$$\begin{aligned} \mathbb{D}^{(l)}_{1,2} &:= \{F \in L^2(\Omega), F = \mathbb{E}[F] + \sum_{n=1} \sum_{i_1,\ldots,i_n=0,1} L_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}) : \\ &\quad \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=0,1} \sum_{k=1}^{n} 1_{\{i_k=l\}} q_{i_1} \cdots q_{i_{k-1}} q_{i_{k+1}} \cdots q_{i_n} \\ &\times \int_{G_{i_1,\ldots,i_n}} \| f_{i_1,\ldots,i_n}(\ldots,u^l,\ldots) 1_{\{G^{(k)}_{i_1,\ldots,i_n}(t)\}} \|^2_{L^2(G^k_{i_1,\ldots,i_n})} \langle H_{i_k} \rangle (du^l) < \infty\} \end{aligned}$$

and $q_0 = 1$.

Definition 6.2.5 *For $F \in \mathbb{D}^{(l)}_{1,2}$ such that*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=0,1} L_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}) \tag{6.28}$$

we define the derivative of F in the l^{th}-direction ($l = 0, 1$) as the element of $L^2(\Omega \times [0, T])$ given by

$$D_{u^l}^{(l)} F = \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=0,1} \sum_{k=1}^{n} 1_{\{i_k=l\}} L_{n-1}^{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n)}(f_{i_1,\ldots,i_n}(\ldots,u^l,\ldots) 1_{\{G^{(k)}_{i_1,\ldots,i_n}(t)\}}). \tag{6.29}$$

where the convergence of series is in $L^2(\Omega \times [0, T])$ and u^l is appearing in the k^{th} position.

We note that $1_{\{i_k=l\}}$ in the formula (6.29) indicates the direction in which we are taking our derivative from and $1_{\{G^{(k)}_{i_1,\ldots,i_n}(t)\}}$ indicates the interval in which the integral is defined. We observe that the derivative reduces the order of integration by 1. This is also the case for Malliavin derivative for Brownian motion. Here, we are not multiplying by a factor n because we are dealing with non-symmetric functions.
We also note that Definition 6.2.4 ensures that the stochastic derivative $D^{(l)}_{u^l}F$ belongs to $L^2(\Omega \times [0,T])$. We observe, as in the standard situation for Gaussian processes, that $\mathbb{D}^{(l)}_{1,2}$ is dense in $L^2(\Omega)$ since every $F \in L^2(\Omega)$ of the form (6.24) is in $\mathbb{D}^{(l)}_{1,2}$. It can be shown that if we remove the Poisson processes, the space $\mathbb{D}^{(0)}_{1,2}$ coincides with the standard Gaussian space $\mathbb{D}_{1,2}$.

We can define the derivative in the directions W_t and $N_t - \lambda t$ through iterated integrals using Definition 6.2.5. We denote the respective derivatives by $D^{(0)}_t$ and $D^{(1)}_{t,z}$. $D^{(0)}_t$ is the derivative with respect to the Wiener direction and $D^{(1)}_{t,z}$ is the derivative with respect to the Poisson random measure direction. Thus, for the Wiener direction, we have

$$D^{(0)}_t F = \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=0,1} \sum_{k=1}^{n} 1_{\{i_k=0\}} L^{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n)}_{n-1}(f_{i_1,\ldots,i_n}(\ldots,t,\ldots)1_{\{G^{(k)}_{i_1,\ldots,i_n}(t)\}}) \tag{6.30}$$

where t is a parameter occurring in the k^{th} position and for the Poisson random measure direction we have

$$D^{(1)}_{t,z} F = \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=0,1} \sum_{k=1}^{n} 1_{\{i_k=1\}} L^{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n)}_{n-1}(f_{i_1,\ldots,i_n}(\ldots,t,z,\ldots)1_{\{G^{(k)}_{i_1,\ldots,i_n}(t)\}}) \tag{6.31}$$

where (t,z) is kept as a parameter occurring in the k^{th} position.

The integrand φ^n in Corollary 6.1.5 can be given explicitly in terms of the stochastic derivative.

Theorem 6.2.6 *Let $F \in \mathbb{D}_{1,2}$. Then*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \int_0^T \mathbb{E}[D^{(l)}_{u^l}F \mid \mathcal{F}_{t^-}]dH^{(n)}_t. \tag{6.32}$$

In particular, we have the following theorem.

Theorem 6.2.7 *Let $F \in L^2(\Omega)$. Then*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D^{(0)}_t F \mid \mathcal{F}_{t^-}]dW_t + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D^{(1)}_{t,z}F \mid \mathcal{F}_{t^-}]d\widetilde{N}(ds,dz). \tag{6.33}$$

This is the Clark-Haussman-Ocone formula.

Proof

We have

$$\begin{aligned}
&\int_0^T \mathbb{E}[D_t^{(0)}F \mid \mathcal{F}_t]dW_t + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{t,z}^{(1)}F \mid \mathcal{F}_t]d\widetilde{N}(ds,dz) \\
= & \sum_{l=0,1} \int_{u^l} \mathbb{E}[D_{u^l}^{(l)}F \mid \mathcal{F}_t]dH_l(u^l) \\
= & \sum_{l=0,1} \int_{u^l} \mathbb{E}[\sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=0,1} \sum_{k=1}^{n} 1_{\{i_k=l\}} L_{n-1}^{(i_1,\dots,i_{k-1},i_{k+1},\dots,i_n)}(f_{i_1,\dots,i_n}(\dots,u^l,\dots)1_{\{G_{i_1,\dots,i_n}^{(k)}(t)\}}) \mid \mathcal{F}_t]dH_l(u^l).
\end{aligned}$$

We note that the expectation of the integrals that have $t_k > t$, $k = 1,\dots,n-1$ are zero. Thus, the only non-zero integrals are the ones that are adapted, that is, those with $k = n$. As a result the third sum of the chaos expansion vanishes and we have

$$\begin{aligned}
& \sum_{l=0,1} \int_{u^l} \mathbb{E}[\sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=0,1} \sum_{k=1}^{n} 1_{\{i_k=l\}} L_{n-1}^{(i_1,\dots,i_{k-1},i_{k+1},\dots,i_n)}(f_{i_1,\dots,i_n}(\dots,u^l,\dots)1_{\{G_{i_1,\dots,i_n}^{(k)}(t)\}}) \mid \mathcal{F}_t]dH_l(u^l) \\
= & \sum_{l=0,1} \int_{u^l} \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=0,1} 1_{\{i_n=l\}} L_{n-1}^{(i_1,\dots,i_{n-1},i_{n+1},\dots,i_n)}(f_{i_1,\dots,i_n}(\dots,u^l,\dots)1_{\{G_{i_1,\dots,i_n}^{(n)}(t)\}}) \mid \mathcal{F}_t]dH_l(u^l) \\
= & \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=0,1} L_n^{(i_1,\dots,i_n)}(f_{i_1,\dots,i_n}) \\
= & F - \mathbb{E}[F].
\end{aligned}$$

□

6.2.2 Wiener-Poisson space

We recall some basic results on the Wiener-Poisson space $(\Omega, \mathcal{F}, P)$ from [51], [62], [64], [76] and [77]).

Fix $T > 0$. Let $\Omega_W = C_0([0,T])$ be the space of continuous functions defined on $[0,T]$, null at the origin, with the topology of the uniform convergence. Let $\mathcal{F}_W$ be the Borel σ-algebra of Ω_W with respect to which $\{\omega(t),\ t \in [0,T]\}$ are measurable and P_W be the probability measure on $(\Omega_W, \mathcal{F}_W)$ such that

$$W_t(\omega) := \omega(t) \tag{6.34}$$

is a standard Brownian motion.

Let Ω_N be the set of all integer-valued measures ω' on $[0,T] \times \mathbb{R}^n$ such that $\omega'(\{t,z\}) \leq 1$ for any point $(t,z) \in [0,1] \times [0,T] \times \mathbb{R}^n$ and $\omega'(A \times B) < \infty$ when $\nu(A \times B) < \infty$. Let P_N be the probability measure on $(\Omega_N, \mathcal{F})$ such that

$$N(\omega', A \times B) := \omega'(A \times B), \qquad A \times B \in [0,T] \times \mathbb{R}^n \tag{6.35}$$

is a Poisson random measure with intensity measure $\nu(dz)$ where ν is a Lévy measure. Hence, $N(A \times B)$ is a Poisson variable with mean $\nu(A \times B)$ and the variables $N(A_i \times B_i)$ are independent when $A_i \times B_j$ are disjoint.

Let $(\Omega, \mathcal{F})$ be the product measure space, that is, $\Omega := \Omega_W \times \Omega_N$, $\mathcal{F}_t = \mathcal{F}_t^W \times \mathcal{F}_t^N$ where $\omega \in \Omega_W$, $\omega' \in \Omega_N$ and consider the product probability measure $P := P_W \times P_N$ on $(\Omega, \mathcal{F})$. The triple $(\Omega, \mathcal{F}, P)$ is called the *Wiener-Poisson space* with Lévy measure ν.

Since in $\Omega_W \times \Omega_N$ there is a product measure, there exists an isometry

$$L^2(\Omega_W \times \Omega_N) \simeq L^2(\Omega_W; L^2(\Omega_N)) \tag{6.36}$$

where

$$L^2(\Omega_W; L^2(\Omega_N)) = \{F : \Omega_W \to L^2(\Omega_N) : \int_{\Omega_W} \| F(\omega) \|^2_{L^2(\Omega_N)} \, dP_W(\omega) < \infty\}. \tag{6.37}$$

Every $F \in L^2(\Omega_W, L^2(\Omega_N))$ can be considered as a functional $F : \omega \to F(\omega, \omega')$. Therefore, we can define a derivative operator $D_t^{(0)}$ on the space $L^2(\Omega_W; L^2(\Omega_N))$ in the usual Malliavin calculus sense. The derivative operator $D_t^{(0)}$ is closed from $L^2(\Omega_W; L^2(\Omega_N))$ into $L^2(\Omega_W \times [0,T]; L^2(\Omega_N))$. Thus, if $F \in \mathbb{D}_{1,2}$ then

$$\begin{aligned} D_t^{(0)} F \in L^2(\Omega_W; L^2(\Omega_N; L^2([0,T]))) &\simeq L^2(\Omega_W; L^2(\Omega_N \times [0,T])) \\ &\simeq L^2(\Omega_W \times \Omega_N \times [0,T]). \end{aligned}$$

Similarly, the derivative operator $D_{t,z}^{(1)}$ is closed from $L^2(\Omega_N; L^2(\Omega_W))$ into $L^2(\Omega_N \times [0,T]; L^2(\Omega_W))$. Thus, if $F \in \mathbb{D}_{1,2}$ then

$$\begin{aligned} D_{t,z}^{(1)} F \in L^2(\Omega_N; L^2(\Omega_W; L^2([0,T]))) &\simeq L^2(\Omega_N; L^2(\Omega_W \times [0,T])) \\ &\simeq L^2(\Omega_N \times \Omega_W \times [0,T]) \end{aligned}$$

Now, it turns out that through the identification of these spaces, the directional derivative $D_t^{(0)}$ is equivalent to the standard Malliavin derivative D_t and the directional derivative $D_{t,z}^{(1)}$ is equivalent to the standard stochastic derivative $D_{t,z}$ (see [62]). We formally state this result in the following proposition.

Proposition 6.2.8 *Assume that W is a Brownian motion and $\widetilde{N}$ is the compensated Poisson process. Then the directional derivative operator $D_t^{(0)}$ in the direction W coincides with the standard Malliavin derivative D_t for Brownian motion and the directional derivative operator $D_{t,z}^{(1)}$ in the direction $\widetilde{N}$ coincides with the standard stochastic derivative operator $D_{t,z}$ for compensated Poisson processes.*

Proof

The proof is given in [62] on page 211. We omit the details. □

The following chain rule holds (see [76]).

Proposition 6.2.9 *1. Let $F = f(Z, Z^1) \in L^2(\Omega)$ where Z only depends on the Brownian motion $\{W_t\}$ and Z^1 only depends on the Poisson process $N_t^{(1)}$. Assume that $f(x, y)$ is a continuously differentiable function with partial derivatives in the variable x and that $Z \in \mathbb{D}_{1,2}^{(0)}$. Then $F \in \mathbb{D}_{1,2}^{(0)}$ and*

$$D_t^{(0)} F = f'(Z, Z^1) D_t^{(0)} Z \tag{6.38}$$

where $D_t^{(0)} Z$ is the usual Gaussian Malliavin derivative.

2. If $F \in \mathbb{D}_{1,2}^{(1)}$ then

$$D_{t,z}^{(1)} F = F \circ \varepsilon_{t,z}^{+} - F \tag{6.39}$$

where $\varepsilon_{t,z}^{+}$ is a transformation on Ω_N, that implies that we have a jump of size z at time t, given by

$$\varepsilon_{t,z}^{-} \omega'(A \times B) = \omega'(A \times B \cap \{t, z\}^c)$$

and

$$\varepsilon_{t,z}^{+} \omega'(A \times B) = \varepsilon_{t,z}^{-} \omega'(A \times B) + 1_A(t) 1_B(z)$$

where $\{t, z\}^c$ stands for the complement of $\{t, z\}$.

The following assumption enables Davis and Johansson [25] to calculate stochastic weights for jump diffusion models using the chain rule (6.38).

Assumption: (Separability)

Assume that b, σ and α are continuously differentiable functions with bounded derivatives and consider Markov jump diffusion of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \sum_{k=1}^{m} \alpha_k(X_t)(dN_t^{(k)} - \lambda_k dt), \quad X_0 = x \tag{6.40}$$

for which we have a continuously differentiable function f with bounded derivative in the first argument such that

$$X_t = f(X_t^c, X_t^d), \quad X_0^c = x \tag{6.41}$$

where X_t^c denote the continuous part and X_t^d denote the jump part (discontinuous part). Here X_t^c satisfies a stochastic differential equation

$$dX_t^c = b_c(X_t^c)dt + \sigma_c(X_t^c)dW_t, \quad X_0^c = x \tag{6.42}$$

with continuous coefficients b_c and σ_c while X_t^d is adapted to the natural filtration $\mathcal{F}_t^N$ of the Poisson processes $(N_t^{(1)}, \dots, N_t^{(m)})$. In particular, X_t^d does not depend on x. We say that the jump diffusion process is separable.

Example

Consider the stochastic differential equation

$$dS_t = bS_t dt + \sigma S_t dW_t + \alpha S_t(dN_t - \lambda dt), \quad S_0 \text{ given.} \tag{6.43}$$

S_t satisfies the separability assumption. Using the Itô formula the solution of (6.43) is given by

$$S_t = S_0 \exp\{(b - \lambda\alpha - \frac{1}{2}\sigma^2)t + \sigma W_t + \alpha N_t\} \equiv X_t \cdot Y_t \tag{6.44}$$

with

$$dX_t = bX_t dt + \sigma X_t dW_t, \quad X_0 = x \tag{6.45}$$

$$dY_t = \alpha Y_t(dN_t - \lambda dt), \quad Y_0 = y \tag{6.46}$$

and x, y are such that $S_0 = x \cdot y$. Therefore S_t is separable.

The chain rule given in Proposition 6.2.9 is applicable only to random variables that satisfy the separability condition above. We extend the application of the chain rule to include random variables that do not satisfy the separability condition. This is done in the following.

Lemma 6.2.10 *The linear span of random variables of the form*

$$\begin{aligned}\exp\{\int_0^T \sigma h(t)dW_t + \int_0^T \int_{\mathbb{R}_0} h(t)\gamma(z)\widetilde{N}(ds,dz) - \frac{1}{2}\int_0^T \sigma^2 h^2(t)dt \\ - \int_0^T \int_{\mathbb{R}_0} (e^{h(t)\gamma(z)} - 1 - h(t)\gamma(z))\nu(dz)dt\}\end{aligned} \tag{6.47}$$

where $\gamma : \mathbb{R}_0 \to (-1,0) \cup (0,1)$ *is the continuous function introduced in Chapter* 5 *and* $h \in C([0,T])$, *is dense in* $L^2(\mathcal{F}_T, P)$.

Proof

The proof is found in [63]. We omit the details. □

We need a technical result (see [76]).

Lemma 6.2.11 *Let $F \in \mathbb{D}_{1,2}^{(0)}$ and $\{F_{n_k}\}_{k=1}^{\infty}$ be a sequence such that $F_{n_k} \in \mathbb{D}_{1,2}^{(0)}$ and $F_{n_k} \to F$ in $L^2(P)$. Then there exists a subsequence $\{F_{k_m}\}_{k_m=1}^{\infty}$ and a constant $0 < C < \infty$ such that $\| D_t^{(0)} F_{k_m} \|^2_{L^2([0,T]\times\Omega)} < C$ and*

$$D_t^{(0)} F = \lim_{m\to\infty} D_t^{(0)} F_{k_m} \tag{6.48}$$

in $L^2([0,T]\times\Omega)$.

Proof

The proof follows the same steps as in Theorem 5.4.2 (see Chapter 5). We omit the details. □

We now state the chain rule that we will use to compute the Greeks (see [76]).

Theorem 6.2.12 *Let $F \in \mathbb{D}_{1,2}^{(0)}$ and f be a continuously differentiable function and of polynomial growth. Then $f(F) \in \mathbb{D}_{1,2}^{(0)}$ and*

$$D_t^{(0)} f(F) = f'(F) D_t^{(0)} F. \tag{6.49}$$

Proof

Let $F \in \mathbb{D}_{1,2}^{(0)}$, then there exists a sequence $\{F_n\}_{n=0}^{\infty}$ where $F_n \in \mathcal{S}$ for all $n \in \mathbb{N}$ ($F_n = f_n(W(h_1), \ldots, W(h_{n_k}), \omega')$ for f_n continuously differentiable with polynomial growth) that converges to F in $L^2(\Omega)$ as $n \to \infty$. Every term of F_n is in $\mathbb{D}_{1,2}^{(0)}$. By Lemma 6.2.11 there exists a subsequence $\{F_{n_k}\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} D^{(0)} F_{n_k} = D^{(0)} F$ in $L^2([0,T]\times\Omega)$. We note that the elements of the sequence $\{F_{n_k}\}_{k=0}^{\infty}$ are separable, thus we can apply Proposition 6.2.9 to the process $f(F_{n_k})$ and we have

$$D_t^{(0)} f(F_{n_k}) = f'(F_{n_k}) D_t^{(0)} F_{n_k}.$$

f is continuously differentiable and of polynomial growth, hence

$$\lim_{k\to\infty} f(F_{n_k}) = f(F) \quad \text{in} \quad L^2(\Omega).$$

By the dominated convergence theorem we have

$$\lim_{k\to\infty} f'(F_{n_k}) = f'(F) \quad \text{in} \quad L^2(\Omega).$$

Thus, we have

$$\lim_{k\to\infty} f(F_{n_k}) D_t^{(0)} F_{n_k} = f'(F) D_t^{(0)} F \quad \text{in} \quad L^2([0,T]\times\Omega) \quad \text{for all} \quad t \in [0,T].$$

The operator $D_t^{(0)}$ is closable, hence $\lim_{k\to\infty} D_t^{(0)} f(F_{n_k}) = D_t^{(0)} f(F)$ in $L^2([0,T]\times\Omega)$ for all $t \in [0,T]$. This completes the proof. □

6.3 Skorohod integral

Let $\delta^{(l)} : L^2(\Omega \times U_l) \to L^2(\Omega)$ be the adjoint operator of the directional derivative $D^{(l)}_{u^l}$, $l = 0, 1$ where U_l is given in (6.21). We denote by $\mathrm{Dom}(\delta^{(l)})$ the domain of $\delta^{(l)}$.

Definition 6.3.1 *Let $h \in L^2(\Omega \times U_l)$. Then h belongs to Dom$(\delta^{(l)})$ if for all $F \in \mathbb{D}^{(l)}_{1,2}$ we have*

$$| \mathbb{E}[\int_{U_l} D^{(l)}_{u^l} F h(u^l)\langle H_l\rangle(du^l)] | \leq C \parallel F \parallel_{L^2(\Omega)} \tag{6.50}$$

where C is some constant depending on h. For every $h \in$ Dom$(\delta^{(l)})$ we can define the Skorohod integral in the l^{th} direction as

$$\mathbb{E}[\int_{U_l} D^{(l)}_{u^l} F h(u^l)\langle H_l\rangle(du^l)] = \mathbb{E}[F\delta^{(l)}(u^l)] \tag{6.51}$$

for any $F \in \mathbb{D}^{(l)}_{1,2}$.

Proposition 6.3.2 *Let F belongs to $L^2(\Omega)$ and $h \in L^2(U_l)$ with chaos expansion*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=0,1} L_n^{i_1,\dots,i_n}(f_{i_1,\dots,i_n}). \tag{6.52}$$

Then

$$\begin{aligned} \delta^{(l)}(Fh) &= \int_{U_l} \mathbb{E}[F]h(u^l)H_l(du^l) \\ &+ \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=0,1} \sum_{k=1}^{n} \int_{U_{i_n}} \cdots \int_{U_{i_{k+1}}} \int_{U_l} \int_{U_{i_k}} \cdots \int_{U_{i_1}} f_{i_1,\dots,i_n}(u_1^{I_1},\dots,u_n^{i_n})h(u^l)1_{G_{i_1,\dots,i_n}} \\ &\cdot \; 1_{\{t_k<t<t_{k+1}\}} H_{i_1}(du_1^{i_1})\cdots H_{i_k}(du_k^{i_k})H_l(du^l)H_{i_{k+1}}(du_{k+1}^{i_{k+1}})\cdots H_{i_n}(du_n^{i_n}) \end{aligned} \tag{6.53}$$

if the infinite sum converges in $L^2(\Omega)$.

Proof

The proof follows the same arguments as in [25]. We omit the details. □

As in the Brownian motion case, the Skorohod integral and the Itô integral coincide in the case of adapted processes.

Proposition 6.3.3 *Let h be an adapted process such that $\mathbb{E}[\int_{U_l} h^2(u^l)\langle H_l\rangle(du^l)] < \infty$. Then $h \in$ Dom$(\delta^{(l)}$ for $l = 0, 1$ and*

$$\delta^{(l)}(h) = \int_{U_l} h(u^l)H_l(du^l). \tag{6.54}$$

The following proposition gives the relationship between the Skorohod integral $\delta^{(l)}$ and the directional derivative $D^{(l)}_{u^l}$.

Proposition 6.3.4 *Let h be an adapted process such that $\mathbb{E}[\int_{U_l} h^2(u^l)\langle H_l\rangle(du^l)] < \infty$. Then, for $h \in \mathbb{D}^{(0)}_{1,2}$,*

$$D^{(l)}_{u^l} \int_0^T h(s)dW_s = \begin{cases} h(t) + \int_t^T D^{(0)}_t h(s)dW_s & \text{if } \ l = 0 \\ \int_t^T D^{(1)}_{t,z} h(s)dW_s & \text{if } \ l = 1. \end{cases}$$

and, for $h \in \mathbb{D}^{(1)}_{1,2}$,

$$D^{(l)}_{u^l} \int_0^T \int_{\mathbb{R}_0} h(s,\theta)\widetilde{N}(ds,d\theta) = \begin{cases} \int_t^T D^{(0)}_t h(s,\theta)\widetilde{N}(ds,d\theta) & \text{if } \ l = 0 \\ h(t,z) + \int_t^T \int_{\mathbb{R}_0} D^{(1)}_{t,z} h(s,\theta)\widetilde{N}(ds,d\theta) & \text{if } \ l = 1. \end{cases}$$

Proof

The proof follows from the definition of the directional derivative. □

6.4 Greeks for jump diffusion models

Let $\{X_t, 0 \le t \le T\}$ be an n-dimensional process satisfying the following stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}_0} \gamma(X_t, z)\widetilde{N}(dt,dz), \quad X_0 = x, \quad x \in \mathbb{R}^n \tag{6.55}$$

where $\{W_t, 0 \le t \le T\}$ is a d-dimensional Brownian motion, $\widetilde{N}(dt,dz)$ is a compensated Poisson random measure. The coefficient $b : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ represents the deterministic drift, $\sigma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^d$ represents the volatility and $\gamma : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ represents the jump size. We assume that the coefficients b, σ and γ are continuously differentiable with bounded derivatives. The coefficients also satisfy the linear growth condition

$$| b(x) |^2 + \sigma(x)^2 + \int_{\mathbb{R}_0} | \gamma(x,z) |^2 \nu(dz) \le C(1 + | x |^2) \tag{6.56}$$

for each $x \in \mathbb{R}^n$ where C is a positive constant as well as the Lipschitz condition

$$| b(x) - b(y) | + | \sigma(x) - \sigma(y) | + | \gamma(z,x) - \gamma(z,y) | \le K | x - y | \tag{6.57}$$

for all $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}_0$, for a constant $K < \infty$. These conditions ensure the existence of a unique solution $\{X_t, 0 \le t \le T\}$ of Equation (6.55).

We assume that the matrix σ satisfies the uniform ellipticity condition. As in the diffusion case, we introduce the first variation process $\{Y_t,\ 0 \leq t \leq T\}$ associated with the solution $\{X_t, 0 \leq t \leq T\}$. The process $\{Y_t, 0 \leq t \leq T\}$ satisfies the stochastic differential equation

$$dY_t = b'(X_t)Y_t dt + \sum_{i=1}^{n} \sigma_i'(X_t)Y_t dW_t^{(i)} + \int_{\mathbb{R}_0} \gamma'(X_t, z)Y_t \widetilde{N}(dt, dz), \quad Y_0 = I \tag{6.58}$$

where I is the identity matrix of $\mathbb{R}^n$, the primes denote derivatives and σ_i is the i^{th} column vector of σ. We can show that the first variation process $\{Y_t, 0 \leq t \leq T\}$ of $\{X_t, 0 \leq t \leq T\}$ is the derivative of X_t with respect to x, that is,

$$Y_t := \frac{\partial X_t}{\partial x}. \tag{6.59}$$

Assuming that $X_t \in \mathbb{D}_{1,2}$, we can show, following the same steps as for the Brownian motion case, that the Malliavin derivative of X_t is given by

$$D_s^{(0)} X_t = Y_t Y_s^{-1} \sigma(X_s) 1_{\{s \leq t\}} \tag{6.60}$$

We assume that the payoff Φ depends on a finite set of payments dates: $t_1, t_2, \ldots, t_n$, with the convention that $t_0 = 0$ and $t_n = T$. Given $0 < t_1 < \ldots < t_n = T$, the option price, under the risk-neutral probability measure, is given by

$$u(x) = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \mid X_0 = x] \tag{6.61}$$

for some underlying assets $X_{t_1}, \ldots, X_{t_n}$. The payoff $\Phi : \mathbb{R}^n \to \mathbb{R}$ is assumed to be square integrable and of polynomial growth. The following proposition gives the explicit formula for calculating Δ (see [25]).

Proposition 6.4.1 *Let $a \in L^2([0,T])$ be an adapted process such that*

$$\int_0^{t_i} a(t)dt = 1, \quad \textit{for all} \ \ i = 1, .., n. \tag{6.62}$$

Assume that b, σ and γ (in Equation (6.55)) are continuously differentiable with bounded partial derivatives and that the matrix σ satisfies the uniform ellipticity condition. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a function of polynomial growth. Then we have

$$\Delta = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n}) \int_0^T a(t)(\sigma^{-1}(X_t)Y_t)^T dW_t \mid X_0 = x]. \tag{6.63}$$

Proof

We first assume that Φ is continuously differentiable. Then we can calculate the derivative inside the expectation and we get

$$\begin{aligned}\Delta &= \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X_{t_1},\cdots,X_{t_n}) \mid X_0 = x] = \mathbb{E}[e^{-rT}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Phi(X_{t_1},\cdots,X_{t_n})\frac{\partial X_{t_i}}{\partial x} \mid X_0 = x] \\ &= \mathbb{E}[e^{-rT}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Phi(X_{t_1},\cdots,X_{t_n})Y_{t_i} \mid X_0 = x] \qquad (6.64)\end{aligned}$$

where Y_{t_i} is the first variation process of X_{t_i}. As a consequence of Lemma 3.0.2 we have

$$Y_{t_i} = \int_0^T a(t)(D_t^{(0)}X_{t_i})\sigma^{-1}(X_t)Y_t dt \qquad (6.65)$$

for any $a \in L^2([0,T])$. Substituting Equation (6.65) into Equation (6.64) we obtain

$$\Delta = e^{-rT}\mathbb{E}[\int_0^T\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Phi(X_{t_1},\cdots,X_{t_n})D_t^{(0)}X_{t_i}a(t)(\sigma^{-1}(X_t)Y_t)^T dt \mid X_0 = x]. \qquad (6.66)$$

An application of the chain rule gives

$$\Delta = e^{-rT}\mathbb{E}[\int_0^T D_t^{(0)}\Phi(X_{t_1},\cdots,X_{t_n})a(t)(\sigma^{-1}(X_t)Y_t)^T dt \mid X_0 = x]. \qquad (6.67)$$

Since the matrix σ is uniformly elliptic by assumption, we can deduce that $a(t)\sigma^{-1}(X_t)Y_t \in \mathrm{Dom}(\delta^0)$. An application of the relation (2.22) to Equation (6.67) gives

$$\Delta = \mathbb{E}[e^{-rT}\Phi(X_{t_1},\cdots,X_{t_n})\delta^0\left(a(t)(\sigma^{-1}(X_t)Y_t)^T\right) \mid X_0 = x]. \qquad (6.68)$$

The proof is completed by using the Equation (2.24).

The general case is obtained by using limit arguments (see Proposition 3.0.4). We omit details. □

Remark

The expression for Δ is given in terms of the Itô integral. This is because the Itô integrals are in general easy to simulate. We note that the formula for Δ is essentially the same as the one obtained in the Brownian motion case. The only difference is in the underlying assets which in this case consist of jump parts.

We provide an example that illustrates how we apply Proposition 6.4.1 to calculate Δ.

Example

We consider the geometric Lévy process

$$dS_t = bS_t dt + \sigma S_t dW_t + \int_{\mathbb{R}_0} zS_t \widetilde{N}(dt, dz), \quad S_0 = x. \tag{6.69}$$

By Itô formula the solution of Equation (6.69) at final time $t = T$ is given by

$$S_T = x \exp\{(b - \frac{1}{2}\sigma^2)T + \sigma W_T + \int_0^T \int_{\mathbb{R}_0} \{\ln(1+z) - z\}\nu(dz)ds + \int_0^T \int_{\mathbb{R}_0} \ln(1+z)\widetilde{N}(ds, dz)\}. \tag{6.70}$$

We note that

$$Y_T = \frac{\partial S_T}{\partial x} = \frac{S_T}{x}, \quad \sigma^{-1}(X_t) = \frac{1}{\sigma S_T}. \tag{6.71}$$

Choose

$$a(t) = \frac{1}{T}.$$

We therefore have

$$\Delta \;=\; \mathbb{E}[e^{-rT}\Phi(S_T)\int_0^T \frac{1}{T}\frac{1}{\sigma S_T}\frac{S_T}{x}dW_t] = \frac{e^{-rT}}{\sigma x T}\mathbb{E}[e^{-rT}\Phi(S_T)W_T].$$

where

$$W_T = \frac{1}{\sigma}\{\ln\left(\frac{S_T}{x}\right) - (b - \frac{1}{2}\sigma^2)T - \int_0^T \int_{\mathbb{R}_0} \{\ln(1+z) - z\}\nu(dz)ds - \int_0^T \int_{\mathbb{R}_0} \ln(1+z)\widetilde{N}(ds, dz)\}.$$

Next, we compute the derivative of the option price with respect to the drift coefficient. As in the Brownian motion case, we introduce the perturbed process

$$dX_t^\varepsilon = (b(X_t^\varepsilon) + \varepsilon\bar{b}(X_t^\varepsilon))dt + \sigma(X_t^\varepsilon)dW_t + \int_{\mathbb{R}_0} \gamma(z, X_t^\varepsilon)\widetilde{N}(dt, dz), \quad X_0^\varepsilon = x \tag{6.72}$$

where ε is a small parameter and $\bar{b}$ is a bounded function. The corresponding option price has the form

$$u^\varepsilon(x) = \mathbb{E}[e^{-rT}\Phi(X_{t_1}^\varepsilon, \cdots, X_{t_n}^\varepsilon) \mid X_0^\varepsilon = x]. \tag{6.73}$$

The following result gives the derivative of the option price with respect to the parameter ε (see [25]).

Proposition 6.4.2 *Assume that b, $\bar{b}$, σ and γ (in Equation (6.72)) are continuously differentiable with bounded partial derivatives and that the matrix σ satisfies the uniform ellipticity condition. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a function of polynomial growth. Then we have*

$$\rho := \frac{\partial}{\partial\varepsilon}u^\varepsilon(x) \mid_{\varepsilon=0} = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\int_0^T (\sigma^{-1}(X_t)\bar{b}(X_t))^T dW_t \mid X_0 = x]. \tag{6.74}$$

Proof

The proof follows the same steps as in Proposition 3.1.3. We omit the details. □

Next we consider the derivative of the option price with respect to the volatility term. Here, we define the perturbed process $\{X_t^\varepsilon,\ 0 \le t \le T\}$ by

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + (\sigma(X_t^\varepsilon) + \varepsilon\bar{\sigma}(X_t^\varepsilon))dW_t + \int_{\mathbb{R}_0} \gamma(z, X_t^\varepsilon)\widetilde{N}(dt, dz), \quad X_0^\varepsilon = x. \tag{6.75}$$

where ε is a small parameter and $\bar{\sigma}$ is a continuously differentiable function with bounded derivative. The variation process $\{Z_t^\varepsilon, 0 \le t \le T\}$ of $\{X_t^\varepsilon, 0 \le t \le T\}$ satisfies the following stochastic differential equation

$$dZ_t^\varepsilon = b'(X_t^\varepsilon)Z_t^\varepsilon dt + (\sigma'(X_t^\varepsilon) + \varepsilon\bar{\sigma}'(X_t^\varepsilon))Z_t^\varepsilon dW_t + \bar{\sigma}(X_t^\varepsilon)dW_t + \int_{\mathbb{R}_0} \gamma'(z, X_t^\varepsilon)Z_t^\varepsilon \widetilde{N}(dt, dz), \;\; Z_0^\varepsilon = 0$$

where 0 is the column vector in $\mathbb{R}^n$. Further, we recall

$$\widetilde{\Upsilon}_n = \{a(t) \in L^2([0,T]) : \int_{t_{i-1}}^{t_i} a(t)dt = 1 \;\text{ for all } \;\; i = 1, \dots, n\}. \tag{6.76}$$

Proposition 6.4.3 *Assume that b, σ, $\bar{\sigma}$ and γ (in Equation (6.75)) are continuously differentiable with bounded partial derivatives and that the matrix σ is uniformly elliptic. Let $\beta_{t_i} = Z_{t_i}Y_{t_i}^{-1}, \;\; i = 1, \dots, n$ such that $\sigma^{-1}(X_t)Y_t\beta_t$ belongs to the domain of the Skorohod integral for all $t \in [0,T]$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a function of polynomial growth. Then, for any $a(t) \in \tilde{\Upsilon}_n$ and for any Φ of polynomial growth, we have*

$$\mathcal{V} := \frac{\partial}{\partial\varepsilon}u^\varepsilon(x) \mid_{\varepsilon=0} = \mathbb{E}[e^{-rT}\Phi(X_{t_1}, \cdots, X_{t_n})\delta^0\left(\sigma^{-1}(X_t)Y\tilde{\beta}_t\right) \mid X_0 = x] \tag{6.77}$$

where $\tilde{\beta}_t = \sum_{i=1}^n a(t)(\beta_{t_i} - \beta_{t_{i-1}})1_{\{t_{i-1} \le t \le t_i\}}$.

Proof

The proof follows the same steps as in Proposition 3.1.5. We omit the details. □

We denote the derivative of the option price with respect to the amplitude parameter γ by $\mathcal{V}_1$. We define the perturbed process

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sigma(X_t^\varepsilon)dW_t + \int_{\mathbb{R}_0} (\gamma(z, X_t^\varepsilon) + \varepsilon\bar{\gamma}(z, X_t^\varepsilon))\widetilde{N}(dt, dz), \quad X_0^\varepsilon = x. \tag{6.78}$$

where ε is a small parameter and $\bar{\gamma}$ is a continuously differentiable function with bounded derivative. As in the above cases, we introduce the variation process $\{Z_t^\varepsilon, 0 \le t \le T\}$ of

$\{X_t^\varepsilon, 0 \le t \le T\}$ as follows

$$\begin{aligned} dZ_t^\varepsilon &= b'(X_t^\varepsilon)Z_t^\varepsilon dt + \sigma'(X_t^\varepsilon)Z_t^\varepsilon dW_t + \int_{\mathbb{R}_0} (\gamma'(z,X_t^\varepsilon) + \varepsilon\bar{\gamma}'(z,X_t^\varepsilon))\, Z_t^\varepsilon \widetilde{N}(dt,dz) \\ &+ \int_{\mathbb{R}_0} \bar{\gamma}(z,X_t^\varepsilon)\widetilde{N}(dt,dz) \end{aligned}$$

with $Z_0^\varepsilon = 0$. The derivative of the option price with respect to the amplitude is obtained in the same way as $\mathcal{V}$. Therefore we have the following proposition.

Proposition 6.4.4 *Assume that b, σ, γ and $\bar{\gamma}$ (in Equation (6.78)) are continuously differentiable with bounded partial derivatives and that the matrix σ is uniformly elliptic (see 3.6 in Chapter 3). Let $\beta_{t_i} = Z_{t_i}Y_{t_i}^{-1}, \quad i = 1,\dots,n$ be such that $\sigma^{-1}(X_t)Y_t\beta_t$ belongs to the domain of the Skorohod integral for all $t \in [0,T]$. Then, for any $a(t) \in \tilde{\Upsilon}_n$ and for any $\Phi : \mathbb{R}^n \to \mathbb{R}$ of polynomial growth, we have*

$$\mathcal{V}_1 := \frac{\partial}{\partial \varepsilon} u^\varepsilon(x) \mid_{\varepsilon=0} = \mathbb{E}[e^{-rT}\Phi(X_{t_1},\cdots,X_{t_n})\delta^0\left(\sigma^{-1}(X_t)Y\tilde{\beta}_t\right) \mid X_0 = x] \tag{6.79}$$

where $\tilde{\beta}_t = \sum_{i=1}^n a(t)(\beta_{t_i} - \beta_{t_{i-1}})1_{\{t_{i-1}\le t\le t_i\}}$.

Proof

The proof follows the same as in Proposition 3.1.5. We omit the details. □

6.5 Greeks for the Heston model with jumps

We extend the Heston model discussed in Chapter 3 to include jumps in both the stock and the variance. The jumps in the variance being of positive deterministic size. We show how to calculate Δ of the Heston model with jumps. We incorporate the jumps into the stock price as follows

$$dS_t = bS_t dt + \sqrt{v_t}S_t dW_t^{(1)} + \int_{\mathbb{R}} zS_{t^-}\widetilde{N}(dt,dz) \tag{6.80}$$

where $W_t^{(1)}$ is a standard Brownian motion and $\widetilde{N}(dt,dz)$ is a compensated Poisson random measure. Using Equation (3.84) in Chapter 3 and letting $\sigma = \sqrt{v_t}$ we can write Equation (6.80) as

$$dS_t = bS_t dt + \rho\sigma_s S_t dW_t^{(2)} + \sigma_s S_t\sqrt{1-\rho^2}dZ_t + \int_{\mathbb{R}} zS_{t^-}\widetilde{N}(dt,dz). \tag{6.81}$$

We incorporate the jumps into the variance process as follows

$$dv_t = \kappa(\theta - v_t)dt + \nu\sqrt{v_t}dW_t^{(2)} + \int_{\mathbb{R}} \gamma\widetilde{N}(dt,dz). \tag{6.82}$$

where γ is the constant of jump size in the volatility. Recall that

$$\sigma := \sqrt{v_t}.$$

Applying Itô's formula on $\sigma = \sqrt{v_t}$ we obtain

$$\begin{aligned}
d\sigma_t &= 0 + \frac{1}{2}v_t^{-\frac{1}{2}}[\kappa(\theta - v_t)dt + \nu\sqrt{v_t}dW_t^{(2)}] + \frac{1}{2}\nu^2 v_t(-\frac{1}{4})v_t^{-\frac{3}{2}}dt \\
&+ \int_{|z|<\mathbb{R}}\{\sqrt{v_t+\gamma} - \sqrt{v_t} - \frac{1}{2}v_t^{-\frac{1}{2}}\gamma\}\nu(dz)dt + \int_{\mathbb{R}}\{\sqrt{v_t+\gamma} - \sqrt{v_t}\}\widetilde{N}(ds,dz) \text{ (since } \sqrt{v_t} > 0 \ a.s.) \\
&= \left(\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\sqrt{v_t}} - \frac{\kappa\sqrt{v_t}}{2} + \int_{|z|<\mathbb{R}}\{\sqrt{v_t+\gamma} - \sqrt{v_t} - \frac{1}{2}v_t^{-\frac{1}{2}}\gamma\}\nu(dz)\right)dt \\
&+ \frac{\nu^2}{2}dW_t^{(2)} + \int_{\mathbb{R}}\left(\sqrt{v_t+\gamma} - \sqrt{v_t}\right)\widetilde{N}(ds,dz) \\
&= \left(\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\sigma_t} - \frac{\kappa\sigma_t}{2} + \int_{|z|<\mathbb{R}}\{\sqrt{\sigma_t^2+\gamma} - \sigma_t - \frac{\gamma}{2\sigma_t}\}\nu(dz)\right)dt \\
&+ \frac{\nu}{2}dW_t^{(2)} + \int_{\mathbb{R}}(\sqrt{\sigma_t^2+\gamma} - \sigma_t)\widetilde{N}(ds,dz). \qquad (6.83)
\end{aligned}$$

As in the case without jumps, we work with logarithmic price $X_t = \log S_t$ rather than the actual price. Applying Itô's formula on $X_t = \log S_t$ we obtain

$$dX_t = (b - \frac{1}{2}\sigma_t^2)dt + \sqrt{1-\rho^2}\sigma_t dZ_t + \rho\sigma_t dW_t^{(2)} + \int_{\mathbb{R}} z\widetilde{N}(dt,dz). \qquad (6.84)$$

The two Equations (6.83) and (6.84) can be thought of as a single two dimensional stochastic differential equation

$$\begin{aligned}
\begin{pmatrix} X_t^x \\ \sigma_t \end{pmatrix} &= \begin{pmatrix} \log x \\ \sigma_0 \end{pmatrix} + \int_0^t \begin{pmatrix} (b - \frac{1}{2}\sigma_s^2) \\ \{\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\sigma_s} - \frac{1}{2}\kappa\sigma_s + \int_{|z|<\mathbb{R}}\{\sqrt{\sigma_s^2+\gamma} - \sigma_s - \frac{\gamma}{2\sigma_s}\}\nu(dz)\} \end{pmatrix} ds \\
&+ \int_0^t \begin{pmatrix} \sqrt{1-\rho^2}\sigma_s & \rho\sigma_s \\ 0 & \frac{\nu}{2} \end{pmatrix}\begin{pmatrix} dZ_s \\ dW_s^{(2)} \end{pmatrix} + \int_0^t \int_{\mathbb{R}} \begin{pmatrix} z \\ \sqrt{\sigma_s^2+\gamma} - \sigma_s \end{pmatrix}\widetilde{N}(ds,dz).
\end{aligned}$$

We assume that σ_t is Malliavin differentiable. The arguments on how to compute Greeks in the case without jumps in Chapter 3 Section 3.5 can be extended in a straight-forward way to computing Greeks for Heston models with jumps. Therefore, by applying Proposition 6.4.1 with

$$a(t) = \frac{1}{T}, \quad Y_t = \begin{pmatrix} \frac{1}{x} \\ 0 \end{pmatrix} \quad \text{and} \quad \sigma^{-1}(X_t) = \begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}\sigma_s} & -\frac{2\rho}{\nu\sqrt{1-\rho^2}} \\ 0 & \frac{2}{\nu} \end{pmatrix}$$

we obtain the following result:

$$\Delta = \mathbb{E}[e^{-rT}\Phi(X_T)\int_0^T \frac{1}{xT\sqrt{1-\rho^2}\sigma_s}dZ_s] \qquad (6.85)$$

where $x = S_0$ is the initial stock price.

6.6 Greeks for Lévy process

In the last stages of writing this thesis we came across a recent paper [12] that contains similar results to those that are presented in this section. We mention that the authors consider the case where the Lévy process X_t is separable and the payoff functional belonging to $L^2(\Omega)$. Apart from the use of Malliavin calculus, the paper considers the Fourier approach which is not used in our case.

We are motivated by ideas in [4] where the authors studied the approximation of small jumps in a Lévy process by a Brownian motion. We extend the ideas and consider computation of Greeks.

Let $\{X(t), 0 \leq t \leq T\}$ be a Lévy process. From the Lévy-Itô decomposition, $X(t)$ can be represented as a sum of a deterministic drift, a Brownian motion, a compound Poisson process and an almost sure limit of compensated compound Poisson process (see [23])

$$X(t) = bt + \sigma W(t) + X^l(t) + \lim_{\varepsilon \downarrow 0} \tilde{X}^\varepsilon(t) \tag{6.86}$$

with

$$X^l(t) := \sum_{s \leq t} \Delta X(s) 1_{|\Delta X(s)| \geq 1} = \int_{|z| \geq 1} z N(t, dz) \tag{6.87}$$

and

$$\tilde{X}^\varepsilon(t) := \sum_{s \leq t} \Delta X(s) 1_{\varepsilon \leq |\Delta X(s)| < 1} - t \int_{\varepsilon \leq |z| < 1} z \nu(dz) = \int_{\varepsilon \leq |z| < 1} z N(t, dz) - t \int_{\varepsilon \leq |z| < 1} z \nu(dz) \tag{6.88}$$

where $\{W(t), 0 \leq t \leq T\}$ is a standard Brownian motion, $N(dt, dz)$ is a Poisson process and $\widetilde{N}(dt, dz)$ is a compensated Poisson random measure of $X(t)$ and $b, \sigma \in \mathbb{R}$ are two constants with $\sigma > 0$. $W(t)$, $X^l(t)$ and $\tilde{X}^\varepsilon(t)$ are independent of each other. The convergence on the right hand side of Equation (6.86) holds a.s uniformly in $t \in [0, T]$. The term $\sum_{s \leq t} \Delta X(s) 1_{|\Delta X(s)| \geq 1}$ in (6.87) describes all the jump sizes greater than or equal to 1 while the term $\lim_{\varepsilon \downarrow 0} \tilde{X}^\varepsilon(t)$ in Equation (6.86) describes all jump sizes less than 1. We can rewrite Equation (6.86) as

$$X(t) = bt + \sigma W(t) + \int_{\varepsilon \leq |z| < 1} z[N(t, dz) - t\nu(dz)] + \int_{|z| \geq 1} z N(t, dz). \tag{6.89}$$

We want to find an approximation of a Lévy process $\{X^\varepsilon(t), 0 \leq t \leq T\}$ which approximates the Lévy process $\{X(t), 0 \leq t \leq T\}$ in some sense to be specified below. There are several ways of doing this (see [4]). Here we focus on the approximation by a Brownian motion.

Define

$$\sigma^2(\varepsilon) := \int_{|z|<\varepsilon} z^2 \nu(dz). \tag{6.90}$$

The function $\sigma(\varepsilon)$ represents the standard deviation of the jumps smaller than ε of the Lévy process $X(t)$. By dominated convergence theorem, $\sigma^2(\varepsilon)$ converges to 0 as ε tends to 0. We approximate the small jumps of a Lévy process $X(t)$ by a Brownian motion as follows

$$X^\varepsilon(t) = \mu_\varepsilon t + (\sigma^2 + \sigma^2(\varepsilon))^{\frac{1}{2}} W^\varepsilon(t) + \int_{\varepsilon \le |z| < 1} z[N(t, dz) - t\nu(dz)] + \int_{|z| \ge 1} zN(t, dz) \tag{6.91}$$

where

$$\mu_\varepsilon = b - \int_{\varepsilon \le |z| < 1} z\nu(dz) \quad \text{and} \quad W^\varepsilon(t) = \frac{1}{\sqrt{\sigma^2 + \sigma^2(\varepsilon)}} \{\sigma W(t) + \sigma(\varepsilon) B(t)\} \tag{6.92}$$

with $B(t)$ a new Brownian motion independent of $W(t)$ and $\sigma^2(\varepsilon)$ is given in Equation (6.90). The drift term μ_ε keeps the overall mean of X_t unchanged, that is, $\mathbb{E}[X^\varepsilon(t)] = \mathbb{E}[X(t)]$ for all t. We note that a Brownian motion appears even if the original process does not have one ($\sigma = 0$).
The following proposition says that the approximation of a Lévy process $X^\varepsilon(t)$ converges in distribution to the original Lévy process $X(t)$ (see [13]).

Proposition 6.6.1 *Let the Lévy processes $X(t)$ and $X^\varepsilon(t)$ be defined as in Equations (6.86) and (6.91) respectively. Then, for every t, we have*

$$\lim_{\varepsilon \to 0} X^\varepsilon(t) = X(t) \qquad in \;\; L^1. \tag{6.93}$$

Proof
We have

$$\mathbb{E}[| \, X^\varepsilon(t) - X(t) \, |] = \mathbb{E}[| \, (\sigma^2 + \sigma^2(\varepsilon))^{\frac{1}{2}} W^\varepsilon(t) - \sigma W(t) - t \int_{\varepsilon \le |z| < 1} z\nu(dz) \, |].$$

From the approximation (6.92), we have

$$(\sigma^2 + \sigma^2(\varepsilon))^{\frac{1}{2}} W^\varepsilon(t) = \sigma W(t) + \sigma(\varepsilon) B(t).$$

Therefore, we have

$$\mathbb{E}[| \, X^\varepsilon(t) - X(t) \, |] = \mathbb{E}[| \, \sigma(\varepsilon) B(t) - t \int_{\varepsilon \le |z| < 1} z\nu(dz) \, |].$$

Using the triangle and Cauchy-Schwartz inequalities we have

$$\begin{aligned}
\mathbb{E}[\mid X^{\varepsilon}(t) - X(t) \mid] &\leq \sigma(\varepsilon)\mathbb{E}[\mid B(t) \mid] + \mathbb{E}[\mid t \int_{\varepsilon\leq|z|<1} z\nu(dz) \mid] \\
&\leq \sigma(\varepsilon)\mathbb{E}[\mid B^2(t)]^{\frac{1}{2}} + \mathbb{E}[\left(\int_0^t \int_{\varepsilon\leq|z|<1} z\nu(dz)dt\right)^2]^{\frac{1}{2}} \\
&\leq \sigma(\varepsilon)\sqrt{t} + \sqrt{t}\int_{\varepsilon\leq|z|<1} z^2\nu(dz) \\
&= 2\sigma(\varepsilon)\sqrt{t}.
\end{aligned}$$

As $\varepsilon \to 0$, $\sigma^2(\varepsilon) \to 0$ and hence $\mathbb{E}[\mid X^{\varepsilon}(t) - X(t) \mid] \to 0$. This proves the convergence in L^1. □

In [4] a rigorous discussion is presented of when the approximation (6.91) is valid. It turns out that the approximation is valid if and only if for each $\kappa > 0$

$$\lim_{\varepsilon\to 0} \frac{\sigma(\kappa\sigma(\varepsilon) \wedge \varepsilon)}{\sigma(\varepsilon)} = 1. \tag{6.94}$$

A sufficient condition for (6.94) is

$$\lim_{\varepsilon\to 0} \frac{\sigma(\varepsilon)}{\varepsilon} = \infty. \tag{6.95}$$

It is proven that the two conditions (6.94) and (6.95) are equivalent if the Lévy measure ν has no atoms in some neighborhood of the origin (see [4]).

We consider the jump diffusion stochastic differential equation of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}_0} \gamma(X_t, z)\widetilde{N}(dt, dz) \tag{6.96}$$

where W_t is a Brownian motion and $\widetilde{N}$ is the compensated Poisson random measure. We assume that the coefficients b, σ and γ are continuously differentiable with bounded derivatives. In addition the coefficients b, σ and γ satisfy the linear growth condition and the Lipschitz condition.

In [12] the authors consider γ to be of the form $\gamma(x, z) = \gamma_1(x)g(z)$ $x \in \mathbb{R}$, $z \in \mathbb{R}_0$ where $\gamma_1(x)$ has a linear growth and is Lipschitz continuous and the factor $g(z)$ satisfies

$$\int_{\mathbb{R}_0} g^2(z)\nu(dz) < \infty.$$

We approximate the small jumps of the jump diffusion by a Brownian motion as follows

$$\begin{aligned} dX_t^\varepsilon &= b(X_t^\varepsilon)dt + \sigma(X_t^\varepsilon)dW_t + \sigma(\varepsilon)dB_t + \int_{|z|\geq\varepsilon} \gamma(X_t^\varepsilon, z)\widetilde{N}(dt,dz) \\ &= b(X_t^\varepsilon)dt + (\sigma^2(X_t^\varepsilon) + \sigma^2(\varepsilon))^{\frac{1}{2}} dW_t^\varepsilon + \int_{|z|\geq\varepsilon} \gamma(X_t^\varepsilon, z)\widetilde{N}(dt,dz) \end{aligned} \tag{6.97}$$

where $\sigma(\varepsilon)$ is defined in Equation (6.90).

The solutions X_t and X_t^ε exist and are unique (see [50] Theorem 9.1). Assume that X_t and X_t^ε belong to $\mathbb{D}_{1,2}$. The first variation process Y_t^ε of X_t^ε satisfies the stochastic differential equation

$$dY_t^\varepsilon = b'(X_t^\varepsilon)Y_t^\varepsilon dt + \frac{\sigma(X_t^\varepsilon)\sigma'(X_t^\varepsilon)Y_t^\varepsilon}{\sqrt{\sigma^2(X_t^\varepsilon)+\sigma^2(\varepsilon)}} dW_t^\varepsilon + \int_{|z|\geq\varepsilon} \gamma'(X_t^\varepsilon, z)Y_t^\varepsilon \widetilde{N}(dt,dz) \tag{6.98}$$

where the primes denote the derivative.

We denote by Δ_ε derivative of the option price with the underlying asset $X^\varepsilon(T)$ with respect to the initial price. Applying the Malliavin calculus approach discussed in the previous sections we have the following result.

Theorem 6.6.2 *Let $a \in L^2([0,T])$ be an adapted process such that*

$$\int_0^{t_i} a(t)dt = 1 \quad \textit{for all} \;\; i = 1,..,n. \tag{6.99}$$

Assume that b, σ and γ (in Equation (6.97)) are continuously differentiable with bounded partial derivatives and that the matrix σ satisfies the uniform ellipticity condition. Then, for any $\Phi : \mathbb{R} \to \mathbb{R}$ of polynomial growth, we have

$$\Delta_\varepsilon = \mathbb{E}[e^{-rT}\Phi(X_T^\varepsilon)\int_0^T a(t)\left(\frac{Y_t^\varepsilon}{\sqrt{\sigma^2(X_t^\varepsilon)+\sigma^2(\varepsilon)}}\right)^T dW_t^\varepsilon \mid X_0 = x]. \tag{6.100}$$

Proof

The proof follows the same arguments as in Proposition 6.4.1. We omit the details. □

In particular, we consider the stochastic differential equation of the form

$$dX_t^\varepsilon = X_t^\varepsilon[bdt + (\sigma^2+\sigma^2(\varepsilon))^{\frac{1}{2}}dW_t^\varepsilon + \int_{\mathbb{R}_0} z\widetilde{N}(dt,dz)], \quad X_0^\varepsilon = x \tag{6.101}$$

where b and σ are constants. Using Proposition 6.6.2 with $a(t) = \frac{1}{T}$ and $Y_t^\varepsilon = \frac{X_t^\varepsilon}{x}$, we calculate Δ_ε as follows

$$\Delta_\varepsilon = \frac{\partial}{\partial x}\mathbb{E}[e^{-rT}\Phi(X^\varepsilon(T))] = \mathbb{E}[e^{-rT}\Phi(X^\varepsilon(T))\frac{W_T^\varepsilon}{xT\sqrt{\sigma^2+\sigma^2(\varepsilon)}}]. \tag{6.102}$$

We have seen before that Δ is calculated as follows

$$\Delta = \mathbb{E}[e^{-rT}\Phi(X(T))\frac{W_T}{x\sigma T}]. \tag{6.103}$$

The following proposition gives the convergence rate of the approximation Δ_ε to Δ.

Proposition 6.6.3 *Let the Lévy processes $X(t)$ and $X^\varepsilon(t)$ be defined as in Equations (6.86) and (6.91) respectively. Assume that the matrix σ satisfies the uniform ellipticity condition. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be of polynomial growth. Then Δ_ε converges to Δ.*

Proof

Using the triangle and Cauchy-Schwartz inequalities we have

$$\begin{aligned}
\mid \Delta_\varepsilon - \Delta \mid &= \mid \mathbb{E}[e^{-rT}\Phi(X^\varepsilon(T))\frac{W_T^\varepsilon}{xT\sqrt{\sigma^2+\sigma^2(\varepsilon)}}] - \mathbb{E}[e^{-rT}\Phi(X(T))\frac{W_T}{x\sigma T}] \mid \\
&\le \mathbb{E}[\mid e^{-rT}\Phi(X^\varepsilon(T)) - e^{-rT}\Phi(X(T)) \mid\mid \frac{W_T^\varepsilon}{xT\sqrt{\sigma^2+\sigma^2(\varepsilon)}} \mid] \\
&+ \mathbb{E}[\mid e^{-rT}\Phi(X(T)) \mid\mid \frac{W_T^\varepsilon}{xT\sqrt{\sigma^2+\sigma^2(\varepsilon)}} - \frac{W_T}{x\sigma T} \mid] \\
&\le \mathbb{E}[\mid e^{-rT}\Phi(X^\varepsilon(T)) - e^{-rT}\Phi(X(T)) \mid^2]^{\frac{1}{2}}\mathbb{E}[\left(\frac{W_T^\varepsilon}{xT\sqrt{\sigma^2+\sigma^2(\varepsilon)}}\right)^2]^{\frac{1}{2}} \\
&+ \mathbb{E}[(e^{-rT}\Phi(X(T)))^2]^{\frac{1}{2}}\mathbb{E}[\mid \frac{W_T^\varepsilon}{xT\sqrt{\sigma^2+\sigma^2(\varepsilon)}} - \frac{W_T}{x\sigma T} \mid^2]^{\frac{1}{2}}.
\end{aligned}$$

Let C be the polynomial growth constant which we assume to equal to the Lipschitz constant of Φ for convenience. Then, we get

$$\begin{aligned}
\mid \Delta_\varepsilon - \Delta \mid &\le \frac{Ce^{-rT}}{xT\sqrt{\sigma^2+\sigma^2(\varepsilon)}}\mathbb{E}[\mid X^\varepsilon(T) - X(T) \mid^2]^{\frac{1}{2}} + Ce^{-rT}\mathbb{E}[1+ \mid X(T) \mid^2]^{\frac{1}{2}} \\
&\cdot \mathbb{E}[\left(\frac{W_T^\varepsilon}{xT\sqrt{\sigma^2+\sigma^2(\varepsilon)}} - \frac{W_T}{x\sigma T}\right)^2]^{\frac{1}{2}}.
\end{aligned}$$

We recall that $W_T^\varepsilon = \frac{1}{\sqrt{\sigma^2+\sigma^2(\varepsilon)}}\{\sigma W_T + \sigma(\varepsilon)B_T\}$ and that W_T is independent of B_T. Then we can show that

$$\mathbb{E}[\left(\frac{W_T^\varepsilon}{xT\sqrt{\sigma^2+\sigma^2(\varepsilon)}} - \frac{W_T}{x\sigma T}\right)^2]^{\frac{1}{2}} = \frac{\sigma(\varepsilon)}{x\sigma T\sqrt{\sigma^2+\sigma^2(\varepsilon)}}\sqrt{T}$$

and in the proof of Proposition 6.6.1 we have

$$\mathbb{E}[\mid X^\varepsilon(T) - X(T) \mid] \le 2\sigma(\varepsilon)\sqrt{T}.$$

Therefore, we have

$$| \Delta_\varepsilon - \Delta | \quad \leq \quad \frac{2Ce^{-rT}}{xT\sqrt{\sigma^2+\sigma^2(\varepsilon)}}\sigma(\varepsilon)\sqrt{T} + Ce^{-rT}\frac{\sigma(\varepsilon)}{x\sigma T\sqrt{\sigma^2+\sigma^2(\varepsilon)}}\sqrt{T}.$$

As ε goes to 0, $\sigma(\varepsilon)$ goes to 0 and hence

$$| \Delta_\varepsilon - \Delta | \rightarrow 0.$$

Hence, the result follows. □

In the case of pure jump case, that is, where there is no Brownian motion in the original Lévy process, we notice that a Brownian motion term appears in our approximation, that is,

$$X^\varepsilon(t) \approx X(t) + \sigma(\varepsilon)B(t) \tag{6.104}$$

where $B(t)$ is assumed to be independent of $X^\varepsilon(t)$.
This is crucial for the application of standard Malliavin calculus approach. Assume that the approximation $X^\varepsilon(t)$ in Equation (6.104) belongs to $\mathbb{D}_{1,2}$. An application of the Malliavin calculus gives the Greek of the approximation of the Lévy process Δ_ε as

$$\Delta_\varepsilon = \mathbb{E}[e^{-rT}\Phi(X^\varepsilon(T))\frac{B_T}{x\sigma(\varepsilon)T}]. \tag{6.105}$$

Remarks

1. Although $\frac{B_T}{\sigma(\varepsilon)}$ may fail to be bounded for some integrable random variables, we can still obtain convergence.

2. Since we can approximate a Lévy process by a jump diffusion processes, we can apply this method to a wider range of Lévy processes.

Chapter 7

White noise calculus for Lévy Processes and its Application to the Calculations of Greeks

In this chapter we first review the extension of white noise analysis developed in Chapter 4 to pure jump case (see [27] and the references therein). We follow the construction in [27]. The results given here generalize the known results for the Malliavin calculus in the case of Brownian motion. Most of the results given here are known but we think that it is of interest to have unified approach based on the white noise analysis for Lévy processes. We mention that the white noise analysis developed in [27] is mainly used to generalize the Clark-Haussmann-Ocone formula.
Our goal in this chapter is to derive explicit expressions for Greeks in the white noise setting using Malliavin calculus. We make use of the Wick chain rule (to be defined later) and the Donsker delta function.

7.1 Basic concepts of Lévy white noise analysis

As in the Brownian motion case, we let $\Omega = S'(\mathbb{R})$ be the space of tempered distributions equipped with its Borel σ-algebra $\mathcal{F} = \mathcal{B}(\Omega)$. The space $S'(\mathbb{R})$ is the dual of the Schwartz space $S(\mathbb{R})$ of test functions, that is, the rapidly decreasing smooth functions on $\mathbb{R}$. Then we define the Lévy white noise probability measure μ, which exists by the Bochner-Minlos theorem (see [47] Appendix A), as the measure $d\mu$ defined on the Borel σ-algebra $\mathcal{B}(\Omega)$ of subsets of Ω by

$$\int_{\Omega} e^{i\langle\omega,f\rangle} d\mu(\omega) = e^{\int_{\mathbb{R}} \psi(f(y))dy}, \qquad f \in S(\mathbb{R}) \tag{7.1}$$

where $i = \sqrt{-1}$ and $\langle\omega, f\rangle = \omega(f)$ denotes the action of $\omega \in \Omega = S'(\mathbb{R})$ applied to $f \in S(\mathbb{R})$ and ψ is given by

$$\psi(u) = -iau - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iuz} - 1 - iuz1_{\{|z|<1\}})\nu(dz). \tag{7.2}$$

Here $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a Lévy measure on $\mathbb{R}_0$.
We suppose that the Lévy measure $\nu(dz)$ satisfies the condition (5.10), that is, for every $\epsilon > 0$ there exists a $\lambda > 0$ such that

$$\int_{(-\epsilon,\epsilon)^c} e^{\lambda|z|}\nu(dz) < \infty. \tag{7.3}$$

The triple $(\Omega, \mathcal{B}(\Omega), \mu)$ is called the *Lévy white noise probability space.*

Lemma 7.1.1 *Let $f \in S(\mathbb{R})$. Then we have*

$$\mathbb{E}[\langle\cdot, f\rangle] = 0 \quad \textit{and} \quad \mathbb{E}[\langle\cdot, f\rangle^2] = M \int_{\mathbb{R}_0} f^2(y)dy. \tag{7.4}$$

where $M = \int_{\mathbb{R}_0} z^2\nu(dz) < \infty$.

Proof
The proof follows the same arguments as in Lemma 4.1.1. We omit the details. □

In this chapter we will denote the Lévy process by η. Similar to the Brownian motion case we can, using the Lemma 7.1.1, extend the definition of $\langle\omega, f\rangle$ from $f \in S(\mathbb{R})$ to any $f \in L^2(\mathbb{R})$. We can then construct the Lévy process $\eta(t,\omega)$ as the càdlàg version of $\tilde{\eta}(t,\omega)$ where

$$\tilde{\eta}(t,\omega) = \langle\omega, \chi_{[0,t]}(\cdot)\rangle \tag{7.5}$$

which is defined since $\chi_{[0,t]}(\cdot)$ is in $L^2(\mathbb{R})$. This leads to the following theorem (see [27]).

Theorem 7.1.2 *The stochastic process $\{\tilde{\eta}(t), 0 \leq t \leq T\}$ has a càdlàg version denoted by η. The process $\{\eta(t), 0 \leq t \leq T\}$ is a Lévy process with Lévy measure ν.*

The Lévy process η_t admits the following stochastic integral representation

$$\eta_t = at + \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz), \quad 0 \leq t \leq T, \tag{7.6}$$

where $\{W_t, 0 \le t \le T\}$ is the standard Brownian motion and $\widetilde{N}(dt, dz)$ is a compensated Poisson random measure associated with η_t. Assuming the integrability condition (7.3), we consider a Lévy process with no drift and $\sigma = 0$, that is, the representation (7.6) reduces to

$$\eta_t = \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz), \quad 0 \le t \le T. \tag{7.7}$$

This is called the pure jump Lévy process. The Poisson process is the most important representative among the pure jump Lévy processes and it corresponds to the specific case in which the measure ν is a point mass at 1.

7.2 Chaos expansion

Let $\{\mathcal{F}_t\}_{0 \le t \le T}$ be the completed filtration generated by the Lévy process in (7.7). Fix $\mathcal{F} = \mathcal{F}_T$. As in the Brownian motion case, we make use of multi-indices of arbitrary length. Here we let $\mathcal{A}$ be the set of multi-indices $\alpha = (\alpha_1, \alpha_2, \dots)$ which have only finitely non-zero values. Further we set $Index(\alpha) = \max\{i : \alpha_i \neq 0\}$ and $\mid \alpha \mid = \sum_i \alpha_i$ for $\alpha \in \mathcal{A}$.
Next we consider two families of orthogonal polynomials. We follow the construction in [27]. First we consider the complete orthonormal system $\xi_j,\ j = 1, 2, \dots$ of $L^2([0, T])$ consisting of the Laguerre functions of order $\frac{1}{2}$, that is,

$$\xi_j(t) = \left(\frac{\Gamma(j)}{\Gamma(j + \frac{1}{2})}\right)^{\frac{1}{2}} e^{-t} t^{\frac{1}{4}} L_{j-1}^{\frac{1}{2}}(t) 1_{(0,\infty)}(t), \quad t \in [0, T], \quad j = 0, 1, \dots. \tag{7.8}$$

where $\Gamma(\cdot)$ is the Gamma function and $L_j^{\frac{1}{2}},\quad j = 0, 1, 2, \dots$ are the Laguerre polynomials of order $\frac{1}{2}$ defined by

$$e^{-t} t^{\frac{1}{2}} L_j^{\frac{1}{2}}(t) = \frac{1}{j!} \frac{d^j}{dt^j} \left(e^{-t} t^{j+\frac{1}{2}}\right), \quad j = 0, 1, 2, \dots. \tag{7.9}$$

The Laguerre functions satisfy

$$\sup_{t \in [0,T]} \mid \xi_j(t) \mid = O(1). \tag{7.10}$$

Further, let $\{l_m\}_{m \ge 0} = \{1, l_1, l_2, \dots\}$ be the Gram-Schmidt orthogonalization of $\{1, z, z^2, \dots\}$ with respect to the inner product of $L^2(\rho)$ where $\rho(dz) = z^2 \nu(dz)$. Then define the polynomials p_j by

$$p_j(z) := \frac{1}{\parallel z l_{j-1} \parallel_{L^2(\rho)}} z \cdot l_{j-1}(z), \quad j = 1, 2, \dots \tag{7.11}$$

In particular,

$$p_1(z) = \frac{1}{\parallel z l_0 \parallel_{L^2(\rho)}} z l_0(z) = \frac{z}{\parallel z \parallel_{L^2(\rho)}}. \tag{7.12}$$

Therefore $z = m_2 p_1(z)$ where $m_2 =\| z \|_{L^2(\rho)}$. The polynomials $\{p_j\}_{j\geq 1}$ form an orthonormal basis for $L^2(\rho)$ (see [82]). We then define the bijective map

$$\kappa : \mathbb{N} \times \mathbb{N} \to \mathbb{N}; (i,j) \to j + (i+j-2)(i+j-1)/2. \tag{7.13}$$

We note that κ gives the Cantor enumeration of the Cartesian product $\mathbb{N} \times \mathbb{N}$. Moreover, if $k = \kappa(i,j)$ for $i,j \in \mathbb{N}$, set

$$\zeta_k(t,z) = \xi_i(t)p_j(z). \tag{7.14}$$

Now, for any $\alpha \in \mathcal{A}$ with $\max\{i : \alpha_i \neq 0\}$ and $\mid \alpha \mid := \sum_{i=1} \alpha_i = n$ we define the tensor product $\zeta^{\otimes \alpha}$ as

$$\begin{aligned}\zeta^{\otimes\alpha}(t_1,z_1,\dots,t_n,z_n) &:= \zeta_1^{\otimes\alpha_1} \otimes \cdots \otimes \zeta_k^{\otimes \alpha_k}(t_1,z_1,\dots,t_n,z_n) \\ &= \zeta_1(t_1,z_1)\cdots\zeta_1(t_{\alpha_1},z_{\alpha_1})\cdots\zeta_j(t_{\alpha_1+\dots+\alpha_{k-1}+1}, z_{\alpha_1+\dots+\alpha_{k-1}+1}\cdots\zeta_k(t_n,z_n)\end{aligned}$$

with $\zeta_k^{\otimes 0} = 1$. Finally, we denote the symmetrized tensor product of $\zeta^{\otimes\alpha}$ by $\zeta^{\hat{\otimes}\alpha}$:

$$\zeta^{\hat{\otimes}\alpha}(t_1,z_1,\dots,t_n,z_n) = \zeta_1^{\hat{\otimes}\alpha_1}\hat{\otimes}\dots\hat{\otimes}\zeta_j^{\hat{\otimes}\alpha_k}(t_1,z_1,\dots,t_n,z_n).$$

For $\alpha \in \mathcal{A}$ we define

$$K_\alpha := I_{|\alpha|}(\zeta^{\hat{\otimes}\alpha}). \tag{7.15}$$

In particular we note that if $\alpha = \epsilon^{(k)}$ with

$$\epsilon^{(k)}(j) := \begin{cases} 1 & \text{if } \; j = k \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$K_{\epsilon^{(k)}} = I_1(\zeta^{\hat{\otimes}\epsilon^{(k)}}) = I_1(\zeta_k) = I_1(\xi_i(t)p_j(z)) \tag{7.16}$$

where $k = \kappa(i,j)$. It can be proved that $\{K_\alpha\}_{\alpha\in\mathcal{J}}$ are orthogonal in $L^2(\mu)$ and

$$\| K_\alpha \|^2_{L^2(\mu)} = \alpha!$$

We note, similar to the Brownian motion case, that if $\mid \alpha \mid= n$

$$\alpha! =\| K_\alpha \|^2_{L^2(\mu)} = n! \| \zeta^{\hat{\otimes}\alpha} \|^2_{L^2((\lambda\times\nu)^n)} . \tag{7.17}$$

This leads to the chaos representation for orthonormal systems K_α (see [27] page 202).

Theorem 7.2.1 *Every $F \in L^2(\mu)$ admits the unique representation of the form*

$$F = \sum_{\alpha\in\mathcal{A}} a_\alpha K_\alpha \tag{7.18}$$

where $a_\alpha \in \mathbb{R}$ for all $\alpha \in \mathcal{A}$. Moreover, the Itô isometry is valid:

$$\| F \|^2_{L^2(\mu)} = \sum_{\alpha \in \mathcal{A}} a_\alpha^2 \alpha!. \tag{7.19}$$

It is known that any sequence of functions $f_n \in \hat{L}^2((\lambda \times \nu)^n), \quad n \geq 1$ such that $\sum_{n\geq 1} n! \| f_n \|^2_{L^2((\lambda\times\nu)^n)} < \infty$ defines a random variable $F \in L^2(\mu)$ by

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

where by convention $I_0(f_0) = f_0$ (constant) (see [27]). By the construction of $\zeta^{\hat{\otimes}\alpha}$, any $f_n \in \hat{L}^2((\lambda \times \nu)^n)$ can be written as

$$f_n = \sum_{|\alpha|=n} a_\alpha \zeta^{\hat{\otimes}\alpha}. \tag{7.20}$$

Hence

$$I_n(f_n) = \sum_{|\alpha|=n} a_\alpha I_n(\zeta^{\hat{\otimes}\alpha}) = \sum_{|\alpha|=n} a_\alpha K_\alpha.$$

The connection between the expansions

$$F = \sum_{n\geq 0} I_n(f_n) \quad \text{and} \quad F = \sum_{\alpha \in \mathcal{A}\alpha} a_\alpha I_{|\alpha|}(\zeta^{\hat{\otimes}\alpha})$$

is given by

$$f_n = a_\alpha \zeta^{\hat{\otimes}\alpha} \quad \text{with} \quad a_\alpha \in \mathbb{R}.$$

Example

Choose $h \in L^2(\mathbb{R})$ deterministic and let $F(\omega) = \int_0^T h(s) d\eta(s)$. For $\eta(t) = \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz)$ we have $d\eta(t) = \int_{\mathbb{R}_0} z\widetilde{N}(dt, dz)$. Therefore

$$F(\omega) = \int_0^T \int_{\mathbb{R}_0} h(s) z\widetilde{N}(ds, dz) = I_1(hz). \tag{7.21}$$

Since $h(s) = \sum_{i\geq 1} (h, \xi_i)_{L^2([0,T])} \xi_i(s)$, we can write (7.21) as

$$\begin{aligned} F(\omega) &= I_1\left(\sum_{i\geq 1} (h, \xi_i)_{L^2([0,T])} \xi_i z\right) = \sum_{i\geq 1} (h, \xi_i)_{L^2([0,T])} I_1(\xi_i z) \\ &= \sum_{i\geq 1} (h, \xi_i)_{L^2([0,T])} I_1(\xi_i m_2 p_1) \qquad \text{by} \quad (7.12) \\ &= \sum_{i\geq 1} (h, \xi_i)_{L^2([0,T])} K_{\epsilon^{k(i,1)}} m_2 \qquad \text{by} \quad (7.16). \end{aligned}$$

By setting $h = \chi_{[0,t]}$, the random variable $\eta(t) \in L^2(\mu)$ has the expansion

$$\eta(t) = m_2 \sum_{i \geq 1} \left(\int_0^t \xi_i(s) ds \right) \cdot K_{\epsilon^{(i,1)}}(\omega) \tag{7.22}$$

where $m_2 = \| z \|_{L^2(\rho)}$.

7.3 The Hida/Kondratiev spaces

We introduce the Lévy versions of the *Hida/Kondratiev test function space* $(S)_\rho$ and the *Hida/Kondratiev stochastic distribution space* $(S)_{-\rho}$ as in the Brownian motion case.

Definition 7.3.1 *For* $0 \leq \rho \leq 1$ *the Kondratiev test function space* $(S)_\rho$ *consists of all* $F = \sum_{\alpha \in \mathcal{A}} a_\alpha K_\alpha \in L^2(\mu)$ *such that*

$$\| F \|^2_{\rho,k} := \sum_{\alpha \in \mathcal{A}} (\alpha!)^{1+\rho} a^2_\alpha (2\mathbb{N})^{k\alpha} < \infty \quad \textit{for all} \quad k \in \mathbb{N} \tag{7.23}$$

where $(2\mathbb{N})^{k\alpha} := (2 \cdot 1)^{k\alpha_1} (2 \cdot 2)^{k\alpha_2} \cdots (2 \cdot j)^{k\alpha_j}$ *if* $k\alpha = (k\alpha_1, \ldots, k\alpha_j) \in \mathcal{J}$.

Definition 7.3.2 *For* $0 \leq \rho \leq 1$ *the Kondratiev distribution space* $(S)_{-\rho}$ *consists of all formal series* $G = \sum_{\alpha \in \mathcal{A}} b_\alpha K_\alpha \in L^2(\mu)$ *such that*

$$\| G \|^2_{-\rho,-q} := \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} b^2_\alpha (2\mathbb{N})^{-q\alpha} < \infty \quad \textit{for some} \quad q \in \mathbb{N} \tag{7.24}$$

As in the Brownian motion case $(S)_\rho$ is endowed with the projective limit topology and $(S)_{-\rho}$ is endowed with limit topology induced by the above seminorms. We note that for any $f = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in (S)_\rho$ and $F = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha \in (S)_{-\rho}$ the action

$$\langle F, f \rangle := \sum_{\alpha \in \mathcal{J}} a_\alpha b_\alpha \alpha! \tag{7.25}$$

is well defined and thus the space $(S)_{-\rho}$ is the dual of $(S)_\rho$

We note that for general $0 \leq \rho \leq 1$ we have the following inclusions

$$(S)_1 \subset (S)_\rho \subset (S)_0 \subset L^2(\mu) \subset (S)_{-0} \subset (S)_{-\rho} \subset (S)_{-1}. \tag{7.26}$$

The spaces $(S) := (S)_0$ and $(S)^* := (S)_{-0}$ are the Lévy versions of the Hida test function space and the Hida stochastic distribution space, respectively. A useful feature of $(S)^*$ is that it contains the white noise of the pure jump Lévy process $\eta(t)$ and of the compensated Poisson random measure as its elements (see [27]).

Definition 7.3.3 *The Lévy white noise process $\dot{\eta}(t)$ is defined by the formal expansion*

$$\dot{\eta}(t) := m_2 \sum_{i\geq 1} \xi_i(t) K_{\epsilon^{(i,1)}} = m_2 \sum_{i\geq 1} \xi_i(t) I_1(\xi_i p_1) = m_2 \sum_{i\geq 1} \xi_i(t) I_1(\xi_i z). \tag{7.27}$$

Lemma 7.3.4 *We have $\dot{\eta}(t) \in (S)^*$ for all t.*

Proof
We have

$$\| \dot{\eta}(t) \|_{-q}^2 \quad = \quad m_2^2 \sum_{i\geq 1} \xi_i^2(t)(2\mathbb{N})^{-q\epsilon^{k(i,1)}} = m_2^2 \sum_{i\geq 1} \xi_i^2(t) 2^{-q} k(i,1)^{-q}. \tag{7.28}$$

Using the fact that $k(i,1) = 1 + (i-1)i/2 \geq i$ and the well-known estimate function

$$\sup_{t\in\mathbb{R}} | \xi_n(t) | = O(n^{-\frac{1}{12}}) \tag{7.29}$$

we conclude that Equation (7.28) is finite for $q \geq 2$. Hence, $\dot{\eta}(t) \in (S)^*$ for all t. □

Comparing Equations (7.22) and (7.27) we have

$$\dot{\eta}(t) = \frac{d}{dt}\eta(t).$$

Lemma 7.3.5

$$\frac{d}{dt}\eta(t) \quad \textit{exists} \quad \textit{in } (S)^* \textit{ for } \quad \textit{all} \quad t. \tag{7.30}$$

Proof
The proof follows by the same arguments as in the Brownian motion case (see Lemma 4.2.10). We omit the details. □

Definition 7.3.6 *The white noise process $\dot{\widetilde{N}}(t,z)$ of the compensated Poisson random measure $\widetilde{N}(dt,dz)$ is defined by the expansion*

$$\dot{\widetilde{N}}(t,z) = \sum_{i,j\geq 1} \xi_i(t) p_j(z) K_{\epsilon^{(i,1)}}(\omega). \tag{7.31}$$

Lemma 7.3.7 $\dot{\widetilde{N}}(t,z) \in (S)^*$ *for all t, z.*

Proof

The proof follows the same arguments in Lemma 7.3.4. We omit the details. $\square$

Lemma 7.3.8 *$\dot{\eta}(t)$ is related to $\dot{\tilde{N}}(t,z)$ by*

$$\dot{\eta}(t) = \int_{\mathbb{R}} z\dot{\tilde{N}}(t,z)\nu(dz). \tag{7.32}$$

Proof

We have

$$\begin{aligned}\int_{\mathbb{R}_0} z\dot{\tilde{N}}(t,z)\nu(dz) &= \int_{\mathbb{R}_0}\sum_{i,j\geq 1}\xi_i(t)p_j(z)K_{\epsilon^{i,j}}z\nu(dz) = \sum_{i\geq 1}\xi_i(t)I_1\left(\xi_i\sum_{j\geq 1}p_j\int_{\mathbb{R}_0}p_jz\nu(dz)\right)\\ &= \sum_{i\geq 1}\xi_i(t)I_1(\xi_i z) = \dot{\eta}(t) \quad \text{by} \quad (7.27)\end{aligned}$$ $\square$

For any Borel set $\Lambda \in \mathcal{B}(\mathbb{R})$ such that its closure does not contain 0 we have

$$\begin{aligned}\tilde{N}(t,\Lambda) &= I_1(\chi_{[0,t]}(s)\chi_\Lambda(z)) = \sum_{i,j\geq 1}(\chi_{[0,t]},\xi_i)_{L^2(\mathbb{R})}(\chi_\Lambda,p_j)_{L^2(\nu)}I_1(\xi_ip_j)\\ &= \int_0^t\int_\Lambda\left(\sum_{i,j\geq 1}\xi_i(s)p_j(z)K_{\epsilon^{i,j}}(\omega)\right)\nu(dz)ds \quad \text{by} \quad (7.16).\end{aligned}$$

So formally we have

$$\dot{\tilde{N}}(t,z) = \frac{\tilde{N}(dt,dz)}{dt\times\nu(dz)} \tag{7.33}$$

which is similar to the Radon Nikodym derivative in $(S)^*$.

7.4 Lévy Wick product

We define a Lévy Wick product as in Chapter 4 (see [27] page 205).

Definition 7.4.1 *The Wick product $F\diamond G$ of two elements of $(S)_{-1}$*

$$F = \sum_{\alpha\in\mathcal{A}}a_\alpha K_\alpha \quad \textit{and} \quad G = \sum_{\beta\in\mathcal{A}}b_\beta K_\beta, \quad a_\alpha, b_\beta \in \mathbb{R}$$

is defined by

$$F\diamond G = \sum_{\alpha,\beta\in\mathcal{A}}a_\alpha b_\beta K_{\alpha+\beta} = \sum_{\gamma\in\mathcal{A}}\left(\sum_{\alpha+\beta=\gamma}a_\alpha b_\beta\right)K_\gamma. \tag{7.34}$$

The spaces $(S)_1, (S), (S)^*$ and $(S)_{-1}$ are topological algebras with respect to the Wick product (see [47] page 47 for the Brownian motion case).

Remark

Let $f_n = \sum_{|\alpha|=n} c_\alpha \zeta^{\hat{\otimes}\alpha} \in \hat{L}^2(\lambda \times \nu)^n$ and $g_m = \sum_{|\beta|=m} b_\beta \zeta^{\hat{\otimes}\beta} \in \hat{L}^2(\lambda \times \nu)^m$. Then we have

$$f_n \hat{\otimes} g_m = \sum_{|\alpha|=n} \sum_{|\beta|=m} c_\alpha b_\beta \zeta^{\hat{\otimes}(\alpha+\beta)} = \sum_{|\gamma|=n+m} \left(\sum_{\alpha+\beta=\gamma} c_\alpha b_\beta \right) \zeta^{\hat{\otimes}\gamma}$$

in $L^2(\lambda \times \nu)^{n+m}$. Hence

$$I_n(f_n) \diamond I_m(g_m) = I_{n+m}(f_n \hat{\otimes} g_m).$$

Example

Choose $h \in L^2([0,T])$ and define $F = \int_0^T h(t) d\eta(t)$. Then

$$\begin{aligned} F \diamond F &= I_1(h_1 z_1) \diamond I_1(h_2 z_2) = I_2(h_1 h_2 z_1 z_2) \\ &= 2 \int_0^T \int_{\mathbb{R}_0} \left(\int_0^T \int_{\mathbb{R}_0} h(t_1) h(t_2) z_1 z_2 \widetilde{N}(dt_1, dz_1) \right) \widetilde{N}(dt_2, dz_2) \\ &= 2 \int_0^T \left(\int_{\mathbb{R}_0} h(t_1) d\eta(t_1) \right) h(t_2) d\eta(t_2). \end{aligned}$$

By the Itô formula, if we put $X(t) = \int_0^t h(s) d\eta(s)$,

$$d(X(t))^2 = 2X(t)dX(t) + h^2(t) \int_{\mathbb{R}_0} z^2 N(dt, dz).$$

Hence

$$F \diamond F = 2 \int_0^T X(s) dX(s) = X^2(T) - \int_0^T \int_{\mathbb{R}_0} h(s)^2 z^2 N(ds, dz). \tag{7.35}$$

In particular, choosing $h = 1$ we obtain

$$\eta(T) \diamond \eta(T) = \eta^2(T) - \int_0^T \int_{\mathbb{R}_0} z^2 N(ds, dz). \tag{7.36}$$

We define an integral in $(S)^*$ as follows.

Definition 7.4.2 *A function $X : [0,T] \times \mathbb{R}_0 \to (S)^*$ is $(S)^*$-integrable if*

$$\langle X(\cdot), f \rangle \in L^1(\lambda \times \nu) \quad \textit{for all} \quad f \in (S),$$

where the action $\langle \cdot, \cdot \rangle$ is defined in Equation (7.25). Then the $(S)^*$-integral of X, denoted by $\int_0^T \int_{\mathbb{R}_0} X(t,z) \nu(dz) dt$, is the unique element of $(S)^*$ such that

$$\langle \int_0^T \int_{\mathbb{R}_0} X(t,z) \nu(dz) dt, f \rangle = \int_0^T \int_{\mathbb{R}_0} \langle X(t,z), f \rangle \nu(dz) dt \quad \textit{for all} \quad f \in (S). \tag{7.37}$$

It is a result of Proposition 8.1 in [46] that Equation (7.37) defines $\int_0^T \int_{\mathbb{R}_0} X(t,z)\nu(dz)dt$ as an element of $(S)^*$.

The next theorem gives the relation between the Skorohod integral and the Wick product.

Theorem 7.4.3 *Assume that $X(t,z) = \sum_{\alpha\in\mathcal{A}} c_\alpha(t,z)K_\alpha$ is a Skorohod integrable stochastic process with*

$$\int_a^b \int_{\mathbb{R}_0} \mathbb{E}[X(t,z)^2]\nu(dz)dt < \infty \tag{7.38}$$

for some $0 \le a < b$. Then $X(t,z) \diamond \widetilde{N}(t,z)$ is $\nu(dz)dt$-integrable in $(S)^*$ over $[a,b) \times \mathbb{R}_0$ and

$$\int_a^b \int_{\mathbb{R}_0} X(t,z)\widetilde{N}(\delta t, dz) = \int_a^b \int_{\mathbb{R}_0} X(t,z) \diamond \dot{\widetilde{N}}(t,z)\nu(dz)dt. \tag{7.39}$$

Proof

The proof follows the same arguments as in the Brownian motion case (see [47] page 52). We omit the details. □

7.5 Lévy Hermite transform

The Levy Hermite transform is introduced as in the case of Brownian motion (see [27] page 211).

Definition 7.5.1 *Let $F(\omega) = \sum_{\alpha\in\mathcal{A}} c_\alpha K_\alpha(\omega) \in (S)_{-1}$. Then the Lévy Hermite transform of F, denoted by $\mathcal{H}F$ or $\tilde{F}$, is defined by*

$$\mathcal{H}F(z) = \tilde{F}(z) = \sum_{\alpha\in\mathcal{A}} c_\alpha z^\alpha \in \mathbb{C} \tag{7.40}$$

where $z = (z_1, z_2, \ldots,) \in \mathbb{C}^n$ and $z^\alpha = z_1^\alpha, z_2^\alpha, \ldots, z_n^{\alpha_n}, \ldots$ under the assumption that the series in Equation (7.40) converges.

Example

For

$$\dot{\eta}(x) = m_2 \sum_{i\ge1} \xi_i(x) K_{\epsilon^{(i,1)}}$$

we have

$$\mathcal{H}(\dot{\eta}(x))(z) = m_2 \sum_{i\ge1} \xi_i(x) z_{\epsilon^{(i,1)}}.$$

The following proposition is an important property of the Lévy Hermite transform as it transforms the Lévy-Wick product into an ordinary product.

Proposition 7.5.2 *If $F, G \in (S)_{-1}$ then*

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z). \tag{7.41}$$

for all z such that $\mathcal{H}F(z)$ and $\mathcal{H}G(z)$ exist.

We mention that the results related to the Hermite transform in the Brownian motion case are valid for the Lévy Hermite transform with minor modifications.

7.6 Lévy stochastic derivative

We can extend the stochastic derivative operator $D_{t,z}$ to the whole space $(S)^*$ by making use of the chaos expansion

$$F = \sum_{\alpha \in \mathcal{A}} c_\alpha K_\alpha.$$

Definition 7.6.1 *For any $F = \sum_{\alpha \in \mathcal{A}} c_\alpha K_\alpha \in (S)^*$ we define the stochastic derivative $D_{t,z}F$ of F as*

$$D_{t,z}F := \sum_{\alpha \in \mathcal{A}} c_\alpha \sum_{i,j} \alpha_{k(i,j)} K_{\alpha - \epsilon^{k(i,j)}} \xi_i(t) p_j(z). \tag{7.42}$$

Example
Let

$$Y = \int_0^\infty \int_{\mathbb{R}_0} \xi_i(t) p_j(z) \widetilde{N}(dt, dz)$$

for some $i, j \geq 1$. Then

$$D_{t,z}Y = \xi_i(t) p_j(z).$$

It can be proved that $D_{t,z}F \in (S)^*$ $(\lambda \times \nu)$ a.e for all $F \in (S)^*$. Furthermore it can also be shown that if $F = \lim_{n \to \infty} F_n$ in $(S)^*$ then there exists a subsequence $F_{n_k} \in (S)^*$ such that

$$D_{t,z}F = \lim_{n \to \infty} D_{t,z}F_{n_k} \quad \text{in} \quad (S)^* \quad \lambda \times \nu \quad \text{a.e.}$$

Definition 7.6.1 of the stochastic derivative $D_{t,z}F$ coincides with the Definition 5.4.1 if $F \in \mathbb{D}_{1,2} \subset (S)^*$. This follows with the help of the closability of the operator $D_{t,z}$. The stochastic derivative $D_{t,z}$ does not satisfy the usual chain rule as in the case of Malliavin derivative for the Brownian motion setting. Nevertheless a chain rule can still be formulated in terms of the Wick product (see [29]).

Proposition 7.6.2 *Let $F \in (S)^*$ and let $g(z) = \sum_{n\geq 0} a_n z^n$ be an analytic function in the whole complex plane. Then $\sum_{n\geq 0} a_n F^{\diamond n}$ is convergent in $(S)^*$. In addition, for $g^\diamond(F) = \sum_{n\geq 0} a_n F^{\diamond n}$ the following identity holds*

$$D_{s,z} g^\diamond(F) = \left(\frac{d}{dz} g\right)^\diamond (F) \diamond D_{s,z} F. \tag{7.43}$$

Proof

The convergence of $\sum_{n\geq 0} a_n F^{\diamond n}$ in $(S)^*$ can be derived following similar arguments as in Theorem 2.6.12 and Theorem 2.8.1 in [47]. The chain rule can be easily be shown that it holds for polynomials. Then the result follows by the closeness of $D_{t,z}$ and the continuity of the Wick product. □

The following lemma gives an integration by parts formula for $D_{t,z}$ (see [27] page 226).

Lemma 7.6.3 *Let $F \in (S)^*$ and let $\int_0^T \int_{\mathbb{R}} \| X(t,z) \|_{-q}^2 \nu(dz)dt < \infty$ for some $q \geq 0$. Then the integration by parts formula for $D_{1,2}$ is given by*

$$\int_0^T \int_{\mathbb{R}_0} \langle X(t,z), D_{t,z} F\rangle \nu(dz)dt = \langle \int_0^T \int_{\mathbb{R}_0} X(t,z) \diamond \dot{\tilde{N}}(t,z)\nu(dz)dt, F\rangle. \tag{7.44}$$

As in Gaussian white noise case we let $P(x) = \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha, \quad x \in \mathbb{R}^N, \quad c_\alpha \in \mathbb{R}$, be a polynomial where $x^\alpha = (x_1^{\alpha_1} x_2^{\alpha_2}, \ldots)$ and $x_j^0 = 1$. Then we can define its Wick version at $X = (X_1, \ldots, X_n) \in (S)^*$ by

$$P^\diamond(X) = \sum_{\alpha\in\mathcal{A}} c_\alpha X^{\diamond\alpha}.$$

In the following the derivative of a process $X : [0,T] \to (S)^*$ is understood in the sense of the topology of $(S)^*$. We denote the derivative of X_t by $\frac{d}{dt}X_t$. Define

$$X_{i,j}^{(t)} = \int_0^t \int_{\mathbb{R}_0} \xi_i(s) p_j(z) \tilde{N}(ds,dz), \quad i,j \geq 0. \tag{7.45}$$

We can write this as $X_{i,j}^{(t)} = \int_0^t \int_{\mathbb{R}_0} \xi_i(s) p_j(z) \dot{\tilde{N}}(s,z)\nu(dz)ds$. It follows from Lemma 2.8.4 in [47] that the derivative of $X_{i,j}^{(t)}$ exists and

$$\frac{d}{dt} X_{i,j}^{(t)} = \int_{\mathbb{R}_0} \xi_i(t) p_j(z) \dot{\tilde{N}}(t,z)\nu(dz) \tag{7.46}$$

where we have used the Bochner integral with respect to ν. The following Wick chain rule then follows by induction (see [27] page $214 - 215$).

Lemma 7.6.4 *Let $P(x) = \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha$ be a polynomial in $x = (x_1, \ldots, x_n) \in \mathbb{R}_0^n$. Suppose $i_k, j_k \geq 1$ for all $k = 1, 2, \ldots, n$ and let $X^{(t)} = (X^{(t)}_{i_1,j_1}, \ldots, X^{(t)}_{i_n,j_n})$ with $X^{(t)}_{i,j}$ as in Equation (7.45). Then*

$$\frac{d}{dt}P^\diamond(X^{(t)}) = \sum_{i,j=1}^{n} \int_{\mathbb{R}_0} \left(\frac{\partial P}{\partial x_j}\right)^\diamond (X^{(t)}) \diamond \xi_i(s) p_j(z) \dot{\tilde{N}}(t,z)\nu(dz). \tag{7.47}$$

7.7 Donsker delta function of a Lévy process

Put

$$\eta_t = \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz). \tag{7.48}$$

In this section we present the Donsker delta function $\delta_x(\eta_t)$ of η_t. It is a generalized white noise functional (see [46]). Our presentation follow the work in [27].

Definition 7.7.1 *Suppose that $X : \Omega \to \mathbb{R}$ is a random variable belonging to the Lévy-Hida distribution space $(S)_{-1}$. The Donsker delta function of X is a continuous function $\delta_.(X) : \mathbb{R} \to (S)_{-1}$ such that*

$$\int_{\mathbb{R}} h(x)\delta_x(X)dx = h(x) \tag{7.49}$$

for all measurable functions $h : \mathbb{R} \to \mathbb{R}$ for which the integral is well-defined in $(S)_{-1}$.

We want to represent a certain class of pure jump Lévy processes in terms of the Donsker delta function. We assume that the pure jump Lévy process satisfies the condition:
There exists $\varepsilon \in (0, 1)$ such that, for $u \in \mathbb{R}$,

$$\lim_{|u|\to\infty} | u |^{-(1+\varepsilon)} \operatorname{Re}\left(\int_{\mathbb{R}} (e^{iuz} - 1 - iuz)\nu(dz)\right) = \infty \tag{7.50}$$

where $\operatorname{Re}\left(\int_{\mathbb{R}}(e^{iuz} - 1 - iuz)\nu(dz)\right)$ denotes the real part of $\int_{\mathbb{R}}(e^{iuz} - 1 - iuz)\nu(dz)$.

Remark

The condition (7.50) implies that the probability law of $\eta_t, t \geq 0$ is absolutely continuous with respect to the Lebesgue measure (see [27] page 226).

We need the following lemma (see [27] page 227).

Lemma 7.7.2 *Let $u \in \mathbb{R}$ and $t \geq 0$. Then*

$$\exp(iu\eta_t) = \exp^\diamond\left(\int_0^t \int_{\mathbb{R}_0} (e^{iuz} - 1)\widetilde{N}(ds, dz) + t\int_{\mathbb{R}_0} (e^{iuz} - 1 - iuz)\nu(dz)\right). \tag{7.51}$$

Proof

Define

$$Y_t = \exp\left(iu\eta_t - t\int_{\mathbb{R}_0}(e^{iuz} - 1 - iuz)\nu(dz)\right). \tag{7.52}$$

Then an application of Itô's formula gives

$$dY_t = Y_{t^-}\int_{\mathbb{R}_0}(e^{iuz} - 1)\widetilde{N}(dt, dz), \qquad Y_0 = 1.$$

Using Equation (7.39) we have

$$\frac{d}{dt}Y_t = Y_{t^-} \diamond \int_{\mathbb{R}_0}(e^{iuz} - 1)\dot{\widetilde{N}}(t, z)\nu(dz)dt. \tag{7.53}$$

By means of a version of the Wick chain rule, the solution to (7.53) is given by

$$\begin{aligned} Y_t &= \exp^{\diamond}\left(\int_0^t\int_{\mathbb{R}_0}(e^{iuz} - 1)\dot{\widetilde{N}}(t, z)\nu(dz)dt\right) \\ &= \exp^{\diamond}\left(\int_0^t\int_{\mathbb{R}_0}(e^{iuz} - 1)\widetilde{N}(ds, dz)\right). \end{aligned} \tag{7.54}$$

This solution is unique. Comparing Equations (7.52) and (7.54) we obtain the desired result. □

We have the following important result (see [27] page 227).

Theorem 7.7.3 *The Donsker delta function $\delta_x(\eta_t)$ of η_t exists in $(S)_{-1}$ and it admits a representation of the form*

$$\delta_x(\eta_t) = \frac{1}{2\pi}\int_{\mathbb{R}_0}\exp^{\diamond}\left(\int_0^t\int_{\mathbb{R}_0}(e^{iuz} - 1)\widetilde{N}(ds, dz) + t\int_{\mathbb{R}_0}(e^{iuz} - 1 - iuz)\nu(dz) - iux\right)du \tag{7.55}$$

for $u \in \mathbb{R}$ and $t \in [0, T]$.

Proof

The proof is based on the application of the Lévy-Hermite transform and the use of Fourier inversion formula. A detailed proof can be found in [27] page 227. We omit the details. □

7.8 Application: Computing Greeks

Suppose we have a financial market, where the bond price $S_0(t)$ and the stock price $S(t)$ are modelled as follows

1. bond price:
$$S_0(t) = 1, \quad 0 \le t \le T \tag{7.56}$$

2. stock price:
$$dS(t) = S(t)d\eta_t, \quad S(0) = x > 0, \quad 0 \le t \le T \tag{7.57}$$

where η_t is a Lévy process of the form (7.48). Assume that $z > -1+\epsilon$ for a.a z with respect to ν for some $\epsilon > 0$. This ensures that $S(t) > 0$ for all $0 \le t \le T$.
Using the Itô formula for Lévy processes the solution to Equation (7.57) is given by

$$S(t) = x\exp\{\int_0^t \int_{\mathbb{R}_0} (\log(1+z) - z)\nu(dz)ds + \int_0^t \int_{\mathbb{R}_0} \log(1+z)\widetilde{N}(ds,dz)\}. \tag{7.58}$$

Here we apply the concept of the white noise analysis together with the Donsker delta function to compute Δ of a digital option. We consider the digital option of the form

$$\chi_{[K,\infty)}(S_T) \tag{7.59}$$

with strike price K. Similar to the pure Brownian motion case, we apply the concepts of the white noise analysis together with the Donsker delta function of the Lévy process S_t. We will only illustrate the computation of Δ.
As in the pure Brownian motion case, we represent f in terms of the Donsker delta function

$$\delta_x(S_T) = \frac{1}{2\pi}\int_{\mathbb{R}_0} \exp^{\diamond}\left(\int_0^T \int_{\mathbb{R}_0} (e^{iuz}-1)\widetilde{N}(ds,dz) + T\int_{\mathbb{R}_0}(e^{iuz}-1-iuz)\nu(dz) - iux\right)du$$

as

$$\begin{aligned} f(S_T) &= \int_{\mathbb{R}_0} f(y)\delta_y(S_T)dy \\ &= \int_{\mathbb{R}_0} \frac{1}{2\pi}\left(\int_{\mathbb{R}_0} f(y)\exp(-iuy)dy\right)\exp^{\diamond}\{\int_0^T \int_{\mathbb{R}_0}(e^{iuz}-1)\widetilde{N}(ds,dz) \\ &+ T\int_{\mathbb{R}_0}(e^{iuz}-1-iuz)\nu(dz)\}du. \end{aligned}$$

We mention that, for $f \in L^1(\mathbb{R})$ with compact support the integral above converges in the distribution space $(S)_{-1}$. Thus, the option price of the digital option takes the form

$$\begin{aligned} u(x) &= \mathbb{E}[e^{-rT}f(S_T)] \\ &= \mathbb{E}[e^{-rT}\int_{\mathbb{R}_0} \frac{1}{2\pi}\left(\int_{\mathbb{R}_0} f(y)\exp(-iuy)dy\right)\exp^{\diamond}\{\int_0^T \int_{\mathbb{R}_0}(e^{iuz}-1)\widetilde{N}(ds,dz) \\ &+ T\int_{\mathbb{R}_0}(e^{iuz}-1-iuz)\nu(dz)\}du]. \end{aligned}$$

Using Lemma 7.7.2 we can write the option price as follows

$$u(x) = \mathbb{E}[e^{-rT} \int_{\mathbb{R}_0} \frac{1}{2\pi} \left(\int_{\mathbb{R}_0} f(y) \exp(-iuy) dy \right) \exp(iuS_T) du]. \tag{7.60}$$

We now state the following result.

Theorem 7.8.1 *Let f be a function of polynomial growth and let the integral $\int_{\mathbb{R}_0} f(y) \exp(-iuy) dy$ belong to L^1. Then*

$$\begin{aligned} & \frac{d}{dx} \mathbb{E}[e^{-rT} \int_{\mathbb{R}_0} \frac{1}{2\pi} \left(\int_{\mathbb{R}_0} f(y) \exp(-iuy) dy \right) \exp(iuS_T) du] \\ = \; & \mathbb{E}[e^{-rT} \int_{\mathbb{R}_0} \frac{1}{2\pi} \left(\int_{\mathbb{R}_0} f(y) \exp(-iuy) dy \right) \exp(iuS_T) iu \frac{S_T}{x} du]. \end{aligned}$$

Proof
The proof follows the same arguments as in Theorem 4.8.1. We omit the details. □

Remark
We mention that similar results in the case of jump diffusion were obtained in [28]. However, in this paper the authors did not use the Malliavin calculus; they use ideas related to the likelihood method.
We also mention that we have only compute Δ in this section, the other Greeks are left for future work. This is because it requires more time.

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