On one class of extensions of Lie algebras and the Casimir functions related to it

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Declaration

I know the meaning of plagiarism and declare that all of the work in the document, save for that which is properly acknowledged, is my own.

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Contents

Introduction

Let us start with a few definitions; let $\mathcal{G}$ be a Lie algebra over a field $\mathbb{K}$ (here we only consider the fields $\mathbb{R}$ or $\mathbb{C}$), and consider the vector space $\mathcal{G}^n$ of $n-$tuples of elements from $\mathcal{G}$. Construction of a Lie bracket for the vector space $\mathcal{G}^n$ using the bracket from $\mathcal{G}$ was considered in [1]. More explicitly, the brackets have the form

$$[\vec{x}, \vec{y}]_\lambda = \sum_{i,j=1}^{n} W_\lambda^{ij}[x_i, y_j] \tag{1}$$

where $W_\lambda^{ij} \in \mathbb{K}$ are fixed, $[,]$ is the bracket for $\mathcal{G}$ and $1 \leq \lambda \leq n$. In this way, the construction is the same for all $\mathcal{G}$ and so it is called a "universal" extension and the corresponding brackets, "universal". The existence and classification of such structures (in the case where $\mathbb{K} = \mathbb{C}$) is given in [1] and explained later (up to $n = 4$).

One of the main motivations to study these universal extensions is in their applications to the theory of Poisson structures (a Lie bracket on the space of smooth $\mathbb{K}-$valued functions on a fixed differential manifold satisfying the derivation property). Also, it is shown that the corresponding Poisson brackets define the Hamiltonian structures of some models of classical hydro-dynamics and magneto-hydrodynamics (see [2]). Consequently stability of these systems were analyzed in [1] using the related Casimir invariants. More specifically, those that are linear and quadratic in form.

Now, in [2], an alternative approach to the universal extension and the related Casimir invariants was introduced and shows the one-to-one correspondence between these universal extensions and commutative associative algebras.

The correspondence is shown more explicitly as the tensor product (over the field $\mathbb{K}$) of the algebra $\mathcal{G}$ and the corresponding associative commutative algebra (which will be made clearer later). In this new approach, the cohomology groups and corresponding cocycles formed in [1] (used for the classification of the extensions) becomes clearer. Within this new view-point, [2] further clarifies the related Casimir invariants studied in [1]. It does so, up to a certain point, where the matrix $W_{(n)}$ (where $(W_{(n)})_{ij} = W_n^{ij}$) has a non-degenerate sub matrix.

In this thesis, we reproduce the approach introduced in [2] and attempt to further clarify the work done in [1], including the final case, where $W_{(n)}$ is degenerate. We start with some background theory on cohomologies of Lie algebras culminating in the important and well known Levi-decomposition theorem. Then we reproduce some of the notions in [1] and [2] and finally attempt to approach the question about the Casimir functions. More clearly;

In chapter 1, we introduce the extensions and cohomologies of Lie algebras, followed by the lemmas of Whitehead and finally the Levi-decomposition theorem.
In chapter 2, we introduce the “universal” extensions of the Lie algebras as originally given in [1] and classify them up to some point (where $n \leq 4$ as given in [1]).
In Chapter 3, an alternative approach to these “universal” extensions is given, that seen in [2] and the two approaches are compared and finally;
In Chapter 4, the related Casimir Invariants are introduced. We attempt to further understand their algebraic properties developing the ideas of [1] and [2].
In the appendices we give some additional information that is needed to understand the text.

Chapter 1

Cohomologies

1.1 Complexes and Cohomologies.

Consider a sequence (d_s) of homomorphisms of abelian groups C^s:

$$\cdots \longrightarrow C^{n-2} \xrightarrow{d_{n-2}} C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \cdots$$

Definition 1.1.1. *If, in the above sequence, we have that $d_n \circ d_{n-1} = 0$ for all n (or $Imd_{n-1} \subseteq kerd_n$), then we say that the sequence of homomorphisms, (d_n), and spaces (C^n), define an open complex which we denote by (C, d), or more simply, C.*

Definition 1.1.2. *The homomorphisms d_k, are called coboundary operators and the property, $d_k \circ d_{k-1} = 0$, is called the coboundary property.*

- *The elements of C^n are called n-cochains.*
- *The elements of $Z^n := Kerd_n$ are called n-cocycles.*
- *The elements of $B^n := Imd_{n-1}$ are called n-coboundaries.*
- *We call the abelian group $H^n := Z^n/B^n$, the $n-th$ cohomology group of the complex.*

Clearly the sequence is exact whenever H^n is trivial for all n. Hence, we see the cohomology groups as some measure of "$in-exactness$" of the sequence. Note that if we have the sequence

$$C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} C^n$$

with $d_{i+1} \circ d_i = 0$ (for $1 \leq i \leq n-2$), we can create an open complex by simply adding zero spaces on either side of the sequence, i.e.

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} C^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Remark 1.1.3. *It is convention to speak of the operators d, instead of d_k and so the coboundary property becomes $d^2 = 0$.*

Definition 1.1.4. *Let $\mathcal{G}$ be a Lie algebra and V, a vector space. Let $gl(V)$ be the usual general Lie algebra on V, i.e the space of linear maps $h : V \to V$ with bracket given by $[g, h] = g \circ h - h \circ g$. Let $f : \mathcal{G} \to gl(V)$ be a Lie algebra homomorphism. We call the triple $(\mathcal{G}, f, V)$ a representation of $\mathcal{G}$.*

Definition 1.1.5. *Two representations $(\mathcal{G}, f_1, V_1)$ and $(\mathcal{G}, f_2, V_2)$ of the same algebra $\mathcal{G}$, are said to be equivalent, if there exists a linear isomorphism $A : V_1 \to V_2$ such that the diagram*

$$\begin{array}{ccc} V_1 & \xrightarrow{A} & V_2 \\ {\scriptstyle f_1(x)}\downarrow & & \downarrow{\scriptstyle f_2(x)} \\ V_1 & \xrightarrow[A]{} & V_2 \end{array}$$

commutes for all $x \in \mathcal{G}$

Remark 1.1.6. *Clearly the definition above defines an equivalence relation, $\sim$, on the set of representations on a given algebra $\mathcal{G}$.*

Examples 1.1.7.

- *$(\mathcal{G}, ad, \mathcal{G})$ is a representation, where $ad_x(y) := [x, y]$.*
- *It is easy to see that the map $T : gl(V) \to gl(V^*)$ (where V^* is the usual dual space) defined by $T(A) = -A^*$, $A^*(\alpha)(x) = \alpha(A(x))$ is a homomorphism and so given a representation $(\mathcal{G}, f, V)$, it follows that $(\mathcal{G}, f^* := T \circ f, V^*)$ is also a representation called the co-representation of the two. If we applied this construction to the first example, the adjoint representation, we get what we call, the co-adjoint representation.*
- *If $(\mathcal{G}, f_1, V_1)$ and $(\mathcal{G}, f_2, V_2)$ are representations, then $(\mathcal{G}, f_1 \oplus f_2, V_1 \times V_2)$ is also a representation, where $(f_1 \oplus f_2(x))(v_1, v_2) := (f_1(x)v_1, f_2(y)v_2)$. It is called the sum of the two representations.*
- *If $(\mathcal{G}, f_1, V_1)$ and $(\mathcal{G}, f_2, V_2)$ are representations, then $(\mathcal{G}, f_1 \otimes id_{V_2} + id_{V_1} \otimes f_2, V_1 \otimes V_2)$ is a representation, called the tensor product of the two representations. It is usually denoted by $(\mathcal{G}, f_1 \otimes f_2, V_1 \otimes V_2)$.*
- *The construction of the previous representation (the tensor product) may be applied to the representation $(\mathcal{G}, f_1, V_1)$ and $(\mathcal{G}, f_2^*, V_2^*)$ of $(\mathcal{G}, f_2, V_2)$ to obtain $(\mathcal{G}, f_1 \otimes id_{V_2^*} + id_{V_1} \otimes f_2^*, V_1 \otimes V_2^*)$ usually denoted by $(\mathcal{G}, f_1 \otimes f_2^*, V_1 \otimes V_2^*)$. Since $V_1 \otimes V_2^* \sim Hom(V_2, V_1)$ and the map establishing the isomorphism commutes with the representation maps (easily checked), the resulting representation can be considered as a representation in the vector space $Hom(V_2, V_1)$. It is carried over by the following construction; for $x \in \mathcal{G}$, define $F : \mathcal{G} \to gl(Hom(V_2, V_1))$ by*

  $$f(x)s = f_1(x) \circ s - s \circ f_2(x)$$

  where $s \in Hom(V_2, V_1)$. So that $(\mathcal{G}, F, Hom(V_2, V_1)) \sim (\mathcal{G}, f_1 \otimes f_2^*, V_1 \otimes V_2^*)$
- *If $(\mathcal{G}, f, V)$ is a representation, and if W is some invariant subspace of V (i.e. $f(\mathcal{G})W \subseteq W$), then we can naturally define a new representation $(\mathcal{G}, f_{V/W}, V/W)$ where $f(x)(v + W) = f(x)v + W$. It is clearly well defined and we call this, the factor representation.*

Definition 1.1.8. *Let $(\mathcal{G}, f, V)$ be a representation of a Lie algebra $\mathcal{G}$. We define a sequence of cochain spaces as follows;*

- $C^0(\mathcal{G}, f, V) = V$. *Equivalently, we could define it as the set of all constant functions* $h : \mathcal{G} \to V$
- $C^k(\mathcal{G}, f, V) = A^k(\mathcal{G}, V)$, *where* $A^k(\mathcal{G}, V)$ *is the set of all* k*-linear, skew symmetric maps* $h : \mathcal{G}^k \to V$.
- *For* $v \in C^0(\mathcal{G}, f, V)$, *we define* $d_0(v) \in C^1(\mathcal{G}, f, V)$ *by*

$$(d_0(v))(x) := f(x)(v)$$

- *For* $S^k \in C^k(\mathcal{G}, f, V), k \geq 1$ *we define* $d_k(S^k)$ *by*

$$\begin{aligned}(d_k(S^k))(x_1, x_2, \ldots, x_{k+1}) \quad := \quad & \sum_{i=1}^{k+1} (-1)^{i+1} f(x_i) S^k(x_1, x_2, \ldots, \hat{x}_i, \ldots, x_{k+1}) + \\ & \sum_{1 \leq i < j \leq k} (-1)^{i+j} S^k([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1})\end{aligned}$$

where " ˆ " indicates an omission and $x_i \in \mathcal{G}$.

With the above definitions, we see that

- for $S \in C^1(\mathcal{G}, f, V)$, $x_1, x_2 \in \mathcal{G}$

$$d_1 S(x_1, x_2) = f(x_1)S(x_2) - f(x_2)S(x_1) - S([x_1, x_2]) \tag{1.1}$$

- for $S^2 \in C^2(\mathcal{G}, f, V)$, $x_1, x_2, x_3 \in \mathcal{G}$

$$d_2 S^2(x_1, x_2, x_3) \quad = \quad f(x_1)S^2(x_2, x_3) - f(x_2)S^2(x_1, x_3) + f(x_3)S^2(x_1, x_2) - \tag{1.2}$$

$$S^2([x_1, x_2], x_3) + S^2([x_1, x_3], x_2) - S^2([x_2, x_3], x_1) \tag{1.3}$$

Remark 1.1.9. *The abelian group structure of the cochain spaces are simply the underlying abelian groups of the vector spaces they form and so the operators* d_k *are clearly abelian group homomorphisms.*

We now aim to show that the sequence of spaces form an open complex, equivalently, $d_{k+1} \circ d_k = 0$. But first, a construction. Given a representation $(\mathcal{G}, f, V)$, define $d_{(0)} : \mathcal{G} \to gl(V)$ with

$$d_{(0)} = f \tag{1.4}$$

Define for $n \geq 1$ $d_{(n)} : \mathcal{G} \to gl((C^n(\mathcal{G}, f, V)))$ by

$$d_{(n)}(y)(h^n)(x_1, x_2, \ldots, x_n) = d_{(0)}(y)h^n(x_1, \ldots, x_n) - \sum_{i=1}^{n} h^n(x_1, \ldots, [y, x_i], \ldots, x_n) \tag{1.5}$$

where $h^n \in C^n(\mathcal{G}, f, V)$ is a skew symmetric map. Clearly $d_{(n)}(y)h^n \in C^n(\mathcal{G}, f, V)$, so that the map $d_{(n)}$ is well defined. Moreover;

Proposition 1.1.10. $d_{(n)}$ *defines a representation on* $\mathcal{G}$. ([5])

Proof.

$$\begin{aligned}
& d_{(n)}(y)d_{(n)}(z)h^n(x_1,\dots x_n) \\
= \; & d_{(0)}(y)d_{(n)}(z)h^n(x_1,\dots,x_n) - \sum_{i=1}^{n} d_{(n)}(z)h^n(x_1,\dots,[y,x_i],\dots,x_n) \\
= \; & d_{(0)}(y)d_{(0)}(z)h^n(x_1,\dots,x_n) - \sum_{i=1}^{n} d_{(0)}(z)h^n(x_1,\dots,[y,x_i],\dots,x_n) - \\
& \sum_{i=1}^{n}\Big(d_{(0)}(y)h^n(x_1,\dots,[z,x_i],\dots,x_n) + \sum_{1\le j<i} h^n(x_1,\dots,[z,x_j],\dots,[y,x_i],\dots,x_n) + \\
& \sum_{i<j\le n} h^n(x_1,\dots,[y,x_i],\dots,[z,x_j],\dots,x_n) + h^n(x_1,\dots[z,[y,x_i]],\dots,x_n)\Big)
\end{aligned}$$

hence we get that

$$\begin{aligned}
& d_{(n)}(y)d_{(n)}(z)h^n(x_1,\dots,x_n) - d_{(n)}(z)d_{(n)}(y)h^n(x_1,\dots,x_n) \\
= \; & (d_{(0)}(y)d_{(0)}(z) - d_{(0)}(z)d_{(0)}(y))h^n(x_1,\dots,x_n) + \sum_{i=1}^{n} h^n(x_1,\dots,[z,[y,x_i]],\dots,x_n) \\
& - \sum_{i=1}^{n} h^n(x_1,\dots,[y,[z,x_i]],\dots,x_n) \\
= \; & d_{(0)}([y,z])h^n(x_1,\dots,x_n) - \sum_{i=1}^{n} h^n(x_1,\dots,[[y,z],x_i]],\dots,x_n) \\
= \; & d_{(n)}([y,z])h^n(x_1,\dots,x_n)
\end{aligned}$$

hence

$$d_{(n)}(y)d_{(n)}(z) - d_{(n)}(z)d_{(n)}(y) = d_{(n)}([y,z]) \tag{1.6}$$

□

Next, we introduce a contraction operator, for $y \in \mathcal{G}$, $n \geq 1$ $p_n(y) : C^n(g,f,V) \to C^{n-1}(g,f,V)$ is defined by

$$(p_n(y)h^n)(x_1,\dots,x_{n-1}) = h^n(y,x_1,\dots,x_{n-1}) \tag{1.7}$$

Note that $p_n : \mathcal{G} \to Hom(C^n(\mathcal{G},f,V), C^{n-1}(\mathcal{G},f,V))$ is a linear map and

Proposition 1.1.11. *If $h^n \in C^n(\mathcal{G},f,V)$, then $h^n = 0$ if and only if $p_n(y)h^n = 0$ for all $y \in \mathcal{G}$.*

Proof. $h^n = 0 \Leftrightarrow h^n(y,x_1,\dots x_{n-1}) = 0$ for all $y,x_1,\dots x_{n-1} \in \mathcal{G} \Leftrightarrow (p_n(y)h^n)(x_1,\dots,x_{n-1}) = 0$ for all $y,x_1,\dots x_{n-1} \in \mathcal{G} \Leftrightarrow p_n(y)h^n = 0$ for all $y \in \mathcal{G}$. □

Now, using equations (1.1) and (1.2) above, we have

$$\begin{aligned}
d_0 h^0(x) &= d_{(0)}(x)h^0 \\
d_1 h^1(x,y) &= d_{(0)}(x)h^1(y) - d_{(0)}(y)h^1(x) - h^1([x,y]) \\
&= (d_{(1)}(x)h^1)(y) - d_{(0)}(y)p_1(x)h^1 \\
&= (d_{(1)}(x)h^1)(y) - d_0 p_1(x)h^1(y) \\
d_2 h^2(x,y,z) &= d_{(0)}(x)h^2(y,z) - d_{(0)}y h^2(x,z) + d_0(z)h^2(x,y) \\
&\quad -h^2([x,y],z) - h^2(y,[x,z]) + h^2(x,[y,z]) \\
&= (d_{(2)}(x)h^2)(y,z) - d_{(1)}(y)p_2(x)h^2(z) + d_{(0)}(z)p_1(y)p_2(x)h^2 \\
&= (d_{(2)}(x)h^2)(y,z) - d_1 p_2(x)h^2(y,z)
\end{aligned}$$

This suggests the following recursive definition

$$\begin{aligned}
p_1(x)d_0 &= d_{(0)}(x) \\
p_{n+1}(x)d_n + d_{n-1}p_n(x) &= d_{(n)}(x) \qquad n \geq 1
\end{aligned}$$

Lemma 1.1.12. *Let $y, z \in \mathcal{G}$, then*

$$p_n(z)d_{(n)}(y) - d_{(n-1)}(y)p_n(z) = -p_n([y,z])$$

([5])

Proof.

$$\begin{aligned}
& (p_n(z)d_{(n)}(y) - d_{(n-1)}(y)p_n(z))h^n(x_1,\ldots,x_{n-1}) \\
= \;& d_{(n)}(y)h^n(z,x_1,\ldots x_{n-1}) - d_{(n-1)}(y)p_n(z)h^n(x_1,\ldots,x_{n-1}) \\
= \;& d_{(0)}(y)h^n(z,x_1,\ldots,x_{n-1}) - h^n([y,z],x_1,\ldots,x_{n-1}) - \sum_{i=1}^{n} h^n(z,x_1,\ldots,[y,x_i],\ldots,x_{n-1}) \\
& -d_{(0)}(y)h^n(z,x_1,\ldots,x_n) + \sum_{i=1}^{n-1} h^n(z,x_1,\ldots,[y,x_i],\ldots,x_{n-1}) \\
= \;& -h^n([y,z],x_1,\ldots,x_{n-1}) \\
= \;& -p_n([y,z])h^n(x_1,\ldots,x_{n-1})
\end{aligned}$$

□

Lemma 1.1.13. *Let $y \in \mathcal{G}$, then for $n \geq 0$,*

$$d_{(n+1)}(y)d_n = d_n d_{(n)}(y)$$

([5])

Proof. If $n = 0$, then

$$\begin{aligned}
d_{(1)}(x)d_0 a(y) - d_0 d_{(0)}(x)a(y) &= d_{(0)}(x)d_{(0)}(y)a - d_{(0)}([x,y])a - d_{(0)}(y)d_{(0)}(x)a \\
&= 0
\end{aligned}$$

Now note that

$$\begin{aligned}
& p_{n+1}(z)(d_{(n+1)}(y)d_n - d_n d_{(n)}(y)) \\
= \;& -p_{n+1}([y,z])d_n + d_{(n)}(y)p_{n+1}(z)d_n - d_{(n)}(z)d_{(n)}(y) + d_{n-1}p_n(z)d_n(y) \\
= \;& -p_{n+1}([y,z])d_n + d_{(n)}(y)(d_{(n)}(z) - d_{n-1}p_n(z)) \\
& -d_{(n)}(z)d_{(n)}(y) + d_{n-1}(d_{(n-1)}(y)p_n(z) - p_n([y,z])) \\
= \;& -d_n([y,z]) + d_{(n)}(y)d_{(n)}(z) - d_{(n)}(z)d_{(n)}(y) + (d_{n-1}d_{(n-1)}(y) - d_{(n)}d_{n-1})p_n(z) \\
= \;& (d_{n-1}d_{(n-1)}(y) - d_{(n)}(y)d_{n-1})p_n(z)
\end{aligned}$$

Since $z \in \mathcal{G}$ was arbitrary, the statement holds by induction and Proposition (1.1.11). □

Now note that

$$\begin{aligned}
p_{n+2}(y)d_{n+1}d_n &= d_{(n+1)}(y)d_n - d_n p_{n+1}(y)d_n \\
&= d_n d_{(n)}(y) - d_n(d_{(n)}(y) - d_{n-1}p_n(y)) \\
&= d_n d_{n-1}p_n(y)
\end{aligned}$$

Then since $d_1 d_0 = 0$, and since $y \in \mathcal{G}$ was arbitrary, by induction and Proposition (1.1.11), we have

$$d_{n+1}d_n = 0$$

So that (d_n) are coboundaries.

Corollary 1.1.14. *$(C(\mathcal{G}, f, V), d)$ defines an open complex associated to the representation, $(\mathcal{G}, f, V)$. ([5])*

1.2 Lie algebra extensions

Definition 1.2.1. *Let $\mathcal{G}, \mathcal{H}, \mathcal{W}$ be Lie algebras. We say that $\mathcal{H}$ is an extension of $\mathcal{G}$ by $\mathcal{W}$, if there exists a sequence of Lie algebra homomorphisms*

$$0 \longrightarrow \mathcal{W} \xrightarrow{\;i\;} \mathcal{H} \xrightarrow{\;\pi\;} \mathcal{G} \longrightarrow 0$$

that is exact, i.e. $ker(\pi) = Im(i)$ and i is injective and π, a surjection. We denote the extension by the pair (i, π).

Definition 1.2.2. *Two extensions defined by (i, π) and (i', π') are said to be equivalent if there exists an isomorphim $J : \mathcal{H} \to \mathcal{H}$ such that the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{W} & \xrightarrow{\;i\;} & \mathcal{H} & \xrightarrow{\;\pi\;} & \mathcal{G} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle J} & & \| & & \\
0 & \longrightarrow & \mathcal{W} & \xrightarrow{\;i'\;} & \mathcal{H} & \xrightarrow{\;\pi'\;} & \mathcal{G} & \longrightarrow & 0
\end{array}$$

commutes. The double line in the diagram indicates the identity map.

Remark 1.2.3. *We denote the above equivalence by $\sim$. Clearly this defines an equivalence relation on the set of extensions of a fixed Lie algebra $\mathcal{G}$ by another fixed algebra $\mathcal{W}$.*

Note that for an extension (i, π), $i(\mathcal{W}) = \ker \pi$ is an ideal in $\mathcal{H}$ and that

$$\mathcal{G} \approx \mathcal{H} \Big/ i(\mathcal{W})$$

Suppose we have an extension (i, π) and a fixed section, i.e. a linear map $t : \mathcal{G} \to \mathcal{H}$ such that $\pi \circ t = id_{\mathcal{G}}$ (a right inverse of π)

$$0 \longrightarrow \mathcal{W} \xrightarrow{\;i\;} \mathcal{H} \underset{t}{\overset{\pi}{\rightleftarrows}} \mathcal{G} \longrightarrow 0$$

As t is not assumed to be a Lie algebra homomorphism, $t(\mathcal{G})$ is not (generally) a subalgebra in $\mathcal{H}$. Using cohomologies, we can measure how much t differs from a homomorphism. First, we introduce a map. Define $\rho : \mathcal{G} \to gl(\mathcal{W})$ by

$$\rho(x)(z) = i^{-1}([t(x), i(z)]) \tag{1.8}$$

for $x \in \mathcal{G}, z \in \mathcal{W}$. Since $i(\mathcal{W})$ is an ideal in $\mathcal{H}$, $[t(x), i(z)] \in \mathcal{W}$ so that i^{-1} is well defined. Also,

$$\begin{aligned}
([\rho(x), \rho(y)] - \rho[x, y])(z) &= \rho(x)\rho(y)(z) - \rho(y)\rho(x)(z) - \rho([x, y])(z) \\
&= \rho(x)i^{-1}[t(y), i(z)] - \rho(y)i^{-1}[t(x), i(z)] - i^{-1}[t[x, y], i(z)] \\
&= i^{-1}[t(x), [t(y), i(z)]] - i^{-1}[t(y), [t(x), i(z)]] - i^{-1}[t[x, y], i(z)] \\
&= i^{-1}\big([t(x), [t(y), i(z)]] - [t(y), [t(x), i(z)]] - [t[x, y], i(z)]\big) \\
&= i^{-1}[[t(x), t(y)], i(z)] - i^{-1}[t[x, y], i(z)] \\
&= i^{-1}[[t(x), t(y)] - t[x, y], i(z)]
\end{aligned}$$

hence

$$([\rho(x), \rho(y)] - \rho[x, y])(z) = i^{-1}[[t(x), t(y)] - t[x, y], i(z)] \tag{1.9}$$

Note that since $\pi t = id_{\mathcal{G}}$,

$$\pi([t(x), t(y)] - t([x, y])) = [\pi t(x), \pi t(y)] - \pi t[x, y] = 0 \tag{1.10}$$

Hence $[t(x), t(y)] - t([x, y]) \in ker(\pi) = i(\mathcal{W})$. If $\mathcal{W}$ were abelian, it follows then that $i(\mathcal{W})$ is abelian and we get that

$$([\rho(x), \rho(y)] - \rho[x, y])(z) = i^{-1}[[t(x), t(y)] - t[x, y], i(z)] = 0 \tag{1.11}$$

so that $\rho : \mathcal{G} \to gl(\mathcal{W})$ is a representation.

We remark that if $t : \mathcal{G} \to \mathcal{H}$ were a homomorphism, then we also get that ρ will be a representation. In this case, if $h \in \mathcal{H}$ then

$$\pi(h - t\pi(h)) = \pi(h) - \pi t\pi(h) = 0$$

hence it follows that $h - t\pi(h) \in ker\pi = i(\mathcal{W})$. And since $h = (h - t\pi(h)) + t\pi(h)$, we see that $\mathcal{H} = i(\mathcal{W}) + t(\mathcal{G})$. If further $x \in i(\mathcal{W}) \cap t(\mathcal{G})$ then $x \in ker\pi \cap t(\mathcal{G})$ so that there exists $y \in \mathcal{G}$ such that $x = t(y)$. Hence

$$0 = \pi(x) = \pi t(y) = y$$

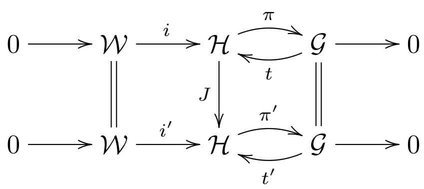
and we get that $x = t(y) = t(0) = 0$. Hence $x \in i(\mathcal{W}) \cap t(\mathcal{G}) = \{0\}$ and we conclude that $\mathcal{H} = i(\mathcal{W}) \oplus t(\mathcal{G})$ (as vector spaces). But then it follows from t being a homomorphism that $\mathcal{H}$ is a semi-direct product

$$\mathcal{H} = i(\mathcal{W}) \times_\rho t(\mathcal{G}) \tag{1.12}$$

which is not interesting as then all extensions of $\mathcal{G}$ by $\mathcal{W}$ will be equivalent. So we assume the first case; $\mathcal{W}$ is abelian and $(\mathcal{G}, \rho, \mathcal{W})$ is a representation of $\mathcal{G}$. The corresponding spaces of co-chains/cycles/boundaries and cohomologies will be denoted by $C^k(\mathcal{G}, \rho, \mathcal{W}), Z^k(\mathcal{G}, \rho, \mathcal{W}), B^k(\mathcal{G}, \rho, \mathcal{W})$ and $H^k(\mathcal{G}, \rho, \mathcal{W})$. Let $w : \mathcal{G} \times \mathcal{G} \to \mathcal{W}$ be defined by

$$w(x, y) = i^{-1}([t(x), t(y)] - t[x, y]) \tag{1.13}$$

By equation (1.10), it is clearly well defined. It is easily checked that w is a two-cocycle, i.e. $w \in Z^2(\mathcal{G}, \rho, \mathcal{W})$. Suppose that we have two equivalent extensions (i, π) and (i', π') and that each have fixed sections, t and t' respectively. Since $(i, \pi) \sim (i', \pi')$, let $J : \mathcal{H} \to \mathcal{H}$ denote the isomorphism that defines the equivalence.



Clearly $J^{-1} \circ t'$ is a fixed section for the extension (i, π). Define $v : \mathcal{G} \to i(\mathcal{W})$ by

$$v = J^{-1} \circ t' - t \tag{1.14}$$

we have that $\pi v = \pi J^{-1} \circ t' - \pi t = \pi' t' - \pi t = 0$, hence $v(\mathcal{G}) \subseteq ker\pi = i(\mathcal{W})$. Now,

$$\begin{aligned}
(\rho(x) - \rho'(x))(z) &= i^{-1}[t(x), i(z)] - (i')^{-1}[t'(x), i'(z)] \\
&= i^{-1}[t(x), i(z)] - (Ji)^{-1}[t'(x), Ji(z)] \\
&= i^{-1}[t(x), i(z)] - i^{-1}J^{-1}[t'(x), Ji(z)] \\
&= i^{-1}[t(x), i(z)] - i^{-1}[J^{-1}t'(x), J^{-1}Ji(z)] \\
&= i^{-1}[t(x), i(z)] - i^{-1}[J^{-1}t'(x), i(z)] \\
&= i^{-1}[t(x) - J^{-1}t'(x), i(z)] \\
&= -i^{-1}[v(x), i(z)]
\end{aligned}$$

and therefore, since $\mathcal{W} \sim i(\mathcal{W})$ is abelian and $v(x) \in i(\mathcal{W})$, $\rho = \rho'$. Note that this then implies that $v \circ t'$ and t define the same representation and in particular, letting $J = id|_{\mathcal{H}}$, we see that ρ is independent on the section t.

We are now able to have an explicit expression for the bracket in $\mathcal{H}$. Identify $\mathcal{W}$ and $i(\mathcal{W})$ and let $\hat{\mathcal{G}}$ be a vector space so that $\mathcal{H} = \hat{\mathcal{G}} \oplus \mathcal{W}$. Then clearly $\pi|_{\hat{\mathcal{G}}}$ will be an isomorphism and $t = \pi|_{\hat{\mathcal{G}}}^{-1}$ will be a fixed section. We identify $\mathcal{G}$ and $\hat{\mathcal{G}}$ by t and write $\mathcal{H} = \mathcal{G} \oplus \mathcal{W}$. Thus if $z \in \mathcal{H}$, we write

$$z = (z_1, z_2);\ z_1 \in \mathcal{G},\ z_2 \in \mathcal{W} \qquad \pi(z) = z_1,\ t(z_1) = (z_1, 0), \quad i(z_2) = (0, z_2).$$

Let $z = (z_1, z_2), h = (h_1, h_2) \in \mathcal{H}$, then

$$[z, h] = [(z_1, 0), (h_1, 0)] + [(0, z_2), (0, h_2)] + [(z_1, 0), (0, h_2)] + [(0, z_2), (h_1, 0)] \tag{1.15}$$

and by definition of w

$$[(z_1, 0), (h_1, 0)] = [t(z_1), t(h_1)] = t([z_1, h_1]_{\mathcal{G}}) + w(z_1, h_1) \tag{1.16}$$

and by definition of ρ

$$[(z_1, 0), (0, h_2)] \quad = \quad [t(z_1), h_2] = i\rho(z_1)(h_2) \tag{1.17}$$

$$([(0, z_2), (h_1, 0)]) \quad = \quad -i\rho(h_1)(z_2) \tag{1.18}$$

So that

$$[z, h] = ([z_1, h_1]_{\mathcal{G}}, \rho(z_1)(h_2) - \rho(h_1)(z_2) + i^{-1}w(z_1, h_1)) \tag{1.19}$$

Conversely, if w is a 2-cocycle defined by a representation ρ of $\mathcal{G}$ in $\mathcal{W}$, we can define a bracket on $\mathcal{G} \oplus \mathcal{W}$ by the above formula. Clearly the bracket defined will be skew-symmetric and the Jacobi identity follows from the fact that w is a 2-cocycle. Next, we check that two cocycles differing by a coboundary yield equivalent extensions. This can be seen by the following argument. Denote the bracket defined by w by $[z, h]_{\mathcal{H}}$ (as is given by equation (1.19)) and the one defined by $w + d\alpha$ by $[z, h]_{\mathcal{H}}^{\alpha}$, we have

$$[z, h]_{\mathcal{H}}^{\alpha} = ([z_1, h_1]_{\mathcal{G}}, \rho(z_1)(h_2 + \alpha(h_1)) - \rho(h_1)(z_2 + \alpha(z_1)) + w(z_1, h_1) - \alpha([z_1, h_1]))$$

hence

$$[\Phi(z), \Phi(h)]_{\mathcal{H}} = \Phi([z, h]_{\mathcal{H}}^{\alpha}) \tag{1.20}$$

where

$$\Phi(z) = \Phi(z_1, z_2) = (z_1, z_2 + \alpha(z_1)) \tag{1.21}$$

Note that if $w = 0$, then the obtained extension is the semi-direct product $\mathcal{G} \times_{\rho} \mathcal{W}$. The above can be summarized into

Theorem 1.2.4. *Suppose that $\mathcal{W}$ is a vector space, considered as an abelian Lie algebra and $(\mathcal{G}, \rho, \mathcal{W})$ is a representation of $\mathcal{G}$. Then the elements of $H^2(\mathcal{G}, \rho, \mathcal{W})$ are in a one-to-one correspondence with the non-equivalent extensions of $\mathcal{G}$ by $\mathcal{W}$ and the extension with a zero cocycle correspond to the semi-direct product $\mathcal{G} \times_{\rho} \mathcal{W}$.([5])*

1.3 Reducibility

Let $(\mathcal{G}, f, V)$ be a representation of a Lie algebra $\mathcal{G}$. Suppose $U \subseteq V$ is invariant ($f(\mathcal{G})U \subset U$)) and assume that W is complimentarily to U in V;

$$V = U \oplus W \tag{1.22}$$

Now assume there exist linear maps

$$f_1 : \mathcal{G} \to gl(U)$$

$$f_2 : \mathcal{G} \to gl(W)$$

$$h : \mathcal{G} \to Hom(W, U) \tag{1.23}$$

such that for $x \in \mathcal{G}$, $u \in U, w \in W$

$$f(x)u = f_1(x)u$$

$$f(x)w = h(x)w + f_2(x)w \tag{1.24}$$

Clearly then, we need that;

- $(\mathcal{G}, f_1, U)$ and $(\mathcal{G}, f_2, W)$ are representations.
- for $x, y \in \mathcal{G}$

$$h([x, y]) = f_1(x) \circ h(y) - h(y) \circ f_2(x) - f_1(y) \circ h(x) + h(x) \circ f_2(y) \tag{1.25}$$

Remark 1.3.1.
- *Note that, given f_1, f_2, h, with the above properties, we can construct f, and that U would be an invariant space for $f(\mathcal{G})$.*
- *f_2 is equivalent to the quotient representation $f_{V/U}$.*

Let W' be a subspace complementary to U. Then there exists $g \in Hom(W, U)$ such that

$$W' = (id_W + g)W \tag{1.26}$$

Proposition 1.3.2. *For $g \in Hom(W, V)$, the subspace $(id_W + g)W$ is invariant with respect to $f(\mathcal{G})$ if and only if for arbitrary $x \in \mathcal{G}$*

$$g \circ f_2(x) - f_1(x) \circ h = h(x)$$

([5])

Proof. Let $W' = (id_W + g)W$ be invariant. For $w \in W$, $x \in \mathcal{G}$

$$f(x)(w + g(w)) = h(x)w + f_1(x)g(w) + f_2(x)w$$

Since $h(x)w + f_1(x)g(w) \in U$, then $f(x)(w + g(w)) \in W'$ if

$$h(x)w + f_1(x)g(w) = gf_2(x)w$$

The reverse is clearly obvious. □

Now if $(\mathcal{G}, f_1, U)$ and $(\mathcal{G}, f_2, W)$ are two representations, recall the representation, $(\mathcal{G}, F, Hom(W, U))$, where

$$F(x)g = f_1(x) \circ g - g \circ f_2(x) \qquad g \in Hom(W, U) \tag{1.27}$$

Consequently, we have

Proposition 1.3.3. *The map h introduced in* (1.24) *satisfies condition* (1.25) *if and only if h is a cocycle with respect to the representation $(\mathcal{G}, F, Hom(W, U))$, i.e. $h \in Z^1(\mathcal{G}, F, Hom(W, U))$.* ([5])

Proof.

$$\begin{aligned} dh(x,y) &= F(x)h(y) - F(y)h(x) - h([x,y]) \\ &= f_1(x)\circ h(y) - h(y)\circ f_2(x) - f_1(y)\circ h(x) + h(x)\circ f_2(y) - h([x,y]) \end{aligned}$$

and so by (1.25), that is equivalent to $dh = 0$. □

Similarly

Proposition 1.3.4. *Let $h \in Hom(\mathcal{G}, Hom(W,U))$. The map g introduced at equation (1.3.2) exists if and only if h is a coboundary, i.e. $h \in B^1(\mathcal{G}, F, Hom(W,U))$.* ([5])

Consequently we obtain

Theorem 1.3.5. *Let $(\mathcal{G}, f, V)$ be a representation of $\mathcal{G}$ and $U \subseteq V$ be an invariant subspace. Let $f_1 = f|_U$ and $f_2 = f_{V/U}$. If $H^1(\mathcal{G}, F, Hom(W, V/W)) = 0$, then $(\mathcal{G}, f, V)$ splits into a direct sum of two representations, equivalent to f_1 and f_2.* ([5])

1.4 Whitehead Lemmas

Definition 1.4.1. *$Q \in End(V) := Hom(V,V)$ is called a Casimir of a representation $(\mathcal{G}, f, V)$ if for all $x \in \mathcal{G}$, $Q \circ f(x) = f(x) \circ Q$ (equivalently $[Q, f(x)] = 0$ for all $x \in \mathcal{G}$).*

Definition 1.4.2. *Let $\mathcal{G}$ be a Lie algebra over $\mathbb{K}$. A symmetric bilinear form $B : \mathcal{G} \times \mathcal{G} \to \mathbb{K}$ on $\mathcal{G}$ is called invariant, if for any $x, y, z \in \mathcal{G}$ we have*

$$B([x,y],z) = B(x,[y,z])$$

Note that this means that $B(ad_x(y), z) = -B(y, ad_x(z))$ i.e. for $x \in \mathcal{G}$, the operator ad_x is skew-symmetric with respect to B. We also say that B defines an invariant inner product.

Definition 1.4.3. *For a subspace $V \subseteq \mathcal{G}$ we define*

$$V^{\perp} = \{x \in \mathcal{G}; B(x,y) = 0, \forall y \in V\}$$

We say also that B is non-degenerate whenever $\mathcal{G}^{\perp} = \{0\}$.

An easily proved but important example is seen in the next theorem.

Theorem 1.4.4. *Let $(\mathcal{G}, f, V)$ be some representation of a Lie algebra $\mathcal{G}$. Define for $x, y \in \mathcal{G}$*

$$B_f(x,y) = tr(f(x)f(y))$$

then, B_f is a symmetric bilinear form on $\mathcal{G}$

Definition 1.4.5. *The bilinear form obtained from the above constructions are called trace forms and the one obtained in particular, through the adjoint representation;*

$$B_{\mathcal{G}}(x,y) := B_{ad}(x,y) = tr(ad_x ad_y)$$

is called the Cartan-Killing form of $\mathcal{G}$.

Two well known results about the trace form and semisimple Lie algebras is seen in the next two theorem which we will not prove here as these are standard results. The first one concerns the Killing form. But first we include a definition.

Definition 1.4.6. *A representation $(\mathcal{G}, f, V)$ of a Lie algebra G, is said to be faithful, whenever $f : \mathcal{G} \to gl(V)$ is injective.*

Theorem 1.4.7. *A Lie algebra $\mathcal{G}$ is semisimple if and only if its Cartan-Killing form is non-degenerate.([3])*

Theorem 1.4.8. *Let $\mathcal{G}$ be a semisimple algebra and $(\mathcal{G}, f, V)$ be a faithful representation. Then the trace form B_f is non-degenerate.([3])*

Now we are ready to introduce a special Casimir element. Suppose that we have a semisimple Lie algebra $\mathcal{G} \subseteq gl(V)$ with a basis $\{x_i\}_{i=1}^n$. By the non-degeneracy of B_{Incl} (where $Incl$ is the inclusion map) we can construct a dual basis $\{y^j\}_{j=1}^n$ with respect to $B := B_{Incl}$, i.e. $B(x_i, y^j) = \delta_i^j$.

Now since

$$B(x - \sum_{i=1}^{n} B(x, y^i)x_i, y^j) = 0$$

for all $y^j \in \mathcal{G}$, we see that

$$x = \sum_{i=1}^{n} B(x, y^i)x_i \tag{1.28}$$

If $[x, x_i] = \sum_j c_{ij}(x)x_j$ and $[x, y^i] = \sum_j d^{ij}y^j$, then by the invariance of B, we obtain that $c_{ij} = -d^{ij}$.

Now, define

$$\Gamma = \sum_{i=1}^{n} x_i y^i \tag{1.29}$$

Since $c_{ij} = -d^{ij}$, we get that

$$[x, \Gamma] = \sum_{i=1}^{n}([x, x_i]y^i + x_i[x, y^i]) = 0$$

which shows that the linear transformation Γ commutes with all elements of $\mathcal{G}$ i.e. Γ is a Casimir element of the representation $(G, Incl, V)$. Note that the same construction may be carried out on any faithful represention $(\mathcal{G}, f, V)$ whenever $\mathcal{G}$ is semisimple and so we shall call Γ the *Casimir endomormphism* of the representation

Note that for the same representation;

$$tr(\Gamma) = \sum_{i=1}^{n} tr(x_i, y^i) = \sum_{i=1}^{n} B_f(x_i, y^i) = n \tag{1.30}$$

which shows that in particular, Casimir endomorphisms (of this type) are not nilpotent.

Definition 1.4.9. *The maximal solvable ideal of an algebra $\mathcal{G}$ is called the radical of $\mathcal{G}$ and denoted by $\mathcal{R}(\mathcal{G})$ or simply $\mathcal{R}$. If $\mathcal{R} = 0$, then the algebra is called semisimple. An algebra $\mathcal{G}$ is called simple if $dim(\mathcal{G}) > 1$ and if it does not contain non-trivial ideals.*

Let us recall some basic facts about semisimple Lie algebras.

- If the algebra $\mathcal{G}$ is simple, then it is semisimple. ([3])
- A Lie algebra $\mathcal{G}$ is semisimple if and only if $\mathcal{G}$ does not contain nontrivial abelian ideals. ([3])
- All one dimensional representations of a semisimple Lie algebra are trivial. ([3])

Theorem 1.4.10. *Let $\mathcal{G}$ be a semisimple Lie algebra and $(\mathcal{G}, f, V)$ be a representation of it. Then ([5])*

$$V = \bigcap_{x\in\mathcal{G}} Ker(f(x)) + \sum_{x\in\mathcal{G}} Im(f(x)) \tag{1.31}$$

Proof. We use induction on the dimension of V. If $dim(V) = 1$, then the represention is trivial and so clearly

$$V = \bigcap_{x\in\mathcal{G}} Ker(f(x)) \tag{1.32}$$

Assume then that $dim(V) > 1$. If f is trivial then again $V = \bigcap_{x\in\mathcal{G}} Ker(f(x))$ and we are done, so assume f is not trivial. Now $f(\mathcal{G}) \approx \mathcal{G}/Ker(f)$ is a semisimple Lie algebra (since $Ker(f)$ is an ideal in $\mathcal{G}$). Thus we can assume with out loss of generality that $\mathcal{G} \subseteq gl(V)$.

Let Γ be the Casimir element of $(\mathcal{G}, inc_{\mathcal{G}}, V)$, where $inc_{\mathcal{G}}$ is just the usual inclusion map. The space V splits into $V_0 \oplus V_1$ where V_0 is the generalized eigenspace corresponding to 0 and V_1 is the sum of the remaining generalized eigenspaces.(If zero is not an eigenvalue then we define $V_0 = \{0\}$). Since h commutes with all the elements of $\mathcal{G}$, V_0 and V_1 are invariant spaces for all $x \in \mathcal{G}$. If both spaces are not zero then, simply use the inductive hypothesis for the restrictions of the elements of $\mathcal{G}$ onto V_0 and V_1 and the result follows. The case where $V_1 = 0$ leads to h being nilpotent, which is impossible. If V_0 is zero, then this means that h is invertible and that $V = Im(h) = \sum_i Im(x_i)$ where $\{x_i\}_{i=1}^n$ forms a basis for $\mathcal{G}$, hence we get

$$V = \sum_{x\in\mathcal{G}} Im(f(x)) \tag{1.33}$$

□

Theorem (First Whitehead lemma) 1.4.11. *If $\mathcal{G}$ is a semisimple Lie algebra over a field of characteristic zero, and if $(\mathcal{G}, f, V)$ is a representation, then $H^1(f) := H^1(\mathcal{G}, f, V) = 0$.([5])*

Proof. Let $g \in Z^1(f)$. Hence for $x, y \in \mathcal{G}$ we have $dg(x, y) = 0$ i.e.

$$f(x)g(y) - f(y)g(x) - g([x, y]) = 0 \tag{1.34}$$

Consider the representation $\hat{f} = f \otimes ad^*$ on $V \otimes \mathcal{G}^*$ or the equivalent representation $(\mathcal{G}, \hat{f}, Hom(\mathcal{G}, V))$ where for $h \in Hom(\mathcal{G}, V)$

$$(\hat{f}(x)h)(y) = f(x)h(y) - h([x, y]) \tag{1.35}$$

Since $dg = 0$, we get (by equation (1.34))

$$(\hat{f}(x)g)(y) = f(y)g(x) \tag{1.36}$$

Note that $f(y)g(x) = d(g(x))(y)$, hence if $g \in Z^1(f)$, then $\hat{f}(x)g \in B^1(f)$. It follows then that $Z^1(f)$ is $\hat{f}-$invariant and $\hat{f}$ induces a representation $x \mapsto \bar{f}(x) = \hat{f}(x)|_{Z^1(f)}$ on it. Then, by the above theorem, we get that

$$Z^1(f) = \bigcap_{x\in\mathcal{G}} Ker(\bar{f}(x)) + \sum_{x\in\mathcal{G}} Im(\bar{f}(x)) \tag{1.37}$$

It follows from above that $\sum_{x\in\mathcal{G}} Im(\bar{f}(x)) \in B^1(f)$. Hence to show that $H^1 = 0$, it suffices to show that

$$\bigcap_{x\in\mathcal{G}} Ker(\bar{f}(x)) \in B^1(f) \tag{1.38}$$

Now if $g \in \bigcap_x Ker(\bar{f}(x))$ then $(\bar{f}(x)g)(y) = 0$ for all $x, y \in \mathcal{G}$ and $dg(x, y) = 0$. It follows then that $g([x, y]) = 0$. Since $\mathcal{G}$ is semisimple, $\mathcal{G} = \mathcal{G}'$ and so $g = 0$, hence $g \in B^1(f)$ so that $H^1(f) = 0$.

□

Consequently we have

Theorem (Weyl) 1.4.12. *Let $\mathcal{G}$ be a semisimple Lie algebra and let $(\mathcal{G}, f, V)$ be a representation of $\mathcal{G}$. Then $(\mathcal{G}, f, V)$ is completely reducible, i.e. V can be split into a sum of invariant subspaces*

$$V = \bigoplus_{i=1}^{k} V_i \tag{1.39}$$

and for every i, the induced representation $(\mathcal{G}, f|_{V_i}, V_i)$, is irreducible.([5])

Remark 1.4.13. *If $(\mathcal{G}, f|_{V_i}, V_i)$ is irreducible and trivial, then it is clearly one-dimensional. We also recall that the converse is true so that in the above decomposition, the trivial representations and only them are one-dimensional.*

Corollary 1.4.14. *Let $\mathcal{G}$ be a semisimple Lie algebra and let $(\mathcal{G}, f, V)$ be a representation of it. Then*

$$V = \bigcap_{x\in\mathcal{G}} Ker(f(x)) \bigoplus \sum_{x\in\mathcal{G}} Im(f(x)) \tag{1.40}$$

([5])

Proof. Assume that f is irreducible. If f is not trivial, then the second term of the sum is an invariant non-zero subspace, so it must coincide with V and we are done. If, however, f is trivial, then the first part coincides with V. For an arbitrary representation, we use the Weyl decomposition given above. The sum of the irreducible components in which f is not zero, coincides with the sum of the images of the operators, $f(x)$, while $\bigcap_{x\in\mathcal{G}} Ker(f(x))$ is the space on which all $f(x)$ are zero, i.e. it is the sum of all irreducible one dimensional subspaces in the decomposition. □

Theorem (Second Whitehead lemma) 1.4.15. *If $\mathcal{G}$ is a semisimple Lie algebra over a field with characteristic zero and $(\mathcal{G}, f, V)$ is a representation, then $H^2(f) = 0$.([5])*

Proof. Consider the representation $\hat{f}$ related to f defined on $V \otimes C^2$, where C^2 is the space of skew-symmetric bilinear maps $g : \mathcal{G} \times \mathcal{G} \to V$. It is given by

$$(\hat{f}(x)g)(y, z) = f(x)g(y, z) - g([x, y], z) - g(y, [x, z]) \tag{1.41}$$

It follows then, that

$$dg(x, y, z) = (\hat{f}(x)g)(y, z) - d(g_x)(y, z) \tag{1.42}$$

where $g_x(z) = g(x, z)$ and g_x for $x \in \mathcal{G}$ is fixed, is considered here as a co-chain from $C^1(f)$. Now, similarly as in the proof of the first Whitehead lemma, we see that for arbitrary $x \in \mathcal{G}$, we again get

$$\hat{f}(x)g \in B^2(f) \subseteq Z^2(f) \tag{1.43}$$

This means that $Z^2(f)$ is $\hat{f}-$invariant, hence we obtain an induced representation on $Z^2(f)$, which we denote again by $\bar{f}$. Again, in a similar way as in the proof of the first Whitehead Lemma, together with the above corollary, we have

$$Z^2(f) = \bigcap_{x\in\mathcal{G}} Ker(\bar{f}(x)) \bigoplus \sum_{x\in\mathcal{G}} Im(\bar{f}(x)) \tag{1.44}$$

Note that the second term is in $B^2(f)$. Let $g \in R$ where

$$R := Z^2(f) \bigcap_{x\in\mathcal{G}} Ker(\bar{f}(x)) \tag{1.45}$$

It follows that $dg_x = 0$ and so by the first Whitehead lemma, there exists $v(x) \in V$ such that $g_x = d(v(x))$, i.e. for each $y \in \mathcal{G}$, we have $g(x,y) = f(y)v(x)$. We first show that $x \mapsto v(x)$ is linear, i.e. $v \in C^1(f)$.

If $g \in R$, we have that for all $x, y, z \in \mathcal{G}$, $(\bar{f}(x)g)(y,z) = 0$. Now we simply take two similar equations that are obtained from this one by making a permutation on the elements x, y, z, sum these equations and using $dg(x,y,z) = 0$ we get that

$$g([x,y],z) + g(y,[x,z]) - g(x,[y,z]) = 0 \tag{1.46}$$

equivalently

$$-(\bar{f}(x)g)(y,z) + f(x)g(y,z) - g(x,[y,z]) = 0 \tag{1.47}$$

and since $g \in R$, we have that $f(x)g(y,z) = g(x,[y,z])$. Now consider the splitting

$$V = V_k \oplus V_I := \bigcap_{x\in\mathcal{G}} Ker(f(x)) \oplus \sum_{x\in\mathcal{G}} Im(f(x)) \tag{1.48}$$

where V_k and V_I have obvious meanings. The relation obtained above shows that for $g \in R$, $g(x,[y,z]) \in V_I$ for all $x, y, z \in \mathcal{G}$. Now, since $\mathcal{G}$ is semisimple, it coincides with $\mathcal{G}'$ so that clearly $g(x,y) \in V_I$ for arbitrary $x, y \in \mathcal{G}$. Since V_I is $\mathcal{G}-$invariant, f induces a representation, f_I. When $g \in R$, the map g_x will be a one cocycle for f_I and so by the first Whitehead lemma, $v(x) \in V_I$.

Suppose that for $x \in \mathcal{G}$ we have $v_1(x), v_2(x) \in V_I$ with

$$f(y)v_1(x) = f(y)v_2(x) = g(x,y) \tag{1.49}$$

for arbitrary $y \in \mathcal{G}$. Then $f(y)(v_1(x) - v_2(x)) = 0$, hence $v_1(x) - v_2(x) \in V_K \cap V_I$ so that $v_1(x) = v_2(x)$. Hence $v(x)$ is uniquely determined element of V_I. From this uniqueness and the relation $f(y)v(x) = g(x,y)$, we now obtain that $x \mapsto v(x)$ is linear. Now, the relation $f(x)g(y,z) = g(x,[y,z])$ can also be written as

$$f(x)f(z)v(y) = f(x)g(y,z) = -g([y,z],x) = f(x)v([z,y]) \tag{1.50}$$

so that $f(x)(f(z)v(y) - v[z,y]) = 0$, hence $f(z)v(y) - v[z,y] \in V_K$. As this element also belongs to V_I, it must be equal to zero and we get that $f(z)v(y) = v([z,y]) = g(y,z)$ for all $z, y \in \mathcal{G}$. Now

$$d(v)(x,y) = f(x)v(y) - f(y)v(x) - v([x,y]) = v([x,y]) = -g(x,y) \tag{1.51}$$

so that $d(-v) = g$ and we are done. □

Levi decomposition theorem 1.4.16. *Let $\mathcal{G}$ be a Lie algebra and let $\mathcal{R}(\mathcal{G})$ be its radical. Then there exists a subalgebra $\mathcal{S}$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{S} \times_s \mathcal{R}(\mathcal{G})$ (semidirect product). Consequently $\mathcal{S} \sim \mathcal{G}/\mathcal{R}(\mathcal{G})$ and the algebra $\mathcal{S}$ is semisimple.* ([5])

Proof. If $\mathcal{R}(\mathcal{G})$ is abelian, the result follows from the fact that the second cohomology groups of the semisimple Lie algebra are trivial, since $\mathcal{G}$ is an extension with abelian kernel of $\mathcal{S} = \mathcal{G}/\mathcal{R}(\mathcal{G})$. So assume that $\mathcal{R}(\mathcal{G})$ is not abelian, i.e. $\mathcal{R}'(\mathcal{G}) \neq 0$. Consider the quotient $\mathcal{G}/\mathcal{R}'(\mathcal{G})$. We know that

$$(\mathcal{G}/I)/(J/I) = \mathcal{G}/J \tag{1.52}$$

so that it follows that the radical of $\mathcal{G}/\mathcal{R}'(\mathcal{G})$ is $\mathcal{R}(\mathcal{G})/\mathcal{R}'(\mathcal{G})$. Now $\mathcal{R}(\mathcal{G})/\mathcal{R}'(\mathcal{G})$ is clearly abelian and so for $\mathcal{G}/\mathcal{R}'(\mathcal{G})$, the theorem holds true. There exists a semisimple subalgebra $\mathcal{S}_0$ of $\mathcal{G}/\mathcal{R}'(\mathcal{G})$ such that

$$\mathcal{G}/\mathcal{R}'(\mathcal{G}) = \mathcal{S}_0 \times_s (\mathcal{R}(\mathcal{G})/\mathcal{R}'(\mathcal{G})) \tag{1.53}$$

Let π be the natural projection onto $\mathcal{G}/\mathcal{R}'(\mathcal{G})$. Then $H := \pi^{-1}(\mathcal{S}_0)$ is a subalgebra of $\mathcal{G}$ containing $\mathcal{R}'(\mathcal{G})$ and $H + \mathcal{R}(\mathcal{G}) = \mathcal{G}$. Also, since $H/\mathcal{R}'(\mathcal{G}) = \mathcal{S}_0$ and $\mathcal{R}'(\mathcal{G})$ is solvable, it follows that the algebra $\mathcal{R}'(\mathcal{G})$ is the radical of H.

Hence we are left with two possibilities

1. $\mathcal{R}'(\mathcal{G})$ is abelian, i.e. $\mathcal{R}^{(2)}(\mathcal{G}) = 0$
2. $\mathcal{R}'(\mathcal{G})$ is not abelian, i.e. $\mathcal{R}^{(2)}(\mathcal{G}) \neq 0$

In the first case, H splits into $\mathcal{S}_1 \times_s \mathcal{R}'(\mathcal{G})$ and so $\mathcal{G}$ splits into $\mathcal{S}_1 \times_s \mathcal{R}(\mathcal{G})$ proving the theorem.

In the second case, we have an algebra H with a radical $\mathcal{R}'(\mathcal{G})$ and we need to show that H splits into a semidirect product of its radical and some algebra. But this is clearly the same situation at the beginning with $\mathcal{R}'(\mathcal{G})$ replacing $\mathcal{R}(\mathcal{G})$. It is clear now that we can proceed as in the above and after k steps, if we encounter only the second alternative, we again split the algebra W with a radical $\mathcal{R}^{(k)}(\mathcal{G})$. Since the radical is a solvable algebra, we have for some n that $\mathcal{R}^{(n)}(\mathcal{G}) = 0$, i.e. $\mathcal{R}^{(n-1)}(\mathcal{G})$ is abelian. Clearly then we cant have the second case after $n-1$ steps so that we would have the first case and consequently the necessary splitting. □

Chapter 2

"Universal" Extensions

Herein, we introduce "universal" extensions, as they were originally introduced in [1] and with the help of cohomology of the Lie algebras, a classification of such extensions is given for low orders ([1]). Let us start with a few definitions;

Let $\mathcal{G}$ be a Lie algebra over a fixed field $\mathbb{K}$. Consider the vector space $\mathcal{G}^n = \{\vec{x} = (x_1, x_2, \ldots, x_n) : x_i \in \mathcal{G}\}$, with Lie bracket $[\ ,\ ]_W : \mathcal{G}^n \times \mathcal{G}^n \to \mathcal{G}^n$, defined as

$$([\vec{x}, \vec{y}]_W)_t = \sum_{a,b=1}^{n} W_t^{ab}[x_a, y_b]$$

where $W_t^{ab} \in \mathbb{K}$ are fixed and $[\ ,\ ] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is the bracket for $\mathcal{G}$.
Since $[\ ,\ ]_W$ is a Lie bracket for $\mathcal{G}^n$, we must have that $[\vec{x}, \vec{y}]_W = -[\vec{y}, \vec{x}]_W$, which is equivalent to the identity

$$\sum_{a,b=1}^{n} (W_t^{ab} - W_t^{ba})[x_a, y_b] = 0 \tag{2.1}$$

for all $t \in \{1, 2, \ldots n\}$. We also require that the *Jacobi* identity be satisfied. First note that

$$([\vec{x}, [\vec{y}, \vec{z}]_W]_W)_t = \sum_{a,b,c,d=1}^{n} W_t^{ab} W_b^{cd}[x_a, [y_c, z_d]]$$

Hence we require

$$0 = ([\vec{x}, [\vec{y}, \vec{z}]_W]_W)_t + ([\vec{y}, [\vec{z}, \vec{x}]_W]_W)_t + ([\vec{z}, [\vec{x}, \vec{y}]_W]_W)_t$$

for all $t \in \{1,2,\ldots n\}$. It follows that for all $t \in \{1,2,\ldots n\}$,

$$\begin{aligned}
0 &= \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[x_a,[y_c,z_d]] + \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[y_a,[z_c,x_d]] + \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[z_a,[x_c,y_d]] \\
0 &= \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[x_a,[y_c,z_d]] + \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}(-[z_c,[x_d,y_a]] - [x_d,[y_a,z_c]]) + \\
&\quad \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[z_a,[x_c,y_d]] \qquad \text{(by the Jacobi identity in } \mathcal{G}) \\
&= \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[x_a,[y_c,z_d]] - \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[x_d,[y_a,z_c]] + \\
&\quad \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[z_a,[x_c,y_d]] - \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[z_c,[x_d,y_a]] \\
&= \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[x_a,[y_c,z_d]] - \sum_{a,b,c,d=1}^{n} W_t^{cb}W_b^{da}[x_a,[y_c,z_d]] + \\
&\quad \sum_{a,b,c,d=1}^{n} W_t^{ab}W_b^{cd}[z_a,[x_c,y_d]] - \sum_{a,b,c,d=1}^{n} W_t^{db}W_b^{ac}[z_a,[x_c,y_d]]
\end{aligned}$$

$$0 = \sum_{a,b,c,d=1}^{n} (W_t^{ab}W_b^{cd} - W_t^{cb}W_b^{da})[x_a,[y_c,z_d]] + \sum_{a,b,c,d=1}^{n} (W_t^{ab}W_b^{cd} - W_t^{db}W_b^{ac})[z_a,[x_c,y_d]] \tag{2.2}$$

Now it is clear that in order to obtain the Lie bracket desired, we require the identities (2.1) and (2.2), above, to be satisfied. Since this must hold for an arbitrary Lie Algebra $\mathcal{G}$, we deduce that we must choose W so that

1. $W_k^{ij} = W_k^{ji}$ (2.3)
2. $\sum_{b=1}^{n}(W_t^{ab}W_b^{cd} - W_t^{cb}W_b^{da}) = 0 \qquad \forall i,j,k,a,b,c,d \in \{1,2,\ldots,n\}.$ (2.4)

Notice that if we define a set of n matrices $W^{(1)}, W^{(2)}, \ldots, W^{(n)}$ with $(W^{(k)})_{ij} := W_i^{kj}$ then, the property given by 2 is equivalent to

$$W^{(a)}W^{(d)} = W^{(d)}W^{(a)} \tag{2.5}$$

Hence we obtain

Proposition 2.1. *Given a Lie algebra $\mathcal{G}$ over a field $\mathbb{K}$ and n commuting matrices $W^{(1)}, W^{(2)}, \ldots, W^{(n)}$ satisfying $(W^{(i)})_{kj} = (W^{(j)})_{ki}$ for $i,j,k \in \{1,2,\ldots,n\}$, we obtain a lie algebra structure on $\mathcal{G}^n$, denoted by $\mathcal{G}^n_W$. The bracket, $[\ ,\ ]_W : \mathcal{G}^n \times \mathcal{G}^n \to \mathcal{G}^n$, is defined as follows. For $\vec{x}, \vec{y} \in \mathcal{G}^n$,*

$$([\vec{x},\vec{y}]_W)_t := \sum_{a,b=1}^{n} W_t^{ab}[x_a, y_b]$$

where $W_t^{ab} := (W^{(a)})_{tb}$ and $a,b \in \{1,2,\ldots,n\}$. We call $\mathcal{G}^n_W$ a Lie algebra extension (for reasons that will become clear later) and n, the order of the extension.[2]

As an example, if we choose the n matrices $W^{(1)}, W^{(2)}, \ldots, W^{(n)}$ to be such that $W_t^{ab} = \delta_t^a \delta_t^b$, then we obtain the usual Lie bracket on $\mathcal{G}^n$ corresponding to the direct sum structure,

$$\mathcal{G}^n = \bigoplus_{k=1}^{n} \mathcal{G}$$

Canonical Form

In the case $\mathbb{K} = \mathbb{C}$, we know that a finite set of matrices that commute, can, by a similarity transformation, be simultaneously block diagonalized, each sub-block being lower triangular with the corresponding eigenvalue on the diagonal. The transformation is defined by a non-singular matrix X on W_t^{ij} as

$$W_{t'}^{i'j'} = \sum_{i,j,t=1}^{n} ((X^{-1})_{t'}^{t} W_t^{ij} X_i^{i'}) X_j^{j'} \tag{2.6}$$

Consequently the matrices $W^{(i)}$ transform to

$$W^{(i')} = \sum_{i=1}^{n} (X^{-1} W^{(i)} X) X_i^{i'} \tag{2.7}$$

with the block structure and symmetry in the upper indices preserved, i.e. preserving the identities, (2.3) and (2.4). Clearly each sub-block corresponds to an ideal of the algebra, hence the block structure corresponds to a splitting of the algebra $\mathcal{G}_W^n$ into a direct sum. Hence we assume we are in a basis whereby the n commuting matrices are lower triangular with a single eigenvalue on the diagonal and that the tensors W_k^{ij} are symmetric in their upper indices.

Now, note that the eigenvalue of $W^{(j)}$ $(j > 1)$ is $(W^{(j)})_{11} = (W^{(1)})_{1j} = 0$. Hence we have that the $n-1$ matrices $W^{(2)}, \ldots, W^{(n)}$ are lower triangular with zeroes on the diagonal (nilpotent). We call the form of the matrices $W^{(a)}$ thus obtained, the canonical form.

Let $A = \{a = (0, a_2, \ldots, a_n); a_i \in \mathcal{G}\}$. Then for $h = (h_1, h_2, \ldots, h_n) \in \mathcal{G}_W^n, a = (0, a_2, \ldots, a_n) \in A$

$$\begin{aligned} [a, h]_1 &= \sum_{i,j=1}^{n} W_1^{ij} [a_i, h_j] \\ &= \sum_{i,j=1}^{n} W_1^{ji} [a_i, h_j] \\ &= \sum_{j=1}^{n} W_1^{j1} [a_1, h_j] \\ &= 0 \end{aligned}$$

hence $[a, h] \in A$, so that A is an ideal $\mathcal{G}_W^n$. We now have a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} \mathcal{G}_W^n \xrightarrow{\pi} G \longrightarrow 0$$

where $G = \mathcal{G}$ and i and π are the usual injection and projections respectively. If we assume the case where $W_1^{11} \neq 0$ then see appendix A, a coordinate transformation that makes $W^{(1)} = I_n$, the identity matrix.

Let $t : G \to \mathcal{G}_W^n$ be defined by $t(x) = (x, 0, \dots, 0)$. Then we have a fixed section, t, together with a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} \mathcal{G}_W^n \underset{t}{\overset{\pi}{\rightleftarrows}} G \longrightarrow 0$$

Recall the cocycle $w : G \times G \to A$ defined at (1.13) for $x, y \in G$;

$$w(x, y) = i^{-1}([t(x), t(y)] - t[x, y]) \tag{2.8}$$

hence for $x, y \in G$

$$\begin{aligned} iw(x, y) &= [t(x), t(y)] - t[x, y] \\ &= [(x, 0, \dots, 0), (y, 0, \dots, 0)] - ([x, y], 0, \dots, 0) \\ &= (W_1^{11}[x, y], 0, \dots, 0) - ([x, y], 0, \dots, 0) \\ &= 0 \end{aligned}$$

Hence t is a homomorphism so that the extension splits and $\mathcal{G}_W^n$ is a semidirect sum. The coordinate transformation that made $W^{(1)}$ the identity, removed a coboundary making the above cocycle vanish.

2.1 Solvable extension $(W_1^{11} = 0)$

Above, it was assumed that W_1^{11} was non-zero. If $W_1^{11} = 0$ then the algebra $\mathcal{G}_W^n$ is solvable and we call this case $(W_1^{11} = 0)$, the solvable case.

We assume now that $W_1^{11} = 0$, so that $\mathcal{G}_W^n$ is a solvable Lie Algebra and the matrices $W^{(i)}(1 \le i \le n)$ are all lower triangular with zeroes on the diagonal.

Let $A = \{(0, \dots, 0, x); x \in \mathcal{G}\}$. If $(0, \dots, 0, x) \in A$, $(y_1, y_2, \dots, y_n) \in \mathcal{G}_W^n$ then

$$\begin{aligned} [(0, \dots, 0, x), (y_1, y_2, \dots y_n)]_k &= \sum_{i,j=1}^{n} W_k^{ij}[x_i, y_j] \\ &= \sum_{j=1}^{n} W_k^{nj}[x_n, y_j] \\ &= 0 \end{aligned}$$

Hence $A \subseteq Z(\mathcal{G}_W^n)$, the centre of $\mathcal{G}_W^n$ and, in particular, is an abelian ideal in $\mathcal{G}_W^n$.

Definition 2.2. *Let $M = \{(0, 0, \dots, 0, b_1, \dots, b_k); b_i \in \mathcal{G}\} \subseteq \mathcal{G}_W^n$. We call the order of M, the number k.*

Now assume $\mathcal{G}_W^n$ has an abelian ideal A of order $n - m$ (note that $n - m \ge 1$). and let G be the algebra of m-tuples with bracket defined by

$$[(x_1, x_2, \dots, x_m), (y_1, y_2, \dots, y_m)]_k = \sum_{i,j=1}^{m} W_k^{ij}[x_i, y_j] \tag{2.9}$$

for $1 \le k \le m$. It is clear that

$$0 \longrightarrow A \xrightarrow{i} \mathcal{G}^n_W \underset{t}{\overset{\pi}{\rightleftarrows}} G \longrightarrow 0$$

is a short exact sequence where

$$\begin{aligned} i(x_{m+1},\ldots,x_n) &= (0,\ldots,0,x_{m+1},\ldots,x_n) \\ \pi(x_1,\ldots,x_n) &= (x_1,\ldots,x_m) \\ t(x_1,\ldots,x_m) &= (x_1,\ldots,x_m,0\ldots,0) \end{aligned}$$

and t is a fixed section. From the definition of the representation, (1.8), $\rho : G \to gl(A)$ is defined by

$$\begin{aligned} i\rho(x_1,\ldots,x_m)(y_{m+1},\ldots y_n) &= [t(x_1,\ldots,x_m), i(y_{m+1},\ldots y_n)] && (2.10) \\ &= [(x_1,\ldots,x_m,0,\ldots,0),(0,\ldots,0,y_{m+1},\ldots,y_n)] && (2.11) \\ &= \sum_{i=1}^{m}\sum_{j=m+1}^{n-1}(0,\ldots,0,W^{ij}_{m+2}[x_i,y_j],\ldots,W^{ij}_n[x_i,y_j]) && (2.12) \end{aligned}$$

The cocycle $w : G \times G \to A$ as defined at (1.13) for $\vec{x} = (x_1,\ldots,x_m), \vec{y} = (y_1,\ldots,y_m) \in G$;

$$\begin{aligned} iw(\vec{x},\vec{y}) &= [t(\vec{x}),t(\vec{y})] - t[\vec{x},\vec{y}] && (2.13) \\ &= [(x_1,\ldots,x_m,0,\ldots,0),(y_1,\ldots,y_m,0\ldots,0)] - t[(x_1,\ldots,x_m),(y_1,\ldots,y_m)] && (2.14) \\ &= \sum_{i,j=1}^{m}(0,\ldots,0,W^{ij}_{m+1}[x_i,y_j],\ldots,W^{ij}_n[x_i,y_j]) && (2.15) \end{aligned}$$

It is clear that, the parts of $(W^{(i)})_{1\le i\le n}$ which contribute to the action and cocycle can be seen if we write

$$W_{(k)} = \left(\begin{array}{c|c} \mathbf{w}_k & \mathbf{r}_k \\ \hline \mathbf{r}^t_k & 0 \end{array}\right)$$

with $m+1 \le k \le n$ and $\mathbf{w}_k$ are symmetric $m \times m$ matrices and determine the cocycle w (where $(\mathbf{w}_{(k)})_{ij} = W^{ij}_k$). $\mathbf{r}_k$ are $m \times (n-m)$ that determine the action ρ. The zero $(n-m)\times(n-m)$ matrix is a consequence of the abelian ideal A.

The algebra G is completely characterized by $(W_{(k)})_{1\le k\le m}$, hence we can repeat the same procedure on G, using the abelian ideal of maximal order of G.

Recall that associated to the short exact sequence,

$$0 \longrightarrow A \xrightarrow{i} \mathcal{G}^n_W \xrightarrow{\pi} G \longrightarrow 0$$

a 1-cochain is just a linear mapping $w^{(1)} : G \to A$, which can be represented by

$$(w^{(1)}(x_1,\ldots,x_m))_k = -\sum_{t=1}^{m} R^t_k x_t \qquad (2.16)$$

$(m+1 \leq k \leq n)$ for some set of constants R_k^t. Hence the form of a 2-coboundary can be found by acting on a 1-cochain with the coboundary operator $w^{cob} : G \times G \to A$, using (2.12) and (1.1), we get that

$$\begin{aligned}
(w^{cob}(\vec{x},\vec{y}))_t &= dw^{(1)}(\vec{x},\vec{y}) && (2.17)\\
&= \rho(\vec{x})(w^{(1)}(\vec{y})) + \rho(\vec{x})(w^{(1)}(\vec{x})) - w^{(1)}([\vec{x},\vec{y}]) && (2.18)\\
&= \sum_{i=1}^{m}\sum_{j=m+1}^{n} W_t^{ij}[x_i, w_j^{(1)}(\vec{y})] - \sum_{i=1}^{m}\sum_{j=m+1}^{n} W_t^{ij}[y_i, w_j^{(1)}(\vec{x})] && (2.19)\\
&\quad + \sum_{i,j,k=1}^{m} R_t^k W_k^{ij}[x_i, y_j] && (2.20)
\end{aligned}$$

Using (2.16) in (2.20) we obtain the general 2-coboundary form

$$(w^{cob}(\vec{x},\vec{y}))_t = \sum_{i,j,k=1}^{m} V_t^{ij}[x_i, y_j] \qquad (2.21)$$

for $m+1 \leq t \leq n$ and

$$V_t^{ij} := \sum_{q=1}^{m} R_t^q W_q^{ij} - \sum_{k=m+1}^{n} (R_k^i W_t^{jk} + R_k^j W_t^{ik}) \qquad (2.22)$$

To see how coboundaries are removed, consider the lower triangular coordinate transformation

$$M = \left(\begin{array}{c|c} \mathbf{I}_{m\times m} & \mathbf{0} \\ \hline \mathbf{k} & c\mathbf{I} \end{array}\right)$$

This transformation changes W to $\bar{W}$ where

$$\bar{W}_{(k)} = \begin{cases} W_{(k)} & 1 \leq k \leq m \\ \left(\begin{array}{c|c} c^{-1}(\mathbf{w}_k - V_k) & \mathbf{r}_k \\ \hline \mathbf{r}_k^t & 0 \end{array}\right) & m+1 \leq k \leq n \end{cases}$$

and $c \neq 0$. Since by (2.15) the block in the upper left characterizes the cocycles. It follows that the transformed cocycle is the cocycle characterized by $\mathbf{w}_{(k)}$ minus the coboundary $V_{(k)}$.

The special case, encountered often, is when the maximal abelian ideal of $\mathcal{G}_W^n$ is simple, corresponds to the subalgebra of elements of the form $(0,\ldots,0,x)$. For this case, $m = n-1$ we have $W_n^{in} = 0$ and so the action is trivial. The cocycle w is thus determined only by $W_{(n)}$ and the form of the coboundary is reduced to

$$V_n^{ij} = \sum_{t=1}^{n-1} R_n^t W_t^{ij} \qquad (2.23)$$

a linear combination of the first $n-1$ matrices.

2.2 Additional Coordinate transformations

Above, we restricted to lower triangular coordinate transformations, which preserved the lower triangular structure of the matrices $W^{(i)}$. But there are non-lower triangular coordinate transformations that preserve the lower triangular structure. Clearly this is outside the scope of cohomology theory, which is restricted to transformations that preserve the exact form of the action ρ, the algebra G and the ideal A. As we shall see (from [2]) the cohomologies are not the most relevant here, since there is another algebraic structure that is responsible for the extension we consider (see chapter 3).

We discuss a particular class of such transformations that is usefull in the classification of solvable extensions.

Consider the case where both the algebra of $(n-1)$-tuples G and of $1-$tuples A are abelian. The possible (solvable) extensions, in lower triangular form, are characterized by $W_{(k)} = 0$ $(1 \leq k \leq n-1)$.

Applying a coordinate transformation to $W_{(k)}$

$$M = \left(\begin{array}{c|c} \mathbf{m} & 0 \\ \hline 0 & c \end{array}\right)$$

(where $\mathbf{m}$ is $(n-1) \times (n-1)$ invertible matrix and c, a non-zero scalar.) we obtain $\bar{W}$ where

$$\bar{W}_{(k)} = \begin{cases} 0 & 1 \leq k \leq n-1 \\ \left(\begin{array}{c|c} c^{-1}\mathbf{m}^t \mathbf{w}_k \mathbf{m} & 0 \\ \hline 0 & 0 \end{array}\right) & k = n \end{cases}$$

Hence this transformation preserves the lower triangular form of the extension, even when $\mathbf{m}$ is not lower triangular.

2.3 Appending a Semisimple part

By the Levi decompostion theorem, we only needed to classify the solvable part of the extension. When we do have a semisimple part ($W_1^{11} \neq 0$), we shall label the matrices $W^{(0)}, W^{(1)}, \ldots, W^{(n)}$ each of size $(n+1) \times (n+1)$ and $W^{(0)} = I_{n+1}$. Hence the matrices $(W^{(i)})_{1 \leq i \leq n}$ correspond to the solvable case

Definition 2.3.1. *If the extension has a semisimple part ($W_1^{11} \neq 0$), we shall refer to it as semidirect.*

Given a solvable algebra of n-tuples, the inverse (in some sense) procedure can be applied by appending a semisimple part to the extension. More explicitly we append a column and row of zeroes to each $W^{(i)}$, changing its dimension to $(n+1) \times (n+1)$ and including $W^{(0)} = I_{n+1}$ to the set of matrices.

In this way, a semisimple extension is constructed from a solvable one, which becomes usefull in the classification of the extension.

Definition 2.3.2. *The extension obtained by appending a semisimple part to the abelian algebra of $n-$tuples will be called pure semidirect.*

Clearly pure semidirect extensions are characterized by $W^{(0)} = I$ and $W_k^{ij} = 0$ for $i, j \geq 0$.

2.4 Interchanging Semisimple and Solvable

In terms of the matrices $W^{(i)}$, the process from passing from semisimple to the solvable extension is done in the following way:

Construct the set $\{R^{(t)}\}_{t=1}^{n-1}$ of $n-1$ matrices, where $R^{(t)}$ is constructed from $W^{(t+1)}$ by deleting the first row and column. Hence it follows that it is more convenient to label the matrices using the indices $0, 1, 2, \ldots, n$ for the semisimple case and $1, 2, \ldots, n$ for the solvable case, so that $R^{(t)}$ is obtained from $W^{(t)}$.

If we, however, start with the semisimple case, i.e. the existence of n, $n \times n$ matrices $W^{(s)}$ with $s = 1, 2, \ldots, n$. We can introduce $n+1$, $(n+1) \times (n+1)$ matrices having the form

$$Q^{(0)} = I_{n+1} \qquad Q^{(t)} = \left(\begin{array}{c|c} 0 & 0 \\ \hline \mathbf{e}_t & W^{(t)} \end{array} \right) \qquad 1 \leq t \leq n$$

where $\mathbf{e}_t$ is a column vector with components defined by $(\mathbf{e_t})^s = \delta_t^s$ ([2]). Clearly this corresponds to a semisimple extension whenever $\{W^{(t)}\}_{t=1}^{n}$ corresponds to a solvable one. Hence it follows that the semisimple and solvable extension are in a one-to-one correspondence, hence it suffices to study only one of these cases.

Now let us assume the matrices, $W^{(i)}$, are in canonical form and that we are in the solvable case. The vector space $\mathcal{G}^n$ splits into

$$\mathcal{G}^n = \mathcal{F}_n^{(1)} \oplus \mathcal{G}_n^{(2)} \oplus \ldots \oplus \mathcal{F}_n^{(n)}$$

where

$$\mathcal{F}_n^{(i)} = \{\vec{x} \in \mathcal{G}^n ; (\vec{x})_j = 0 \quad \text{for} \quad j \neq i\}$$

For $1 \leq k \leq n$, define

$$\mathcal{F}[n,k] = \mathcal{F}_n^{(k)} \oplus \mathcal{F}_n^{(k+1)} \oplus \ldots \oplus \mathcal{F}_n^{(n)}$$

and $\mathcal{F}_n^{(k)} = 0$ for $k > n$ ([2]). Then it follows that

- $0 \subseteq \mathcal{F}_n^{(n)} = \mathcal{F}[n,n] \subseteq \mathcal{F}[n,n-1] \subseteq \ldots \subseteq \mathcal{F}[n,1] = \mathcal{G}_W^n$
- $[\mathcal{F}[n,k], \mathcal{F}[n,s]] \subseteq \mathcal{F}[n, max(k,s)+1]$

Also note that $\mathcal{F}_n^{(n)}$ is an abelian ideal. Define S_n^k to be the maximal abelian ideal of the form $\mathcal{F}[n, n-k+1]$.

Since $\mathcal{F}[n,n]$ is an abelian ideal, S_n^k is non zero, hence the short exact sequence,

$$0 \longrightarrow S_n^k \longrightarrow \mathcal{G}_W^n \longrightarrow \mathcal{G}_W^n / S_n^k$$

is obtained. In this way the study of $\mathcal{G}_W^n$ is reduced to $\mathcal{G}_{\bar{W}}^{n-k}$, where $\bar{W}$ is obtained in the obvious way; by simply taking the tensor W_k^{ij} and allowing the indices to run over $1, 2, \ldots, n-k$. This process is termed a "reduction" and since the above reduction can be performed by k reductions of the type $\mathcal{G}_W^n \to \mathcal{G}_{\bar{W}}^{n-1}$, we assume $k = 1$.

2.5 Leibniz Extension

A particular class of extension studied is known as the Leibniz extension ([1]). For the solvable case, this extension has the form

$$W^{(1)} := \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

or $W_k^{i1} = \delta_{k-1}^i$ ($k \geq 1$), i.e. a Jordan block. It is obvious that in order for the other matrices to commute with $W^{(1)}$ we require

$$W^{(i)} := \begin{pmatrix} 0 & & & & & \\ a & 0 & & & & \\ b & a & 0 & & & \\ c & b & a & 0 & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ & \dots & c & b & a & 0 \end{pmatrix}$$

and using symmetry of the upper indices, we must have that

$$W^{(j)} = (W^{(1)})^j \tag{2.24}$$

Equivalently we can characterize the Leibniz extension by

$$W_k^{ij} = \delta_k^{i+j} \tag{2.25}$$

($1 \leq i, j, k \leq n$).To construct the semidirect Leibniz extension, we include $W^{(0)} = I_{n+1}$ to the solvable Leibniz extension.

2.6 Low-order Extensions

We now classify algebra extension of low order according to [1]. As seen before, we only need to classify the solvable case. Classification is done up to order $n = 4$. For each case we write down the most general set of lower triangular matrices $(W^{(i)})_{1 \leq i \leq n}$ with the symmetry condition included, and then use the commutativity of the matrices.

Finally we eliminate coboundaries for each case by the methods above. Due to the lower triangular structure of the extensions, the classification found for an $m-$tuple algebra applies to the first m elements of an $n-$tuple algebra for $n \geq m$.

There are three (generic) cases encountered for any order, they are

1. The Leibniz extension.
2. The extension where $W_{(k)} = 0$ for $1 \leq k \leq n - 1$.
3. The abelian extension, where $W_{(k)} = 0$ ($1 \leq k \leq n$) (a special case of (2) above). When appended to a semidirect part, the abelian extension generates the pure semidirect extension.

We call an order n extension trivial if $W_{(n)} = 0$ since appending the "cocycle" to the order $n - 1$ extension contributes nothing.

2.6.1 n=1

Clearly, this corresponds to the abelian algebra i.e. $W_1^{11} = 0$.

2.6.2 n=2

From the above case, we get that the form of the matrices are

$$W^{(1)} = \begin{pmatrix} 0 & 0 \\ W_2^{11} & 0 \end{pmatrix}$$

and $W^{(2)} = 0$. If $W_2^{11} \neq 0$, then it can be rescaled to unity. Hence, we let $W_2^{11} = \theta_1$, where θ_1 is either zero or unity.

If $\theta_1 = 0$ then this corresponds to an abelian algebra and $\theta_1 = 1$ corresponds to the $n = 2$ Leibniz extension.

2.6.3 n=3

Using (2.6.2) above, we get that the form of the matrices are

$$W^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ \theta_1 & 0 & 0 \\ W_3^{11} & W_3^{21} & 0 \end{pmatrix}$$

$$W^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ W_3^{21} & W_3^{22} & 0 \end{pmatrix}$$

and $W^{(3)} = 0$. The requirement that the matrices commute is equivalent to

$$\theta_1 W_3^{22} = 0 \tag{2.26}$$

The symmetric matrix representing the cocycle is

$$W_{(3)} = \begin{pmatrix} W_3^{11} & W_3^{21} & 0 \\ W_3^{21} & W_3^{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$W_{(2)} = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If $W_3^{21} \neq 0$, then we can rescale it to unity. Hence, assume that $W_3^{21} = \theta_2$ where θ_2 is either zero or unity. Now if $\theta_1 = 1$ then $W_3^{22} = 0$. Then by (2.23), we can remove from $W_{(3)}$, a multiple of $W_{(2)}$, consequently, we may assume $W_3^{11} = 0$. The cocycle representation for the case $\theta_1 = 1$ is thus given by

$$W_{(3)} = \begin{pmatrix} 0 & \theta_2 & 0 \\ \theta_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that $\theta_2 = 1$ corresponds to the Leibniz extension (section (2.5)).

If instead $\theta_1 = 0$, then we use the methods of section (2.2). We can diagonalize and rescale $W_{(3)}$ such that

$$W_{(3)} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $(\lambda_1, \lambda_2) \in \{(1,1),(1,0),(0,0),(1,-1)\}$. Through suitable transformation, the cases $(\lambda_1, \lambda_2) = (1,-1)$ can be transformed so that it corresponds to $(\theta_1, \theta_2) = (0,1)$. Simalarly, $(\lambda_1, \lambda_2) = (1,0)$ corresponds to $(\theta_1, \theta_2) = (1,0)$ and $(\lambda_1, \lambda_2) = (1,1)$ to $(\theta_1, \theta_2) = (0,1)$.

In total, there are thus four extension for $n = 3$ (up to transformation), these correspond to

$$(\theta_1, \theta_2) \in \{(0,0),(0,1),(1,0),(1,1)\} \tag{2.27}$$

2.6.4 n=4

Using sections (2.6.3) and (2.6.2) we get that $W_{(1)}$,$W_{(2)}$ and $W_{(3)}$ are given as in the case $n = 3$, with an extra row and column of zeroes appended and

$$W_{(4)} = \begin{pmatrix} W_4^{11} & W_4^{21} & W_4^{31} & 0 \\ W_4^{21} & W_4^{22} & W_4^{32} & 0 \\ W_4^{31} & W_4^{32} & W_4^{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

commutativity of the matrices lead to

$$\begin{aligned} \theta_2 W_4^{33} &= 0 && (2.28) \\ \theta_2 W_4^{31} &= \theta_1 W_4^{22} && (2.29) \\ \theta_2 W_4^{32} &= 0 && (2.30) \\ \theta_1 W_4^{33} &= 0 && (2.31) \end{aligned}$$

There are hence four cases to look at.

1. $(\theta_1, \theta_2) = (0,0)$. Diagonalizing $W_{(4)}$ we get

$$W_{(4)} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $(\lambda_1, \lambda_2, \lambda_3) \in \{(1,1,1),(1,1,0),(1,0,0),(0,0,0),(1,1,-1),(1,-1,0)\}$ hence there are six different cases. Through a coordinate transformation, the $(1,1,0)$ and $(1,-1,0)$ cases can be both mapped to case 2. The $(1,0,0)$ case can be mapped to case 3a and $(1,1,-1)$ can be mapped to the $(1,1,1)$ case. After transform the $(1,1,-1)$ we are left with

$$W_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. $(\theta_1, \theta_2) = (0,1)$. By (2.31), we are left with $W_4^{33} = W_4^{31} = W_4^{32} = 0$ and we have

$$W_{(4)} = \begin{pmatrix} W_4^{11} & W_4^{21} & 0 & 0 \\ W_4^{21} & W_4^{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

As in the previous case, we can remove W_4^{21} (by using $W_{(3)}$). After rescaling we obtain 4 distinct extensions. These are

$$W_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It turns out that the second case can be mapped to case 3c and the last two, to case 3b. It follows that there is only one independent possibility here, the trivial one, $W_{(4)} = 0$

3. $(\theta_1, \theta_2) = (1,0)$. Here, W_4^{11} can be removed in $W_{(4)}$ and from (2.31) we get that $W_4^{22} = W_4^{32} = 0$. It follows that $W_{(3)} = 0$ and

$$W_{(4)} = \begin{pmatrix} 0 & W_4^{21} & W_4^{31} & 0 \\ W_4^{21} & 0 & 0 & 0 \\ W_4^{31} & 0 & W_4^{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that elements of the form $(0,x,0,y)$ form an abelian ideal with this bracket and hence it follows that $W_4^{33} W_4^{31} = 0$. Applying a suitable upper triangular transformation we can also make $W_4^{21} W^{31} = 0$ and after rescaling, five cases emerge

$$W_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $W_{(4)} = 0$. It turns out that, here, the last case may be mapped to case 4 (where $W_4^{31} = 0$). We shall call the case corresponding to the trivial extension as case 3a and the others, cases 3b, 3c and 3d respectively.

4. $(\theta_1, \theta_2) = (1, 1)$. Again W_4^{11} and W_4^{21} can be removed by a suitable coordinate transformation. From (2.31) we have $W_4^{33} = W_4^{32} = 0$ and $W_4^{22} = W_4^{31}$. Let $\theta_3 = W_4^{22} = W_4^{31}$ so that

$$W_{(4)} = \begin{pmatrix} 0 & 0 & \theta_3 & 0 \\ 0 & \theta_3 & 0 & 0 \\ \theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly $\theta_3 = 1$ corresponds to the Leibniz extension.

Hence we see that there are 9 distinct cases for the $n = 4$ case.

Chapter 3

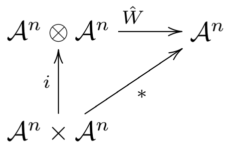
Extensions through Associative Algebras

In Chapter 2, we considered universal extensions from the viewpoint of that set originally in [1]. As shown in [2], there is another approach and we consider it here from this viewpoint, that of commutative associative algebras. It shall be seen that in this new approach the objects introduced in chapter 2 have a clearer algebraic meaning.

Let $\mathcal{A}^n$ be an n-dimensional vector space with a basis $\{e^i\}_{i=1}^n$ over $\mathbb{K}$. Define a binary operation $*$: $\mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}^n$ as follows,

$$e^i * e^j = \sum_{t=1}^{n} W_t^{ij} e^t$$

(where W_t^{ij} has the properties of (2.3) and (2.4)) and extend bi-linearly. By the universal property of tensor products, we obtain a unique linear map $\hat{W} : \mathcal{A}^n \otimes \mathcal{A}^n \to \mathcal{A}^n$ such that, the diagram



commutes (where $i : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}^n \otimes \mathcal{A}^n$ is the usual injection and $*$ the multiplication defined above). It follows that $\hat{W}(e^i \otimes e^j) = \sum W_k^{ij} e^k$. Now, since we have the linear isomorphisms,

$$Hom(\mathcal{A}^n \otimes \mathcal{A}^n, \mathcal{A}^n) \sim \mathcal{A}^n \otimes (\mathcal{A}^n \otimes \mathcal{A}^n)^* \sim \mathcal{A}^n \otimes (\mathcal{A}^n)^* \otimes (\mathcal{A}^n)^*$$

(where V^* denotes the usual dual space of V). We see that for $\hat{W} \in Hom(\mathcal{A}^n \otimes \mathcal{A}^n, \mathcal{A}^n)$, there corresponds the tensor $W \in \mathcal{A}^n \otimes (\mathcal{A}^n)^* \otimes (\mathcal{A}^n)^*$, which can be clearly represented by $\sum_{i,j,k} W_k^{ij} e^k \otimes e_i \otimes e_j$ or more simply determined by the set of constants, (W_k^{ij}). Hence, we now see that W behave as a tensor, which was noted previously and correctly stated in [1], but its exact meaning was overlooked. Note that

$$\begin{aligned} e^i * e^j - e^j * e^i &= \sum_{t=1}^{n} (W_t^{ij} - W_t^{ji}) e^t = 0 \\ (e^i * e^j) * e^k - e^i * (e^j * e^k) &= \sum_{t=1}^{n} (\sum_{\lambda=1}^{n} (W_\lambda^{ij} W_t^{\lambda k} - W_\lambda^{jk} W_t^{i\lambda})) e^t = 0 \end{aligned}$$

It follows that $(\mathcal{A}^n, *)$ is an associative commutative algebra and we denote it by $\mathcal{A}^n_W$. Conversely, given a finite dimensional commutative associative algebra over a field $\mathbb{K}$, then the structural constants W^{ij}_k obtained, satisfy the same properties as in (2.3) and (2.4), hence we obtain

Theorem 3.1. *The n-dimensional "universal" extension $\mathcal{G}^n_W$ over $\mathbb{K}$ defined by the tensors W^{ij}_k are in a one-to-one correspondence with the n-dimensional associative commutative algebras $\mathcal{A}^n_W$ over $\mathbb{K}$. The extension is obtained by setting $\mathcal{G}^n_W := \mathcal{A}^n_W \otimes_{\mathbb{K}} \mathcal{G}$, with the Lie bracket defined for elements $\vec{x} \otimes_{\mathbb{K}} \alpha, \vec{y} \otimes_{\mathbb{K}} \beta$ as*

$$[\vec{x} \otimes_{\mathbb{K}} \alpha, \vec{y} \otimes_{\mathbb{K}} \beta]_W = (\vec{x} * \vec{y}) \otimes_{\mathbb{K}} [\alpha, \beta]$$

and extending bi-linearly. The adjoint representation $T : \mathcal{A}^n_W \to End(\mathcal{A}^n_w)$ described by $a \mapsto T_a$ where $T_a(b) = a * b$ (written in the basis $\{e^i\}^n_{i=1}$), induces a matrix representation $e^i \mapsto W^{(i)} \in Mat(n, \mathbb{K})$ with $(W^{(i)})_{tj} := W^{ij}_t$.[2]

In the above, by a representation, we mean an associative algebra homomorphism $J : \mathcal{A}^n_W \to Mat(n, \mathbb{K})$. Note that it follows that any linear transformation applied to e^i corresponds the transformation in (2.6) of the tensor W^{ij}_t. It is also useful to note that in this basis, $\{e^i\}^n_{i=1}$, T_{e^i} has matrix form, $W^{(i)}$.

Note that the splitting of $\mathcal{A}^n_W$ into a sum of ideals;

$$\mathcal{A}^n_W = \bigoplus_{i=1}^{k} I^{n_i}_{W_i}$$

(where $n_i = dim(I^{n_i}_{W_i})$), corresponds to the splitting of the algebra $\mathcal{G}^n_W$ into a direct sum. This follows from Theorem (3.1) above, and the following identity;

$$\mathcal{G}^n_W = \mathcal{A}^n_W \otimes \mathcal{G} = (\bigoplus_{i=1}^{k} I^{n_i}_{W_i}) \otimes \mathcal{G} \approx \bigoplus_{i=1}^{k} (I^{n_i}_{W_i} \otimes \mathcal{G}) \tag{3.1}$$

It follows that the associative algebra $\mathcal{A}^n_W$ is simple if and only if the Lie algebra $\mathcal{G}^n_W$ is simple.

Recall that the canonical structure of the matrices $W^(i)$ (where $(W^{(i)})_{jk} = W^{ik}_j$) refer to them being lower triangular for all i and futhermore, nilpotent for $i > 1$ and for $i = 1$ a repeated eigenvalue, W^{11}_1 along the diagonal of $W^{(1)}$.

Also, recall that the solvable case (section (2.1)) corresponds to $W^{11}_1 = 0$ and the semisimple case to W^{11}_1 being non-zero. Now it is clear that

Proposition 3.2. *The canonical structure of the matrices $W^{(i)}$ is equivalent to the requirement that in the solvable case, in the basis $\{e^i\}^n_{i=1}$, we have $e^i * e^j = \sum_{t>max(i,j)} W^{ij}_t e^t$ $(1 \leq i, j, n)$ and to extend to the semi-simple case, one more independent element e^0 must be introduced such that for all $0 \leq i \leq n$ $e^0 * e^i = e^i = e^i * e^0$ i.e. a unit. [2]*

This summarizes neatly, into the following: The solvable case corresponds to having all the operators T_a $(a \in \mathcal{A}^n_W)$ nilpotent and the semi-simple case corresponds to the algebras with a unit.
We now prove a result about associative algebras containing nilpotent elements needed later.

Proposition 3.3. *Let A be an associative algebra consisting entirely of nilpotent elements (i.e. $\forall x \in A \exists n \in \mathbb{N}$ such that $x^n = 0$), then $\exists y \in A \setminus \{0\}$ such that $yA = \{0\}$.[8].*

Proof. First note that if $x, y \in A$ such that $xy = x$, then $x = 0$, since there exists a natural number n such that $y^n = 0$, hence it follows that $x = xy = xy^2 = \ldots = xy^n = 0$. Secondly, if $x \in xA$ then $x = 0$, since if $x \in xA$ then $x = xy$ for some $y \in A$, hence $x = 0$. Now finally, let x be such that xA has minimal non-zero dimension (If $xA = \{0\}$ for all x then we are done and in fact the multiplication is clearly trivial). Take $y \in xA \setminus \{0\}$. Note that $y \notin yA$ (by above), but $y \in xA$, hence $yA \subsetneq xA$. Hence by minimality of dimensions of xA, we conclude that $yA = \{0\}$ and we are done. □

Let ($\mathcal{A}$,*) be an n-dimensional commutative associative algebra over $\mathbb{C}$ which can't be split into ideals. Let $a \in \mathcal{A}$; and $T_a \in End(\mathcal{A})$ is the adjoint action, $T_a(b) = a * b$. The Jordan decomposition gives $T_a = S_a + N_a$ (where S_a is diagonalizable and N_a nilpotent). We thus have that $S_a = P(T_a)$, $N_a = Q(T_a)$ (for some polynomials P and Q with coefficients in $\mathbb{C}$). Hence $S_a = P(T_a) = T_b$ and $N_a = Q(T_a) = T_c$ for some $b, c \in \mathcal{A}$ and $S_a N_a = N_a S_a$.

Now assume that for some $a \in \mathcal{A}$, $S_a \neq 0$. As S_a is diagonalizable, $\mathcal{A}$ can be split into eigenspaces invariant under S_a ($\mathcal{A} = \oplus_{i=1}^{k} \mathcal{A}_i$ where $\mathcal{A}_i = \{v \in \mathcal{A} : S_a v = \lambda_i v\}$) and $\{\lambda_i\}_{i=1}^{k}$ is the set of eigenvalues for S_a.

Now if $x \in \mathcal{A}$ then for $v \in \mathcal{A}_i$ we get that

$$S_a(T_x(v)) = T_b T_x(v) = T_x T_b(v) = T_x S_a(v) = \lambda_i T_x(v)$$

Hence $T_x(\mathcal{A}_i) \subseteq \mathcal{A}_i$. That is, for every $x \in \mathcal{A}$, $\mathcal{A}_i$ is invariant under T_x. It follows that the $\mathcal{A}_i$ are ideals, hence there is only one eigenvalue, λ, for S_a (since we assumed no non-trivial ideals for $\mathcal{A}$). Hence $S_a = T_b = \lambda id_{\mathcal{A}}$. Since $S_a \neq 0$ we have that $\lambda \neq 0$ and so setting $e^0 := \lambda^{-1} b$ we obtain a unit for $\mathcal{A}$.

Consequently, the absence of a unit in $\mathcal{A}$ means that $S_a = 0$ for all $a \in \mathcal{A}$, or equivalently that for all $a \in \mathcal{A}$, T_a is nilpotent, i.e. the solvable case. In this case considering $T : \mathcal{A} \to gl(\mathcal{A})$ as a Lie algebra homomorphism (where $(T(a))(b) = T_a(b) = a * b$ and $\mathcal{A}$ is an abelian Lie algebra), then $T(\mathcal{A}) \subseteq gl(\mathcal{A})$ consisting only of nilpotent endomorphisms, it follows that there is a basis for $\mathcal{A}$ in which the matrix representation of these endomorphisms are lower triangular with zeroes on the diagonal. In particular, since $T(\mathcal{A})$ consists only of nilpotent endomorphisms it follows (by proposition (3.3)) that there as a non-zero element $e^n \in \mathcal{A}$ such that $T_a(e^n) = a * e^n = 0$ for all $a \in \mathcal{A}$ that behaves as a zero and is consequently called a pseudo-zero element. Note that this means that $T_{e^n} = 0$ so that $T : \mathcal{A} \to gl(\mathcal{A})$ is not injective.

Now assume there is an unit, e^0. Then for $a \in \mathcal{A}$, $T_a = \lambda_a id_{\mathcal{A}} + N_a = \lambda_a id_{\mathcal{A}} + T_c$ ($\lambda_a \in \mathbb{C}$), hence there is a unique decomposition for $a \in \mathcal{A}$, that is, $a = \lambda_a e^0 + c$ where T_c is nilpotent. This is the semi-simple case. Now let $\mathcal{N} = \{x \in \mathcal{A} : T_x \text{ is nilpotent}\}$. Now by the commutativity of $*$, it follows that N is closed under $*$. Also note that for nilpotent elements $A, B \in End(\mathcal{A})$ that do commute, there exists $n_1, n_2 \in \mathbb{N}$ such that $A^{n_1} = 0 = B^{n_2}$, now

$$(A + B)^{n_1 n_2} = \sum_{k=0}^{n_1 n_2} \binom{n_1 n_2}{k} A^k B^{n_1 n_2 - k}$$

In the sum, if $k \geq n_1$ then $A^k = 0$ if not, then $k < n_1$, hence $k \leq n_1 - 1 \leq n_2(n_1 - 1)$ so that $n_1 n_2 - k \geq n_2$ and $B^{n_1 n_2 - k} = 0$. Hence $(A + B)^{n_1 n_2} = 0$, i.e. $\mathcal{N}$ is closed under addition. Now it is obvious from above that $\mathcal{N}$ is a subalgebra of $\mathcal{A}$ and because of the unique decomposition for elements in $\mathcal{A}$, $\mathcal{N}$ has codimension 1 in $\mathcal{A}$. In other words $\mathcal{A} = Span\{e^0\} \oplus \mathcal{N}$ and we see that $\mathcal{N}$ corresponds to the solvable extension. For this reason we will continue assuming that we are working in the solvable $\mathcal{A}_W^n$

The above discussion reveals that (in the canonical basis) in order to obtain the solvable case, from the semi-simple case, one forgets the unit and conversely to pass from the solvable to the semi-simple extension, one needs to introduce a unit.

More formally, to introduce a new independant element, e^0, to a commutative associative algebra $\mathcal{A}$, such that the new algebra, $\mathcal{Q}$, is also a commutative associative algebra with unital element e^0 with $\mathcal{A}$ a subalgebra of dimension one less than $\mathcal{Q}$ can be done as follows. Let $\mathcal{Q} = \mathbb{K} \times \mathcal{A}$ (the set of pairs) with multiplication $* : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ defined for pairs $(a, \vec{x}), (b, \vec{y}) \in \mathcal{Q}$ as

$$(a, \vec{x}) * (b, \vec{y}) = (ab, a\vec{y} + b\vec{x} + \vec{x} \star \vec{y})$$

where $\star : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is the multiplication on $\mathcal{A}$. Clearly $*$ is commutative whenever $\star$ is and since

$$\begin{aligned}
((a, \vec{x}) * (b, \vec{y})) * (c, \vec{z}) &= (ab, a\vec{y} + b\vec{x} + \vec{x} \star \vec{y}) * (c, \vec{z}) \\
&= (abc, ca\vec{y} + cb\vec{x} + c(\vec{x} \star \vec{y}) + ab\vec{z} + a\vec{y} \star \vec{z} + b\vec{x} \star \vec{z} + \vec{x} \star \vec{y} \star \vec{z}) \\
(a, \vec{x}) * ((b, \vec{y}) * (c, \vec{z})) &= (a, \vec{x}) \star (bc, b\vec{z} + c\vec{y} + \vec{y} \star \vec{z}) \\
&= (abc, ab\vec{z} + ac\vec{y} + a(\vec{y} \star \vec{z}) + bc\vec{x} + b\vec{z} \star \vec{x} + c\vec{y} \star \vec{x} + \vec{y} \star \vec{z} \star \vec{x})
\end{aligned}$$

it follows that

$$((a, \vec{x}) * (b, \vec{y})) * (c, \vec{z}) = (a, \vec{x}) * ((b, \vec{y}) * (c, \vec{z})) \tag{3.2}$$

so that $*$ is associative. Also $(1, 0)$ is now a unit and if $\{e^1, e^2, \ldots, e^n\}$ is a basis for $\mathcal{A}$ (where $n = dim\mathcal{A}$) then clearly $\{(1, 0), (0, e^1), (0, e^2), \ldots, (0, e^n)\}$ forms a basis for $\mathcal{Q}$ and that $E : \mathcal{A} \to \mathcal{Q}$ defined by $E(a) = (0, a)$ is a natural embedding of associative algebras.

Let us introduce an algebra with the following structure ([2]). For $k = (k_1, k_2) \in \mathbb{K}$, let

$$\begin{aligned}
([x, y])_0^{(k)} &= [x_0, y_0] \\
([x, y])_1^{(k)} &= [x_0, y_1] + [x_1, y_0] \\
([x, y])_2^{(k)} &= [x_0, y_2] + [x_2, y_0] + k_1[x_1, y_1] \\
([x, y])_3^{(k)} &= [x_0, y_3] + [x_3, y_0] + k_1([x_1, y_2] + [x_2, y_1]) \\
([x, y])_{n-1}^{(k)} &= [x_0, y_{n-1}] + [x_{n-1}, y_0] + k_1([x_1, y_{n-2}] + \ldots + [x_{n-2}, y_1]) \\
([x, y])_n^{(k)} &= [x_0, y_n] + [x_n, y_0] + k_2([x_1, y_{n-1}] + \ldots + [x_{n-1}, y_1])
\end{aligned}$$

Now it is easily checked that $[,]^{(k)}$ defines a Lie bracket on $\mathcal{G}^{n+1}$. The semisimple part of this extension corresponds to the Leibniz extension (section (2.5)) bracket, defined by

$$\bar{W}_t^{ij} = k_1 \delta_t^{i+j} \qquad 1 \leq i, j, t \leq n - 1 \tag{3.3}$$

The Lie algebra structure defining the Poisson structure of the Compressible Reduced Magneto-hydrodynamic model ([1]) is in fact a particular case of the above construction, since there, it is defined as follows; for the non-zero entries of W_t^{ij} $(0 \leq i, j, t \leq 3)$,

$$W_i^{0i} = 1 \qquad 1 \leq i \leq 3 \qquad W_3^{12} = -\beta_e = W_3^{21}$$

where β_e is a paramater. The corresponding bracket is given by (see [1]), $[,]^{(0,-\beta_e)}$ where $n = 3$.

Let's make the following constructions (recall that we are in the canonical basis and in the solvable case). Set for $1 \leq k \leq n$, $\mathcal{A}_n^{(k)} = span\{e^k\}$ and $\mathcal{A}[n,k] = \mathcal{A}_n^{(k)} \oplus \mathcal{A}_n^{(k+1)} \oplus \ldots \oplus \mathcal{A}_n^{(n)}$. For $k > n$ set $\mathcal{A}[n,k] = \{0\}$. Then clearly

$$\{0\} \subseteq \mathcal{A}_n^{(n)} = \mathcal{A}[n,n] \subseteq \mathcal{A}[n,n-1] \subseteq \ldots \subseteq \mathcal{A}[n,1] = \mathcal{A}_W^n$$

Proposition 3.4. *$\mathcal{A}[n,k] * \mathcal{A}[n,s] \subseteq \mathcal{A}[1 + max\{k,s\}]$ for $1 \leq k, s \leq n$ and $\mathcal{A}_n^{(n)}$ is an ideal.[2]*

Proof. As multiplication by e^n gives zero, it is obvious that $\mathcal{A}_n^{(n)}$ is an ideal. Note that if $1 \leq m \leq k \leq j \leq n$ then $W_m^{js} = (W^{(s)})_{mj} = 0$, hence

$$(\sum_{j=k}^{n} \lambda_j e^j) * (\sum_{t=s}^{n} \alpha_s e^s) = \sum_{j=k}^{n}\sum_{t=s}^{n} \lambda_j \alpha_s \sum_{m=1}^{n} W_m^{js} e^m \in \mathcal{A}[n, 1+k]$$

So that $\mathcal{A}[n,k] * \mathcal{A}[n,s] \subseteq \mathcal{A}[n,1+k]$. Now since $*$ is commutative the result follows. ☐

Now let us consider an extension of $\mathcal{A}_W^n$ to an algebra $\mathcal{A}^{n+1}$ assuming that the matrices, $W^{(i)}$, are put into canonical form. We can simply take our basis $\{e^i\}_{i=1}^n$ and add the independent vector e^{n+1} in such a way so that $I = \mathbb{C}e^{n+1}$ is an ideal and the sequence of associative algebras,

$$0 \longrightarrow I \xrightarrow{i} \mathcal{A}^{n+1} \xrightarrow{\pi} \mathcal{A}_W^n \longrightarrow 0$$

is exact (where i and π are the usual injections and projections). Assuming that the basis $\{e^i\}_{i=1}^{n+1}$ is also in canonical form and that $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is the (associative, commutative) multiplication on $\mathcal{A}^{n+1}$, then it follows from the exactness of the above sequence that;

$$e^i \cdot e^j = \sum_{k=1}^{n} W_k^{ij} e^k + R^{ij} e^{n+1} \quad \text{for } 1 \leq i, j \leq n \tag{3.4}$$

and by assuming that $\{e^i\}_{i=1}^{n+1}$ is in canonical form we see that

$$e^i \cdot e^{n+1} = 0 \quad \text{for } 1 \leq i \leq n+1 \tag{3.5}$$

Commutativity of $\cdot$ is equivalent to $R^{ij} = R^{ji}$ and for associativity, we require;

$$\begin{aligned}
0 &= (e^i \cdot e^j) \cdot e^k - e^i \cdot (e^j \cdot e^k) \\
&= \sum_{m=1}^{n} (W_m^{ij} e^m + R^{ij} e^{n+1}) \cdot e^k - e^i \cdot (\sum_{m=1}^{n} (W_m^{jk} e^m + R^{jk} e^{n+1})) \\
&= \sum_{m=1}^{n} W_m^{ij} (\sum_{t=1}^{n} W_t^{mk} e^t + R^{mk} e^{n+1}) - \sum_{m=1}^{n} W_m^{jk} (\sum_{t=1}^{n} W_t^{im} e^t + R^{im} e^{n+1}) \\
&= \sum_{m=1}^{n} (R^{mk} W_m^{ij} - R^{im} W_m^{jk}) e^{n+1}
\end{aligned}$$

It follows that, the conditions on R that determine such an extension are

$$(1) \qquad R^{ij} = R^{ji} \tag{3.6}$$

$$(2) \qquad \sum_{m=1}^{n} (R^{mk} W_m^{ij} - R^{im} W_m^{jk}) = 0 \tag{3.7}$$

Every element $X \in \mathcal{A}^{n+1}$ can be written uniquely in the form

$$X = \sum_{k=1}^{n} x_k e^k + x_{n+1} e^{n+1} = \vec{x} + x_{n+1} e^{n+1} \tag{3.8}$$

Now,

$$\begin{aligned}
X \cdot Y &= (\sum_{k=1}^{n} x_k e^k + x_{n+1} e^{n+1}) \cdot (\sum_{m=1}^{n} y_m e^m + y_{n+1} e^{n+1}) \\
&= (\sum_{k=1}^{n} x_k e^k) \cdot (\sum_{m=1}^{n} y_m e^m) \\
&= \sum_{k=1}^{n} \sum_{m=1}^{n} x_k y_m (\sum_{t=1}^{n} W_t^{km} e^t + R^{km} x_k y_m e^{n+1}) \\
&= \sum_{t=1}^{n} (\sum_{k=1}^{n} \sum_{m=1}^{n} x_k y_m W_t^{km}) e^t + (\sum_{k=1}^{n} \sum_{m=1}^{n} R^{km} x_k y_m) e^{n+1} \\
&= \vec{x} * \vec{y} + R(\vec{x}, \vec{y}) e^{n+1}
\end{aligned}$$

where $R : \mathcal{A}_W^n \times \mathcal{A}_W^n \to \mathbb{K}$ is a symmetric bilinear form with $R(\vec{x}, \vec{y}) = \sum_{k=1}^{n} \sum_{m=1}^{n} R^{km} x_k y_m$.

Assume now conditions (3.6) and (3.7) above. If $\vec{x} = \sum_{i=1}^{n} x_i e^i$, $\vec{y} = \sum_{j=1}^{n} y_j e^j$, $\vec{z} = \sum_{k=1}^{n} z_k e^k \in \mathcal{A}_W^n$, then

$$\begin{aligned}
R(\vec{x} * \vec{y}, \vec{z}) &= R(\sum_{m=1}^{n} (\sum_{i,j=1}^{n} x_i y_j W_m^{ij}) e^m, \vec{z}) \\
&= \sum_{a,b=1}^{n} R^{ab} (\sum_{i,j=1}^{n} x_i y_j W_a^{ij}) z_b \\
&= \sum_{i,j,a,b=1}^{n} R^{ab} W_a^{ij} x_i y_j z_b \\
&= \sum_{i,j,a,b=1}^{n} R^{ia} W_a^{jb} x_i y_j z_b \\
&= \sum_{a,b=1}^{n} R^{ab} x_a \sum_{j,k=1}^{n} y_j z_k W_b^{jk} \\
&= R(\vec{x}, \sum_{t=1}^{n} (\sum_{j,k=1}^{n} y_j z_k W_t^{jk}) e^t) \\
&= R(\vec{x}, \vec{y} * \vec{z})
\end{aligned}$$

So that $R(\vec{x} * \vec{y}, \vec{z}) = R(\vec{x}, \vec{y} * \vec{z})$. We now define cohomologies related to $\mathcal{A}_W^n$ such that R has the same meaning with respect to the Lie Algebra structure as in [1]. We reintroduce the cohomology complex as in [2] corresponding to the algebra $\mathcal{A}_W^n$.

- The cochains are defined as follows. Let $C^0(\mathcal{A}_W^n) = \mathbb{K}$ and for $s \geq 1$, let $C^s(\mathcal{A}_W^n)$ be the set of all s-linear functions $w_s : (\mathcal{A}_W^n)^s \to \mathbb{K}$.

- The coboundary operators d_i are defined as follows. For $\alpha \in C^0(\mathcal{A}_W^n) = \mathbb{K}$, $d_0(\alpha)(a) = 0$ and for $w_i \in C^i(\mathcal{A}_W^n)$ $(i \geq 1)$, $d_i(w_i) \in C^{i+1}(\mathcal{A}_W^n)$ is defined by

$$d_i w_i(a_1, a_2, \ldots, a_{i+1}) = \sum_{k=1}^{i} (-1)^k w_i(a_1, a_2, \ldots, a_k * a_{k+1}, \ldots, a_{i+1})$$

  for $a_j \in \mathcal{A}_W^n$ with $1 \leq j \leq i+1$.

The cohomology groups obtained will be denoted by $H^i(\mathcal{A}_W^n)$ while $Z^i(\mathcal{A}_W^n)$ and $B^i(\mathcal{A}_W^n)$ correspond to the set of cocycles and coboundaries respectively.

Now since $Kerd_2 = \{w_2 \in C^2(\mathcal{A}_W^n) : w_2(a * b, c) = w_2(a, b * c)\}$, we see that R above defines a cohomology class in $H^2(\mathcal{A}_W^n)$. Also, for $\alpha \in C^1$ a linear map;

$$d_1(\alpha)(x, y) = -\alpha(x * y) \tag{3.9}$$

hence $R = d_1(-\sum_{k=1}^{n} \lambda^k e_k)$, where $\{e_k\}_{k=1}^n$ forms the dual basis in $(\mathcal{A}_W^n)^*$. Hence we see that for the cohomologies, the correspondence between the Lie algebras $\mathcal{G}_W^n$ and the commutative associative algebras $\mathcal{A}_W^n$ holds as the spaces $H^2(\mathcal{A}_W^n)$ and $H^2(\mathcal{G}_W)$ are clearly equivalent.

Let us consider the example given in [2]; We construct the algebra $\mathcal{A}_W^n$ in the following way. Let $T_n = \{1, 2, \ldots, n\}$ and let $\mathcal{M} = \{e^1, e^2, \ldots, e^n\}$ be a commutative monoid with binary operation $* : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ such that

$$e^i * e^j = e^{f(i,j)} \tag{3.10}$$

where $f : T \times T \to T$ such that

1. $f(i, f(j, k)) = f(f(i, j), k)$

2. $f(i, j) = f(j, i)$

Now define $\mathcal{A}_W^n$ to be the vector space generated by $\mathcal{M}$ and define $W_k^{ij} = \delta_k^{f(i,j)}$ to obtain a "universal" extension tensor ([2]).

In particular, let f define the multiplication in Z_n, the ring of integers $modn$, i.e.

$$W_k^{ij} = \delta_k^{ij(modn)} \tag{3.11}$$

The case $n = 2$ is trivial while for $n = 3$, the matrices $W^{(i)}$ are given by

$$W^{(0)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad W^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad W^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Assuming that $W^{(i)}$ correspond to the generators e^i, then applying a transformation defined by;

$$q^0 = e^0 \qquad q^1 = -e^0 + \frac{1}{2}e^1 + \frac{1}{2}e^2 \qquad q^2 = \frac{1}{2}e^1 - \frac{1}{2}e^1 - \frac{1}{2}e^2$$

transforms $W^{(i)}$ into

$$Q^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that the corresponding extension splits; $\mathcal{G}_W^3 = \mathcal{G} \oplus \mathcal{G} \oplus \mathcal{G}$. For $n = 4$, let $q^0 = e^0$, $q^i = e^i - e^0$ $(i > 0)$, then

$$q^i * q^0 = \delta_0^i q^i \qquad q^i * q^j = q^{ijmod4}$$

Then, in this basis, the matrices $W^{(i)}$ have block diagonal form, so that the extension splits into

$$\mathcal{G}_W^4 = \mathcal{G} \oplus \mathcal{G}_Q^3$$

where Q_k^{ij} $(1 \leq i, j, k \leq 3)$ is a new tensor corresponding to $\mathcal{G}_Q^3$. It is defined by the matrices $Q^{(i)}$ where

$$Q^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q^{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The transformation obtained by $f^1 = q^1$, $f^2 = q^1 - q^3$ and $f^3 = q^2$, changes the matrices $Q^{(i)}$ into $F^{(i)}$ with

$$F^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad F^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad F^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Truncating the semisimple part leads to

$$M^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad M^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Chapter 4

Casimir Invariants

The notion of a Casimir operator first appeared in the physical literature related with Quantum Mechanics. The name originates from the paper [9].

The mathematical study of the Casimir operators revealed that for a semisimple Lie algebra $\mathcal{G}$ the universal enveloping algebra, $U(\mathcal{G})$, contains elements that commute with $\mathcal{G}$ (Casimir elements) and that they span a vector space of dimension r, where r is the rank of $\mathcal{G}$.

Remark 1. *The universal enveloping algebra is as well known*

$$U(\mathcal{G}) = \cup_{k=0}^{\infty} G^k, \tag{4.1}$$

where G^k is the space of elements that is spanned by the (noncommutative) polynomials of degree less or equal than k of the elements of $\mathcal{G}$, see [Goto-Grosshans]. The multiplication rule in $U(\mathcal{G})$ is such that $[G^k, G^s] \subset G^{k+s}$.

Best known are the applications of the Casimirs of rank 2.The question of the Casimir elements of an arbitrary Lie algebra is a complicated one and there are results only about some specific types of algebras.

According to the classical Poincaré-Birkgoff-Witt theorem to the universal enveloping algebra, $U(\mathcal{G})$, is closely related a commutative and associative algebra $grU(\mathcal{G})$

$$grU(\mathcal{G}) = \cup_{k=0}^{\infty} P^k, \quad P^k = G^k/G^{k-1}. \tag{4.2}$$

The algebra $grU(\mathcal{G})$ is isomorphic to algebra of all polynomials on $\mathcal{G}^*$ and on it there is also a Lie algebra structure which is defined by: For $\hat{x} = x + P^{k-1}$, $\hat{y} = y + P^{s-1}$ we set

$$\{\hat{x}, \hat{y}\} = [x, y] + P^{k+s-2}. \tag{4.3}$$

It turns out that it is exactly the Poisson-Lie structure we shall define a little later. The Casimirs elements of $U(\mathcal{G})$ give rise to elements belonging to the center of the Poisson-Lie bracket structure (4.3) (also called Casimir functions). For more details see [10].

Since the Casimir functions are of course integrals of motion the interest in them is quite natural. In our context however, we shall be interested in obtaining Casimirs to the universal extensions, that is

Casimirs of the extensions we are considering in this thesis that are constructed starting from Casimirs of a given algebra $\mathcal{G}$. Thus we shall use the term "Casimir" in somewhat different sense we specify below.

We will now introduce the related *Casimir Invariants* to the universal extensions given in [1] and use the algebraic approach developed in chapter 3 to clarify their algebraic meaning. Let us begin with a few definitions;

Definition 4.1. *Let $\mathcal{M}$ be a manifold, we call a $\mathbb{K}$-valued function on $\mathcal{M}$, smooth, if it is infinitely differentiable and $C^\infty(\mathcal{M})$ denotes the space of all such smooth functions on $\mathcal{M}$.*

Definition 4.2. *A Poisson structure on a differential manifold $\mathcal{M}$ is a bilinear map $\{\ ,\ \} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, on the space of smooth $\mathbb{K}$-valued functions on $\mathcal{M}$ such that*

(i) $\{\ ,\ \}$ is a Lie bracket for $C^\infty(\mathcal{M})$.

(ii) $\{\ ,\ \}$ satisfies the derivation property, i.e. for $f, h, g \in C^\infty(\mathcal{M})$

$$\{fh, g\} = f\{h, g\} + h\{f, g\}$$

The bracket $\{\ ,\ \}$, is called a Poisson bracket and $(\mathcal{M}, \{\ ,\ \})$, is called a Poisson manifold.

Let $(\mathcal{H}, [\ ,\ ])$ be a finite dimensional Lie algebra and $\mathcal{H}^*$, its dual space. Define $\{\ ,\ \} : C^\infty(\mathcal{H}^*) \times C^\infty(\mathcal{H}^*) \to C^\infty(\mathcal{H}^*)$ by the following rule: for $f, h \in C^\infty(\mathcal{H}^*)$ and $\alpha \in \mathcal{H}^*$, set

$$\{f, h\}(\alpha) := \langle [df|_\alpha, dh|_\alpha], \alpha\rangle_{\mathcal{H}} \tag{4.4}$$

where the differentials $df|_\alpha, dh|_\alpha \in (\mathcal{H}^*)^* \approx \mathcal{H}$ and $\langle\ ,\ \rangle_{\mathcal{H}}$ denotes the canonical pairing of $\mathcal{H}$ and $\mathcal{H}^*$. It turns out that

Theorem 4.3. *$\{\ ,\ \}$ defines a Poisson structure on $\mathcal{H}^*$. [6]*

See Let $ad : \mathcal{H} \to gl(\mathcal{H})$ and $-ad^* : \mathcal{H} \to gl(\mathcal{H}^*)$ be the adjoint and co-adjoint representations (of the Lie algebra $\mathcal{H}$) respectively, i.e. $ad(x)(y) = ad_x(y) = [x, y]$ and $(-ad^*(x)(\alpha))(y) = -\alpha([x, y]) = \langle ad_x(y), -\alpha\rangle_{\mathcal{H}}$. Hence another formulation of equation (4.4) is

$$\{f, h\}(\alpha) = \langle dh|_\alpha, ad^*_{df|_\alpha}(\alpha)\rangle_{\mathcal{H}} \tag{4.5}$$

We now apply the above construction to $\mathcal{H} = \mathcal{G}^n_W$. Let $\alpha, \beta \in (\mathcal{G}^n_W)^* \backsimeq (\mathcal{G}^*)^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \in \mathcal{G}^*$.

Let $f \in C^\infty((\mathcal{G}^*)^n)$ and let $\partial_i f|_\alpha = (\partial_i f)_\alpha$ be the partial derivative of f with respect to α_i, i.e.

$$(\partial_i f)_\alpha(\beta_i) = df|_\alpha(0, \ldots, 0, \beta_i, 0 \ldots, 0) \qquad \beta_i \in \mathcal{G}^* \tag{4.6}$$

The Poisson bracket for $(\mathcal{G}^n_W)^*$ is constructed as follows; for $f, h \in C^\infty((\mathcal{G}^n_W)^*)$,

$$\begin{aligned}
\{f, h\}(\alpha) &= \langle [df|_\alpha, dh|_\alpha], \alpha\rangle_{\mathcal{G}} \\
&= \sum_{k=1}^{n} \langle [df|_\alpha, dh|_\alpha], \alpha_k\rangle_{\mathcal{G}} \\
&= \sum_{i,j,k=1}^{n} \langle W^{ij}_k [\partial_i f, \partial_j h], \alpha_k\rangle_{\mathcal{G}}
\end{aligned}$$

(where we consider $(\partial_i h)_\alpha \in \mathcal{G}$).

Definition 4.4. *Let $\mathcal{M}$ be a Poisson manifold. If f belongs to the center of the algebra $C^\infty(\mathcal{M})$, then we call it a Casimir function or simply a Casimir.*

It now follows that, $C \in C^\infty((\mathcal{G}^n_W)^*)$ is a Casimir if and only if for all $h \in C^\infty((\mathcal{G}^n_W)^*)$,

$$0 = \sum_{i,j,k=1}^{n} \langle W^{ij}_k [\partial_i C, \partial_j h], \alpha_k \rangle_{\mathcal{G}} = \sum_{i,j,k=1}^{n} W^{ij}_k \langle \partial_j h, ad^*_{\partial_i C}(\alpha_k) \rangle_{\mathcal{G}} \tag{4.7}$$

This is equivalent to

$$\sum_{i,k=1}^{n} W^{ij}_k ad^*_{\partial_i C}(\alpha_k) = 0 \tag{4.8}$$

Now let us consider the case where $\mathcal{G}$ is a (finite dimensional) semi-simple Lie algebra. This is equivalent to (in the case where $char(\mathbb{K}) = 0$) having non-degenerate Killing form $K : \mathcal{G} \times \mathcal{G} \to \mathbb{K}$, $K(x, y) = tr(ad_x ad_y)$ as introduced in chapter 2.

If $\mathcal{G}$ is semi-simple, then the non-degeneracy of K, induces a linear isomorphism $\hat{K} : \mathcal{G} \to \mathcal{G}^*$, $(\hat{K}(x))(y) = K(x, y)$ for $x, y \in \mathcal{G}$. This isomorphism establishes an equivalence between the adjoint and co-adjoint representations of the semi-simple Lie algebra, $\mathcal{G}$. This can be seen by the commutativity of the following diagram;

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\hat{K}} & \mathcal{G}^* \\ {\scriptstyle ad_x}\downarrow & & \downarrow{\scriptstyle -ad^*_x} \\ \mathcal{G} & \xrightarrow{\hat{K}} & \mathcal{G}^* \end{array}$$

Indeed, for $x, y, \alpha \in \mathcal{G}$,

$$\begin{aligned} -ad^*_x(\hat{K}(y))(\alpha) &= \hat{K}(y)[\alpha, x] \\ &= K([\alpha, x], y) \\ &= K(\alpha, [x, y]) \\ &= \hat{K}(ad_x(y))(\alpha) \end{aligned}$$

i.e. $-ad^*_x \circ \hat{K} = \hat{K} \circ ad_x$. From now on, we consider the case where $\mathcal{G}$ is semi-simple, finite dimensional and $char(\mathbb{K}) = 0$.

4.1 Linear Casimirs

Assume that $C^{(1)} \in C^\infty((\mathcal{G}^n_W)^*)$ is a linear Casimir with $C(\alpha) = \sum_{i=1}^n \langle y^i, \alpha_i \rangle$ ($y^i \in \mathcal{G}$). Then by equation (4.7) above, for all $f \in C^\infty((\mathcal{G}^n_W)^*$,

$$\sum_{i,j,k=1}^{n} \langle W^{ij}_k y^i, ad^*_{\partial_j f} \alpha_k \rangle_{\mathcal{G}} = 0 \tag{4.9}$$

(since $(\partial_i C)_\alpha = y^i$). Let $x_k = \hat{K}^{-1}(\alpha_k)$, then

$$\begin{aligned} 0 &= \sum_{i,j,k=1}^{n} ad^*_{\partial_j f}(\hat{K}(x_k))(W^{ij}_k y^i) \\ &= \sum_{i,j,k=1}^{n} \hat{K}(ad_{\partial_j f}(x_k))(W^{ij}_k y^i) \\ &= \sum_{i,j,k=1}^{n} \hat{K}(W^{ij}_k y^i)([\partial_j f, x_k]) \\ &= \sum_{i,j,k=1}^{n} \langle [\partial_j f, x_k], \hat{K}(W^{ij}_k y^i)\rangle \end{aligned}$$

Now since $f \in C^\infty((\mathcal{G}^n_W)^*$ was arbitrary and $\hat{K}$ is a bijection, it follows that $\sum_{i=1}^n W^{ij}_k y^i = 0$ for $1 \leq k, j \leq n$.

As an example, we could take $y_i = p_i x$ (for some $x \in \mathcal{G}, p_i \in \mathbb{K}$) where $(\vec{p})_i := p_i$ forms an eigenvector with zero eigenvalue for all matrices $W^{(\sigma)}$. Such solutions do exist, in particular, if $W^{(\sigma)}$ were in canonical form, then $p_i = \delta^i_n$ is an eigenvector (corresponding to zero eigenvalues) for all $W^{(\sigma)}$ and so $y_i = p_i x$ (for all $x \in \mathcal{G}$) satisfies the condition.

In this formulation $\vec{p}$ defines an element $\vec{p} = \sum_{i=1}^n p_i e^i \in \mathcal{A}^n_W$. Now if $\vec{p}$ is a pseudo zero element (i.e. $T_p = 0$), with $\vec{p} = \sum_{i=1}^n p_i e^i$, then if $\vec{b} = \sum_{j=1}^n b_j e^j \in \mathcal{A}^n_W$ then

$$0 = \vec{p} * \vec{b} = \sum_{i,j=1}^{n} b_j p_i e^j * e^i = \sum_{i,j,k=1}^{n} b_j p_i W^{ij}_k e^k$$

and since $\vec{b}$ was arbitrary, it follows that $\sum_{i=1}^n W^{ij}_k p_i = 0$. Consequently we obtain;

Corollary 4.5. *For every pseudo zero element $\vec{p} \in \mathcal{A}^n_W$ (i.e. $T_{\vec{p}} = 0$) there corresponds to a Casimir function of the type $C^{(1)}$. ([2])*

Note that such vectors do exist in the solvable case, but not in the semi-simple case.

4.2 Quadratic Casimirs

We now look at another class of Casimirs, the quadratic ones. We assume the Casimirs are of the form

$$C^{(2)} = \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\alpha^j(\hat{K}^{-1}(\alpha^i)) \tag{4.10}$$

where $C_{ij} = C_{ji}$ is a symmetric matrix and $\alpha^i \in \mathcal{G}^*$. Now, by equation (4.7), we obtain that

$$\sum_{i=1}^{n}(W^{ji}_k C_{ir} - W^{ji}_r C_{ik}) = 0 \tag{4.11}$$

for $1 \leq j, k, r \leq n$. Define $C : (\mathcal{A}^n_W)^* \times (\mathcal{A}^n_W)^* \to \mathbb{K}$ as a symmetric bilinear form on $(\mathcal{A}^n_W)^*$ as follows; for $x = \sum_{i=1}^n x^i e_i, y = \sum_{j=1}^n y^j e_j \in (\mathcal{A}^n_W)^*$, $C(x, y) = \sum_{i,j=1}^n C_{ij} x^i y^j$. Now define $\hat{C} : (\mathcal{A}^n_W)^* \to (\mathcal{A}^n_W)^{**} \approx \mathcal{A}^n_W$ by

$$\hat{C}(e_i) = \sum_{q=1}^{n} C_{iq} e^q \tag{4.12}$$

and extend linearly (where $\{e_k\}_{k=1}^n$ is dual to the basis $\{e^k\}_{k=1}^n$). Define $T^* : \mathcal{A}_W^n \to gl((\mathcal{A}_W^n)^*)$ by $(T_x^*\alpha)(y) = \alpha(x * y)$. Now, we introduce an identity in order to proceed.

Lemma 4.6.

$$T_{e^i}^* e_j = \sum_{q=1}^{n} W_j^{iq} e_q \tag{4.13}$$

where $1 \leq i, j \leq n$.

Proof. Let $T_{e^i}^* e_j = \sum_{q=1}^n \lambda^q e_q$, then

$$\lambda^t = (\sum_{q=1}^{n} \lambda^q e_q)(e^t) = (T_{e^i}^* e_j)(e^t) = e_j(e^i * e^t) = \sum_{k=1}^{n} e_j(W_k^{it} e^k) = W_j^{it}$$

Hence $T_{e^i}^* e_j = \sum_{q=1}^n W_j^{iq} e_q$. □

Example 4.7. *Let us look at the* $n = 2$*, Leibniz extension case where* $W^{(2)} = 0$ *at section (2.6.1).*

$$W^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Note that for $C : (\mathcal{A}_W^n)^* \times (\mathcal{A}_W^n)^* \to \mathbb{C}$ *(equivalently the* 2×2 *matrix* C*) we require*

$$\sum_{i=1}^{n} W_k^{ji} C_{ir} = \sum_{i=1}^{n} W_r^{ji} C_{ik} \tag{4.14}$$

or equivalently $(W^{(j)}C)_{kr} = (W^{(j)}C)_{rk}$*, i.e.* $W^{(j)}C$ *is a symmetric matrix for all* j*. Now in the above Leibniz case; it is easy to show that*

$$C \in \left\{ \begin{pmatrix} 0 & a \\ b & c \end{pmatrix}; a, b, c \in \mathbb{C} \right\}$$

so that $C(x, y) = \sum_{i,j=1}^2 C_{ij} x^i y^j$ *where* $x = \sum_{i=1}^2 x^i e_i$, $y = \sum_{i=1}^2 y^i e_i \in (\mathcal{A}_W^2)^*$ *are all the Casimirs for this Leibniz extension.*

Now note that if we assume C is a Casimir of the type we are looking for, then

$$T_{e^i}\hat{C}(e_k) = e^i * \hat{C}(e_k) = e^i * \sum_{q=1}^{n} C_{kq} e^q = \sum_{t,q=1}^{n} C_{kq} W_t^{iq} e^t = \sum_{t,q=1}^{n} C_{qt} W_k^{iq} e^t = \hat{C}(\sum_{q=1}^{n} W_k^{iq} e_q) = \hat{C} T_{e^i}^* e_k$$

It follows that the diagram below commutes for all $a \in \mathcal{A}_W^n$,

$$\begin{array}{ccc} (\mathcal{A}_W^n)^* & \xrightarrow{T_a^*} & (\mathcal{A}_W^n)^* \\ \downarrow \hat{C} & & \downarrow \hat{C} \\ \mathcal{A}_W^n & \xrightarrow{T_a} & \mathcal{A}_W^n \end{array}$$

that is;

$$T_a \hat{C} = \hat{C} T_a^* \tag{4.15}$$

In fact, more can be said;

Proposition 4.8. *For every symmetric bilinear form, C, on $\mathcal{A}_W^n$ such that*

$$C(T_a^* x, y) = C(x, T_a^* y) \tag{4.16}$$

(or equivalently, equation (4.15)) $a \in \mathcal{A}_W^n$, $x, y \in (\mathcal{A}_W^n)^*$, there corresponds a Casimir function of the type $C^{(2)}$ and vice-versa.

Proof. By definition for $\alpha, \beta \in (\mathcal{A}_W^n)^*$, $C(\alpha, \beta) =< \hat{C}(\alpha), \beta >$ hence

$$< T_a \hat{C}(\alpha), \beta >=< \hat{C}(\alpha), T_\alpha^* \beta >= C(\alpha, T_a^*, \beta) = C(T_a^* \alpha, \beta) =< \hat{C} T_a^* \alpha, \beta >$$

so that $T_a \hat{C} = \hat{C} T_a^*$. □

Now define $\alpha \odot \beta : (\mathcal{A}_W^n)^* \times (\mathcal{A}_W^n)^* \to (\mathcal{A}_W^n)^*$ by

$$\alpha \odot \beta = T_{\hat{C}(\alpha)}^* \beta \tag{4.17}$$

Proposition 4.9. *C is a Casimir if and only if $\alpha \odot \beta = \beta \odot \alpha$ for all $\alpha, \beta \in (\mathcal{A}_W^n)^*$.*

Proof. For $y \in \mathcal{A}_W^n$ we have

$$\begin{aligned} < \alpha \odot \beta, y > &= < T_{\hat{C}(\alpha)}^* \beta, y > \\ &= < \beta, T_{\hat{C}(\alpha)} y > \\ &= < \beta, T_y \hat{C}(\alpha) > \\ &= < T_y^* \beta, \hat{C}(\alpha) > \\ &= C(\alpha, T_y^* \beta) \\ &= C(T_y^* \alpha, \beta) \\ &= C(\beta, T_y^* \alpha) \\ &= < \beta \odot \alpha, y > \end{aligned}$$

and so the result follows. □

Moreover we have;

Proposition 4.10. *Casimirs of the type $C^{(2)}$ induce a commutative, associative algebra structure on $(\mathcal{A}_W^n)^*$, for which $\hat{C} : (\mathcal{A}_W^n)^* \to \mathcal{A}_W^n$ is a homomorphism of associative algebras. The multiplication is as defined above;*

$$\alpha \odot \beta = T_{\hat{C}(\alpha)}^* \beta \tag{4.18}$$

Proof. For associativity, we see that,

$$(\alpha \odot \beta) \odot z = (T_{\hat{C}(\alpha)}^* \beta) \odot z = T_{\hat{C}(T_{\hat{C}(\alpha)}^* \beta)}^* z = T_{(T_{\hat{C}(\alpha)} \hat{C}(\beta))}^* z = T_{\hat{C}(\alpha) * \hat{C}(\beta)}^* z = T_{\hat{C}(\alpha)}^* (T_{\hat{C}(\beta)}^* z)$$

$$\alpha \odot (\beta \odot z) = T_{\hat{C}(\alpha)}^* (\beta \odot z) = T_{\hat{C}(\alpha)}^* (T_{\hat{C}(\beta)}^* z)$$

hence $(\alpha \odot \beta) \odot z = \alpha \odot (\beta \odot z)$. Note that $\odot$ is clearly bilinear and by proposition (4.9), $\odot$ is a commutative operation and compatibility with addition is obvious, so that $(\mathcal{A}_W^n)^*$ becomes an associative, commutative algebra. Now, note that for $\alpha, \beta \in (\mathcal{A}_W^n)^*$,

$$\hat{C}(\alpha \odot \beta) = \hat{C}(T_{\hat{C}(\alpha)}^* \beta) = T_{\hat{C}(\alpha)} \hat{C}(\beta) = \hat{C}(\alpha) * \hat{C}(\beta)$$

so that $\hat{C} : (\mathcal{A}_W^n)^* \to \mathcal{A}_W^n$ is a homomorphism of associative algebras. □

Corollary 4.11. *Equation* (4.11) *is equivalent to the identity* $A_{ij}^q = A_{ji}^q$*, where* A_{ij}^q *are the structural constants for* $\odot$ *above (which is just the commutativity of* $\odot$*).*

Proof. We have that

$$\sum_{q=1}^{n} A_{ij}^q e_q = e_i \odot e_j = T^*_{\hat{C}(e_i)} e_j = \sum_{t=1}^{n} C_{ti} T^*_{e^t} e_j = \sum_{q,t=1}^{n} C_{ti} W_j^{tq} e_q$$

Hence $A_{ij}^q = \sum_{t=1}^{n} W_j^{tq} C_{ti}$, and so equation (4.11) is equivalent to $A_{ij}^q = A_{ji}^q$. □

We will use this construction, to simplify some results given in [2]

4.3 Extension and reduction of the Casimir Functions

As seen in [2], we shall describe the extension and reduction of the Casimir functions attempted in [1] using equation (4.11) within the new approach, that of the algebras $\mathcal{A}_W^n$. This means that we are going to investigate the relation of the Casimir function of the extension defined by $\mathcal{A}_W^{n+1}$, $\mathcal{A}_{\hat{W}}^n$ and $\mathcal{A}_{\bar{W}}^n$ where $\mathcal{A}_W^{n+1}$ is "semisimple", $\mathcal{A}_{\hat{W}}^n$ is the corresponding "solvable" part and $\mathcal{A}_{\bar{W}}^n$ is the corresponding factor algebra- see section (2.4).

To this end, assume now that we are working in the semisimple case, i.e. there exists a basis $\{e^i\}_{i=0}^n$ for which $e^0 * e^i = e^i$ (for $0 \le i \le n$) and such that $e^n * e^i = \delta_0^i e^n$ $(0 \le i \le n)$. This is equivalent to $W_i^{0i} = \delta_j^i$ and $W_j^{ni} = \delta_0^i \delta_j^n$, or in terms of the co-adjoint action,

$$T^*_{e^0} e_i = e_i \qquad T^*_{e^n} e_i = \delta_i^n e_0 \tag{4.19}$$

Note that, if we have the solvable case, then all that is needed is the introduction of an identity e^0 to the canonical basis. Let us consider the invariance condition (equation (4.16)) of a symmetric bilinear form on $\mathcal{A}_W^{n+1}$.

Since $T^*_{e^0} = id|_{\mathcal{A}_W^{n+1}}$, the invariance with respect to $T^*_{e^0}$ is trivially satisfied. Also, the invariance conditions on the components C_{ii}, are too trivially satisfied. Consequently, we only consider invariance in the case of $C_{ij} (i \neq j)$ with respect to $T^*_{e^s} (s \neq 0)$.

Starting with the assumption that the invariance conditions on the components $C_{0i} (0 \le i \le n-1)$ with respect to $T^*_{e^n}$ are satisfied, we obtain that

$$C_{0j} = C(T^*_{e^n} e_n, e_j) = C(e_n, T^*_{e^n} e_j) = \delta_i^n C_{0n} = 0$$

when $1 \le j \le n-1$ and

$$C_{00} = C(e_0, e_0) = C(e_0, T^*_{e^n} e_n) = C(T^*_{e^n} e_0, e_n) = 0 \tag{4.20}$$

Hence we obtain $C_{0j} = 0$ for $0 \le j \le n-1$. It follows from this, that the invariance condition with respect to $T^*_{e^n}$ for all C_{ij} are satisfied. To see this we have to consider the invariance on the components

(1) $C_{ij} (1 \le i, j \le n-1)$ which follows from $C(T^*_{e^n} e_i, e_j) = 0 = C(e_i, T^*_{e^n} e_j)$

(2) C_{0n} which is a consequence of $C(T^*_{e^n} e_n, e_0) = 0 = C(e_n, T^*_{e^n} e_0)$

(3) $C_{in}(1 \leq i \leq n-1)$ this can be seen by

$$C(T^*_{e^n} e_i, e_n) = \delta^n_i C(e_0, e_n) = 0 = C(e_i, e_0) = C(e_i, T^*_{e^n} e_n) \tag{4.21}$$

Now we move onto the invariance respect to $T^*_{e^s}(1 \leq s \leq n-1)$.

If $1 \leq i, j \leq n-1$ then

$$\begin{aligned} C(T^*_{e^s} e_i, e_j) &= \sum_{k=1}^{n-1} W^{sk}_i C_{kj} + W^{s0}_i C_{0j} + W^{sn}_i C_{nj} \\ &= \sum_{k=1}^{n-1} W^{sk}_i C_{kj} \end{aligned}$$

So it follows that the invariance condition with respect to $T^*_{e^s}$ (with $1 \leq i, j, s \leq n-1$) is satisfied if and only if

$$\sum_{k=1}^{n-1} (W^{sk}_i C_{kj} - W^{sk}_j C_{kj}) = 0 \tag{4.22}$$

Finally, it is easily deduced that, invariance on the components C_{in} with respect to $T^*_{e^s}(1 \leq i, s \leq n-1)$, is satisfied if and only if

$$\sum_{k=1}^{n-1} (W^{sk}_n C_{ki} - W^{sk}_i C_{kn}) = \delta^s_i C_{0n} \tag{4.23}$$

(the case of invariance of C_{0n} with respect to $T^*_{e^s}$ yields nothing).

If we denote the solvable extension corresponding to $\mathcal{A}^{n+1}_W$ by $\mathcal{A}^n_{\hat{W}}$, then we obtain $\mathcal{A}^n_{\hat{W}}$ by setting $e^0 = 0$. Note that $\mathbb{K}e^n$ is an ideal in both $\mathcal{A}^{n+1}_W$ and $\mathcal{A}^n_{\hat{W}}$ and that $\mathcal{A}^{n-1}_{\bar{W}} = \mathcal{A}^n_{\hat{W}}/\mathbb{K}e^n$ with $\bar{W}^{ij}_k = W^{ij}_k$ $(1 \leq i, j, k \leq n-1)$. Define $\bar{C}$ to be the restriction, $C|_{\mathcal{A}^{n-1}_{\bar{W}}}$. Clearly we can assume $\mathcal{A}^{n-1}_{\bar{W}} \subseteq \mathcal{A}^n_{\hat{W}} \subseteq \mathcal{A}^{n+1}_W$ and $(\mathcal{A}^{n-1}_{\bar{W}})^* \subseteq (\mathcal{A}^n_{\hat{W}})^* \subseteq (\mathcal{A}^{n+1}_W)^*$.

From the above discussion, the results about the Casimirs we had in [1] can be reformulated into the form;

Proposition 4.12. *The invariance of C with respect to the co-adjoint action on $(\mathcal{A}^{n+1}_W)^*$ is equivalent to the invariance of $\bar{C}$ with respect to the co-adjoint action of $(\mathcal{A}^{n-1}_{\bar{W}})^*$ and the condition* (4.23)*.([2])*

Corollary 4.13. *$\bar{C}$ is a Casimir for $\mathcal{A}^{n-1}_{\bar{W}}$ whenever C is, for $\mathcal{A}^{n+1}_W$.([2])*

Corollary 4.14. *If C_{ij} $(1 \leq i, j \leq n-1)$ defines a Casimir for the algebra $\mathcal{A}^{n-1}_{\bar{W}}$, then for arbitrary solution X_i $(0 \leq i \leq n-1)$ of the system*

$$\delta^s_i X_0 + \sum_{k=1}^{n-1} W^{sk}_i X_k = \sum_{k=1}^{n-1} W^{sk}_n C_{ki} \qquad (1 \leq i, s \leq n-1) \tag{4.24}$$

and for any number C_{nn}, we obtain a Casimir for $\mathcal{A}^{n+1}_W$, by setting $C(e_n, e_n) = C_{nn}$, $C(e_0, e_i) = 0$ (for $1 \leq i \leq n-1$) and $C(e_n, e_j) = X_j$ for $0 \leq j \leq n-1$. ([2])

Clearly, by equation (4.24), it is natural to consider two cases; $W_{(n)}$ degenerate and $W_{(n)}$ non-degenerate. In coordinate free notation, the assumption that $(W^{ij}_n)_{1 \leq i,j \leq n-1}$ is non-degenerate is equivalent to the map

$$\bar{\psi} : (\mathcal{A}^{n-1}_{\bar{W}}) \to (\mathcal{A}^{n-1}_{\bar{W}})^*, \quad \bar{\psi}(x) = T^*_x e_n$$

being a bijection. In this case we obtain

Proposition 4.15. *If $\bar{\psi}$ is a bijection, then the invariance condition $C(T^*_{e^s}e_i, e_j) = C(e_i, T^*_{e^s}e_j)$ follows from $C(T^*_{e^s}e_n, e_j) = C(e_n, T^*_{e^s}e_j)$ $(1 \leq i, j, s \leq n-1)$.([2])*

Proof. Fix i. Then, since $\bar{\psi}$ is a bijection, there exists an $x_i \in \mathcal{A}^{n-1}_{\bar{W}}$ such that $T^*_{x_i}e_n = e_i$. Then

$$\begin{aligned} C(T^*_{e^s}e_i, e_j) &= C(T^*_{e^s}T^*_{x_i}e_n, e_j) = C(T^*_{e^s*x_i}e_n, e_j) = C(e_n, T^*_{e^s*x_i}e_j) \\ &= C(e_n, T^*_{x_i*e^s}e_j) = C(T^*_{x_i}e_n, T^*_{e^s}e_j) = C(e_i, T^*_{e^s}e_j). \end{aligned}$$

□

In co-ordinate form, this proposition states that, whenever $(W^{ij}_n)_{1\leq i,j\leq n-1}$ is non-degenerate, equation (4.22) follows from (4.23) (see ([1])).

Let $(\bar{g}_{ik})$ be the components of the matrix that define the unique generalized Penrose inverse (see Appendix B) of $(W^{ij}_n)_{1\leq i,j\leq n-1}$, i.e. the unique matrix $(\bar{g}_{ik})$ such that

$$(1) \qquad \sum_{a,b=1}^{n-1} W^{ia}_n \bar{g}_{ab} W^{bj}_n = W^{ij}_n$$

$$(2) \qquad \sum_{a,b=1}^{n-1} \bar{g}_{ia} W^{ab}_n \bar{g}_{bj} = \bar{g}_{ij}$$

Now define $\bar{G} : (\mathcal{A}^{n-1}_{\hat{W}})^* \to \mathcal{A}^{n-1}_{\hat{W}}$ by $\bar{G}(e_i) = \sum_{k=1}^{n-1} \bar{g}_{ik}e^k$. We look at the first case;

4.3.1 $(W^{ij}_n)_{1\leq i,j\leq n-1}$ is non-degenerate

We assume now that $(W^{ij}_n)_{1\leq i,j\leq n-1}$ is non-degenerate or equivalently, $\bar{\psi}$, is invertible. Hence $\sum_{i=1}^{n-1} \bar{g}_{im}W^{it}_n = \delta^t_m$ (for all $m, t \in \{1, 2, \ldots n-1\}$) or $\bar{G} = \bar{\psi}^{-1}$.
As correctly shown in [2], the map $\bar{\psi}$ can be used to transfer the algebraic structure from $\mathcal{A}^{n-1}_{\bar{W}}$ onto the dual space $(\mathcal{A}^{n-1}_{\bar{W}})^*$. Here we take an alternative route and note that more can be said.

Proposition 4.16. *$\bar{G} = \bar{\psi}^{-1}$ defines a Casimir of $\mathcal{A}^{n-1}_{\bar{W}}$.*

Proof. Let $x, y \in \mathcal{A}^{n-1}_{\bar{W}}$ then

$$\bar{\psi}T_y(x) = T^*_{yx}e_n = T^*_yT^*_xe_n = T^*_y\bar{\psi}(x)$$

hence $\bar{\psi}T_y = T^*_y\bar{\psi}$ so that

$$T_y\bar{\psi}^{-1} = \bar{\psi}^{-1}T^*_y$$

and so by Proposition (4.16), we have that $\bar{\psi}^{-1}$ defines a Casimir of $\mathcal{A}^{n-1}_{\bar{W}}$. □

Now it follows from Proposition (4.10), that the bilinear map, $\bar{\odot} : (\mathcal{A}^{n-1}_{\bar{W}})^* \times (\mathcal{A}^{n-1}_{\bar{W}})^* \to (\mathcal{A}^{n-1}_{\bar{W}})^*$ defined by

$$\alpha\bar{\odot}\beta = \bar{T}^*_{\bar{\psi}^{-1}(\alpha)}\beta$$

induces a commutative associative algebra structure on $(\mathcal{A}^{n-1}_{\bar{W}})^*$ for which $\bar{\psi}^{-1} : ((\mathcal{A}^{n-1}_{\bar{W}})^*, \bar{\odot}) \to (\mathcal{A}^{n-1}_{\bar{W}}, \bar{*})$ is a homomorphism of associative algebras (recall that $\bar{T}^*$ is the co-adjoint action on $(\mathcal{A}^{n-1}_{\bar{W}})^*$. It now follows from the invertibility of $\bar{\psi}$, that $\bar{\psi}$ is an isomorphism of associative algebras.

If $1 \le i, j \le n-1$ then we have

$$\begin{aligned} T^*_{e^i} e_j - \bar{T}^*_{e^i} e_j &= \sum_{q=0}^{n-1} W_j^{iq} e_j - \sum_{q=1}^{n-1} W_j^{iq} e_q \\ &= W_j^{i0} e_0 + W_j^{in} e_n \\ &= \delta_j^i e_0 + \delta_j^n \delta_0^i \\ &= \delta_j^i e_0 \\ &= e_j(e^i) e_0 \end{aligned}$$

So $T^*_{e^i} e_j - \bar{T}^*_{e^i} e_j = e_j(e^i) e_0$. Hence, for $x \in \mathcal{A}^{n-1}_{\bar{W}}$, $\alpha \in (\mathcal{A}^{n-1}_{\bar{W}})^*$

$$T^*_x \alpha = \bar{T}^*_x \alpha + \alpha(x) e_0$$

where T^* is the co-adjoint action on $(\mathcal{A}^{n+1}_W)^*$. By proposition (4.11), the structural constants $(\bar{A}^i_{jk})$ of the algebra $(\mathcal{A}^{n-1}_{\bar{W}})^*$ (where $e_i \bar{\odot} e_j = \sum_{k=1}^{n-1} \bar{A}^k_{ij} e_k$) are given by

$$\bar{A}^q_{ij} = \sum_{t=1}^{n-1} W_j^{tq} \bar{g}_{ti}$$

It is now easy to see that for C_{ij} $(1 \le i, j \le n-1)$ we have;

$$\begin{aligned} C_{ij} &= C(e_i, e_j) \\ &= C(\bar{\psi}\bar{\psi}^{-1}(e_i), e_j) \\ &= C(T^*_{\bar{\psi}^{-1}(e_i)} e_n, e_j) \\ &= C(e_n, T^*_{\bar{\psi}^{-1}(e_i)} e_j) \\ &= C(e_n, e_i \bar{\odot} e_j) + C(e_n, e_0)\bar{g}_{ij} \\ &= C(e_n, \sum_{k=1}^{n-1} \bar{A}^k_{ij} e_k) + C(e_n, e_0)\bar{g}_{ij} \end{aligned}$$

so that

$$C_{ij} = \bar{g}_{ij} C_{n0} + \sum_{k=1}^{n-1} \bar{A}^k_{ij} C_{nk}$$

These equations are of course equivalent to (4.23). Indeed, since

$$C_{ij} = C(e_n, T^*_{\bar{\psi}^{-1}(e_i)} e_j) = C(e_i, e_j) \tag{4.25}$$

we have

$$\begin{aligned} C(e_n, T^*_{\bar{\psi}^{-1}(T^*_{e^s} e_i)} e_j) &= C(e_n, T^*_{T_{e^s} \bar{\psi}^{-1}(e_i)} e_j) \\ &= C(e_n, T^*_{e^s \bar{\psi}^{-1}(e_i)} e_j) \\ &= C(e_n, T^*_{\bar{\psi}^{-1}(e_i)} T^*_{e^s} e_j) \end{aligned}$$

But this can be made even more transparent, in order to see it, let us extend $\bar{\psi}$ to $\psi : \mathcal{A}^{n+1}_W \to (\mathcal{A}^{n+1}_W)^*$ by

$$\psi(x) = T^*_x e_n$$

Then $\psi(e^n) = T^*_{e^n}e_n = e_0$ and $\psi(e^0) = T^*_{e^0}e_n = e_n$ and $\psi|_{\mathcal{A}^{n-1}_{\bar{W}}} = \bar{\psi}$. Hence it follows that ψ is non-degenerate whenever $\bar{\psi}$ is. It is now clear that we can define $G : (\mathcal{A}^{n+1}_W)^* \to \mathcal{A}^{n+1}_W$ so that $G = \psi^{-1}$ and $G(e_i) = \sum_{k=0}^{n+1} g_{ik}e^k$ where $g_{ij} = \bar{g}_{ij}$ for $1 \le i, j \le n-1$, so that $G|_{\mathcal{A}^{n-1}_{\bar{W}}} = \bar{G}$.

In exactly the same way as before, it follows that the binary operation $\odot : (\mathcal{A}^{n+1}_W)^* \times (\mathcal{A}^{n+1}_W)^* \to (\mathcal{A}^{n+1}_W)^*$ defined by

$$\alpha \odot \beta = T^*_{\psi^{-1}(\alpha)\beta}$$

$(\alpha, \beta \in (\mathcal{A}^{n+1}_W)^*)$ induces a commutative associative algebra structure on $(\mathcal{A}^{n+1}_W)^*$ with ψ being a homomorphism. Hence ψ establishes an isomorphism between $(\mathcal{A}^{n+1}_W, *)$ and $((\mathcal{A}^{n+1}_W)^*, \odot)$.

Note that in the "new" algebra, the psuedo zero is e_0 and e_n is the unity which can be seen by the identities

$$e_0 \odot e_i = 0 \quad \text{for } 1 \le i \le n \qquad e_n \odot \alpha = \alpha \quad \text{for all } \alpha \in (\mathcal{A}^{n+1}_W)^*.$$

Now let $\alpha, \beta \in (\mathcal{A}^{n+1}_W)^*$, then

$$\alpha = \alpha^0 e_0 + (\sum_{i=1}^{n} \alpha^i e_i) + \alpha^n e_n$$
$$\beta = \beta^0 e_0 + (\sum_{i=1}^{n} \beta^i e_i) + \beta^n e_n$$

Then it follows that

$$\alpha \odot \beta = (\sum_{i,j=1}^{n-1} g_{ij}\alpha^i\beta^j + \alpha^n\beta^0 + \alpha^0\beta^n)e_0 + \sum_{t=1}^{n-1}(\sum_{i,j=1}^{n-1} \alpha^i\beta^j\bar{A}^t_{ij} + \alpha^n\beta^t + \alpha^t\beta^n)e_t + \alpha^n\beta^n e_n$$

To obtain the solvable part, $\mathcal{A}^n_{\hat{W}}$, we put $e_n := 0$ and $\alpha^n, \beta^n := 0$. So that the above becomes

$$\alpha\hat{\odot}\beta = (\sum_{i,j=1}^{n-1} g_{ij}\alpha^i\beta^j)e_0 + \sum_{t=1}^{n-1}(\sum_{i,j=1}^{n-1} \alpha^i\beta^j\bar{A}^t_{ij})e_t$$

or

$$\alpha\hat{\odot}\beta = \sum_{t=0}^{n-1}(\sum_{i,j=0}^{n-1} \alpha^i\beta^j\hat{A}^t_{ij})e_t$$

where

$$\begin{aligned}
\hat{A}^t_{ij} &= \bar{A}^t_{ij} \quad 1 \le i, j, t \le n-1 \\
\hat{A}^j_{0i} &= 0 \quad 0 \le i, j \le n-1 \\
\hat{A}^0_{ij} &= g_{ij} \quad 1 \le i, j \le n-1
\end{aligned}$$

Factoring $(\mathcal{A}^{n+1}_{\bar{W}})^*$ over the ideal, $\mathbb{K}e_0$, we obtain $(\mathcal{A}^{n-1}_{\bar{W}})^*$ with

$$\alpha\bar{\odot}\beta = \sum_{t=1}^{n-1}(\sum_{i,j=1}^{n-1} \alpha^i\beta^j\bar{A}^t_{ij})e_t \quad \alpha, \beta \in (\mathcal{A}^{n-1}_{\hat{W}})*$$

Coming back, all we need is

Proposition 4.17. *The matrix* $(g_{ij})_{1\le i,j\le n-1}$ *defines an extension of* $(\mathcal{A}_{\bar{W}}^{n-1})^*$ *by*

$$\begin{aligned} e_i\hat{\odot}e_j &= e_i\bar{\odot}e_j + g_{ij}e_0 \\ e_i\hat{\odot}e_0 &= 0 \\ e_0\hat{\odot}e_0 &= 0 \quad \text{for } 1\le i,j\le n-1 \end{aligned}$$

To obtain $(\mathcal{A}_W^{n+1})^*$, we simply add the semisimple part (i.e. the identity).

Now note that for $0\le i,j\le n-1$

$$\begin{aligned} C(e_i,e_j) = C(\psi\psi^{-1}(e_i),e_j) &= C(T^*_{\psi^{-1}(e_i)}e_n,e_j) && (4.26)\\ &= C(e_n,T^*_{\psi^{-1}(e_i)}e_j) && (4.27)\\ &= C(e_n,e_i\odot e_j) && (4.28) \end{aligned}$$

so that $C_{ij}=C(e_n,e_i\odot e_j)$, i.e.

$$C_{ij}=\sum_{t=1}^{n-1}\hat{A}^t_{ij}C_{nt} \quad \text{for } 0\le i,j\le n-1$$

The relation at (4.28) explains now why we get all the symmetries needed for the coefficients C_{ij} from the coefficients C_{0n}.

4.3.2 $(W_n^{ij})_{1\le i,j\le n-1}$ is degenerate

We now consider $(W_n^{ij})_{1\le i,j\le n-1}$ degenerate and try to extend the ideas in [2] to further explain this case introduced in [1]. Define, as before

$$\psi:\mathcal{A}_W^{n+1}\to(\mathcal{A}_W^{n+1})^* \tag{4.29}$$

by $\psi(x)=T^*_xe_n$ so that

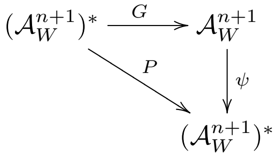
$$\psi(e^0)=e_n \quad \psi(e^n)=e_0 \quad \psi|_{\mathcal{A}_{\bar{W}}^{n-1}}=\bar{\psi}$$

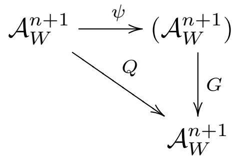
Recall that $\bar{\psi}$ is bijective if and only if ψ is. Clearly we can also extend $\bar{G}$ to

$$G:(\mathcal{A}_W^{n+1})^*\to\mathcal{A}_W^{n+1} \qquad G(e_i)=\sum_{k=0}^{n}g_{ik}e^k \tag{4.30}$$

where $(g_{ik})_{0\le i,k\le n}$ form the Penrose inverse to $(W_n^{ij})_{0\le i,j\le n}$ and $g_{ik}=\bar{g}_{ik}$ for $1\le i,k\le n-1$.

Define the projector $P=\psi G$ and let $Q=G\psi$.



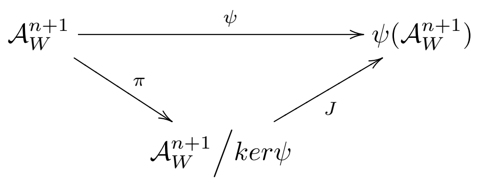


Now consider the onto function $\psi:\mathcal{A}_W^{n+1}\to\psi(\mathcal{A}_W^{n+1})$. Define $\circledast:\psi(\mathcal{A}_W^{n+1})\times\psi(\mathcal{A}_W^{n+1})\to\psi(\mathcal{A}_W^{n+1})$ by

$$\psi(x) \circledast \psi(y) = T^*_{G\psi(x)}\psi(y) = T^*_{Q(x)}\psi(y)$$

It is easily checked that $\circledast$ is a well defined binary operation. Moreover

Proposition 4.18. *$\circledast$ defines an associative commutative algebraic structure on $\psi(\mathcal{A}_W^{n+1})$ with the following diagram commutative.*



where $J(x + ker\psi) = \psi(x)$ forms an isomorphism of associative algebras, π is the natural projection and ψ is an associative algebra homomorphism.

Proof. Clearly the diagram commutes and that J is a well defined bijective linear map (isomorphism theorem for vector spaces). All that is left to show is that $\circledast$ is an associative operation and that ψ and J are homomorphisms. First note that

$$\begin{aligned}
\psi(e^i) \circledast \psi(e^j) &= T^*_{Q(e^i)}\psi(e^j) \\
&= T^*_{G\psi(e^i)}T^*_{e^j}e_n \\
&= \sum_{k=0}^{n} W_n^{ik} T^*_{G(e_k)} T^*_{e^j} e_n \\
&= \sum_{k=0}^{n} W_n^{ik} g_{kt} T^*_{e^t} T^*_{e^j} e_n \\
&= \sum_{k=0}^{n} W_n^{ik} g_{kt} T^*_{e^t * e^j} e_n \\
&= \sum_{k=0}^{n} W_n^{ik} g_{kt} T^*_{\sum_{m=0}^{n} W_m^{tj} e^m} e_n \\
&= \sum_{k=0}^{n}\sum_{m=0}^{n} W_n^{ik} g_{kt} W_m^{tj} \psi(e^m)
\end{aligned}$$

So that

$$\psi(e^i) \circledast \psi(e^j) = \sum_{k=0}^{n}\sum_{m=0}^{n} W_n^{ik} g_{kt} W_m^{tj} \psi(e^m) \tag{4.31}$$

Now

$$\begin{aligned}
(\psi(e^i) \circledast \psi(e^j)) \circledast \psi(e^p) &= \sum_{k,t,m=0}^{n} W_n^{ik} g_{kt} W_m^{tj} (\psi(e^m) \circledast \psi(e^p)) \\
&= \sum_{k,t,m,f,s,q=0}^{n} W_n^{ik} g_{kt} (W_m^{tj} W_n^{mq}) g_{qs} W_f^{sp} \psi(e^f) \\
&= \sum_{k,t,m,f,s,q=0}^{n} W_n^{ik} g_{kt} W_m^{jq} W_n^{mt} g_{qs} W_f^{sp} \psi(e^f) \\
&= \sum_{m,f,s,q=0}^{n} W_n^{im} W_m^{jq} g_{qs} W_f^{sp} \psi(e^f)
\end{aligned}$$

$$\begin{aligned}
\psi(e^i) \circledast (\psi(e^j) \circledast \psi(e^p)) &= \psi(e^i) \circledast (\sum_{k,t,m=0}^{n} W_n^{jk} g_{kt} W_m^{tp} \psi(e^m)) \\
&= \sum_{k,t,m=0}^{n} W_n^{jk} g_{kt} W_m^{tp} (\psi(e^i) \circledast \psi(e^m)) \\
&= \sum_{k,t,m,f,q,s=0}^{n} W_n^{jk} g_{kt} (W_m^{tp} W_n^{iq}) g_{qs} W_f^{sm} \psi(e^f) \\
&= \sum_{k,t,m,f,q,s=0}^{n} (W_n^{jk} g_{kt} W_n^{it}) W_m^{pq} g_{qs} W_f^{sm} \psi(e^f) \\
&= \sum_{m,f,q,s=0}^{n} W_n^{ij} W_m^{pq} g_{qs} W_f^{sm} \psi(e^f) \\
&= \sum_{m,f,q,s=0}^{n} W_n^{im} W_m^{jq} g_{qs} W_f^{sp} \psi(e^f)
\end{aligned}$$

Hence

$$(\psi(e^i) \circledast \psi(e^j)) \circledast \psi(e^p) = \psi(e^i) \circledast (\psi(e^j) \circledast \psi(e^p))$$

It follows that $\circledast$ is associative. The fact that ψ is a homomorphism is deduced from

$$\begin{aligned}
\psi(e^i) \circledast \psi(e^j) &= \sum_{k,t,m=0}^{n} W_n^{ik} g_{kt} W_m^{tj} \psi(e^m) \\
&= \sum_{k,t,m,q=0}^{n} W_n^{ik} g_{kt} W_m^{tj} W_n^{mq} e_q \\
&= \sum_{k,t,m,q=0}^{n} W_n^{ik} g_{kt} W_m^{jq} W_n^{mt} e_q \\
&= \sum_{k,t,m,q=0}^{n} W_n^{im} W_m^{jq} e_q \\
&= \sum_{k,t=0}^{n} W_n^{ik} W_k^{jt} e_t \\
&= \sum_{k,t=0}^{n} W_n^{kt} W_k^{ij} e_t \\
&= \sum_{k=0}^{n} W_k^{ij} \psi(e^k) \\
&= \sum_{k=0}^{n} \psi(W_k^{ij} e^k) \\
&= \psi(e^i * e^j)
\end{aligned}$$

and since

$$\begin{aligned}
J((x + ker\psi) * (y + ker\psi)) = J(x * y + ker\psi) &= \psi(x * y) \\
&= \psi(x) \circledast \psi(y) \\
&= J(x + ker\psi) \circledast J(y + ker\psi)
\end{aligned}$$

we see that J is a homomorphism and so the result is proved. □

Let us look at the operation $\circledast$ more closely. Note that

$$\psi(x) \circledast \psi(y) = \psi(x * y) = T^*_{x*y} e_n = T^*_x T^*_y e_n = T^*_x \psi(y) \tag{4.32}$$

Hence we see that the algebraic structure is independent of the Penrose inverse, G.

Let us extend $\circledast$ to the whole space $(\mathcal{A}_W^{n+1})^*$. Define, for $\alpha, \beta \in \mathcal{A}_W^{n+1}$,

$$\alpha \circledast \beta = T^*_{G(\beta)}(\alpha - P(\alpha)) + T^*_{G(\alpha)} \beta \tag{4.33}$$

or

$$\alpha \circledast \beta = T^*_{G(\beta)} \alpha + T^*_{G(\alpha)} \beta - T^*_{G(\beta)} P(\alpha) \tag{4.34}$$

Note that this parallels the form given in [1], where the coordinate form is given. Also,

$$\begin{aligned} T^*_{G(\beta)}P(\alpha) &= T^*_{G(\beta)}\psi G(\alpha) \\ &= T^*_{G(\beta)}T^*_{G(\alpha)}e_n \\ &= T^*_{G(\beta)*G(\alpha)}e_n \\ &= T^*_{G(\alpha)*G(\beta)}e_n \\ &= T^*_{G(\alpha)}P(\beta) \end{aligned}$$

Hence

$$\alpha \circledast \beta - \beta \circledast \alpha = T^*_{G(\alpha)}P(\beta) - T^*_{G(\beta)}P(\alpha) = 0 \tag{4.35}$$

It follows that $\circledast$ is a commutative binary operation. If we further assume that $\circledast$ is associative and that $\beta = \psi(x) \in \psi(\mathcal{A}_W^{n+1})$ then

$$\begin{aligned} \alpha \circledast \beta &= T^*_{G(\alpha)}\psi(x) \\ &= T^*_{G(\alpha)}T^*_x e_n \\ &= T^*_{G(\alpha)*x} \in \psi(\mathcal{A}_W^{n+1}) \end{aligned}$$

i.e. $\psi(\mathcal{A}_W^{n+1})$ is an ideal in $((\mathcal{A}_W^{n+1})^*, \circledast)$.

Now note that for $0 \leq i, j \leq n$ we have

$$\begin{aligned} e_i \circledast e_j &= T^*_{G(e_j)}(e_i - P(e_i)) + T^*_{G(e_i)}e_j \\ &= \sum_{m=0}^{n} g_{jm}(T^*_{e^m}e_i - \sum_{t,q=0}^{n} g_{it}W_n^{tq}T^*_{e^m}e_q) + \sum_{m=0}^{n} g_{im}T^*_{e^m}e_j \\ &= \sum_{m=0}^{n} g_{jm}(\sum_{f=0}^{n} W_i^{mf}e_f - \sum_{f,t,q=0}^{n} g_{it}W_n^{tq}W_q^{mf}e_f) + \sum_{m,f=0}^{n} g_{im}W_j^{mf}e_f \\ &= \sum_{f=0}^{n}(\sum_{m=0}^{n} g_{jm}W_i^{mf} - \sum_{t,q,m=0}^{n} g_{jm}g_{it}W_n^{tq}W_q^{mf} + \sum_{m=0}^{n} g_{im}W_j^{mf})e_f \end{aligned}$$

Hence we get

Proposition 4.19. *If $((\mathcal{A}_W^{n+1})^*, \circledast)$ is an associative algebra, then the structural constants defined by*

$$e_i \circledast e_j = \sum_{k=0}^{n} A^k_{ij}e_k$$

are given by

$$A^f_{ij} = \sum_{m=0}^{n} g_{jm}W_i^{mf} + \sum_{m=0}^{n} g_{im}W_j^{mf} - \sum_{t,q,m=0}^{n} g_{jm}g_{it}W_n^{tq}W_q^{mf}$$

Note the equivalence in the above proposition and that given in [1]. We shall come back to this but first we will now show that in the degenerate case, $C_{0n} = 0$

Given a Quadratic Casimir C on $(\mathcal{A}_W^{n+1})^*$ and if we look into the algebra $((\mathcal{A}_W^{n+1})^*, \odot)$, (where $\odot$ defines the algebra using C as before as defined in proposition (4.10)), we see that the action $L_{e_n} : (\mathcal{A}_W^{n+1})^* \to (\mathcal{A}_W^{n+1})^*$ defined by

$$L_{e_n}(\alpha) = e_n \odot \alpha = T^*_{\hat{C}(e_n)}\alpha = \psi \circ \hat{C}(\alpha)$$

gives

$$L_{e_n}(e_i) = C_{n0}e_i + \sum_{j=1}^{n} C_{ni}T^*_{e^j}e_i \tag{4.36}$$

Now, since T_{e^j} is nilpotent, $T^*_{e^j}$ is nilpotent, hence L_{e_n} has semi-simple part equal to $C_{0n}id$. Hence

Proposition 4.20. *The algebra $((\mathcal{A}_W^{n+1})^*, \odot)$, is an algebra with unity if and only if $C_{0n} \neq 0$. The pseudo zero of $((\mathcal{A}_W^{n+1})^*, \odot)$ is e_0 and the unity can be written in the form*

$$\gamma_0 = \frac{1}{C_{0n}}e_n - \sum_{j=0}^{n-1} \lambda_k e_k \tag{4.37}$$

Note the action

$$L_{e_n} = \psi \circ \hat{C} \tag{4.38}$$

by above, if $C_{0n} \neq 0$, then

$$n + 1 = rank(L_{e_n}) = rank(\psi \circ \hat{C}) \leq rank(\psi) \leq n \tag{4.39}$$

so that C_{0n} is indeed zero. Hence we have,

Proposition 4.21. *The algebra $((\mathcal{A}_W^{n+1})^*, \odot)$, does not have unity.*

Now, the idea is to generalize proposition 4.15 in accordance with [1]. Now, since ψ is degenerate, we assume the solvability condition (equation 4.11) has a solution and as shown in [1], the solvability condition (within the new framework) is easily seen to be the requirement that $T^*_{\hat{C}(e_n)}e_i \in \psi((\mathcal{A}_W^{n+1}))$ i.e. $\forall i$, $0 \leq i \leq n$ $\exists x_i \in \mathcal{A}_W^{n+1}$ such that

$$T^*_{\hat{C}(x_n)}e_i = \psi(x_i) = T^*_{x_i}e_n \tag{4.40}$$

Then we have

Proposition 4.22. *The invariance condition*

$$C(T^*_{e^k}e_j, e_i) = C(e_i, T^*_{e^k}e_j) \tag{4.41}$$

follows from the condtion that

$$C(e_i, e_j) = C(e_n, \psi(x_i) \circledast \psi(x_j)) = C(e_n, \psi(x_i * x_j)) \tag{4.42}$$

where $0 \leq i, j, s \leq n$ and x_i, x_j are as from equation (4.40) whenever we assume that $T^*_{\hat{C}(e_n)}e_i \in \psi((\mathcal{A}_W^{n+1}))$.

$$T^*_{\hat{C}x_n}e_i = \psi(x_i) = T^*_{x_i}e_n \tag{4.43}$$

Proof. Recall that $C_{0n} = 0$ (the degenerate case), and we have seen that $C_{j0} = 0$ for $0 \leq j \leq n-1$. Note that the formula given in equation (4.42) automatically ensures that $C_{s0} = 0$ for $0 \leq s \leq n$, since for e_0, the element $x_0 = e^n$ and e^n is the pseudo zero element. Also note that since

$$\psi(x_i) \circledast \psi(x_j) = T^*_{G\psi(x_i)}\psi(x_j) \tag{4.44}$$

it follows that the term $C(e_n, \psi(x_i) \circledast \psi(x_j))$ is well defined.

$$C(e_i, T^*_{e^k} e_j) = C(e_n, \psi(x_i) \circledast \psi(x_j^k)).$$

Note that where $\psi(x_j^k) = T^*_{\hat{C}(e_n)} T^*_{e^k} e_j$. Now,

$$\begin{aligned} C(e_i, T^*_{e^k} e_j) &= C(e_n, \psi(x_i) \circledast \psi(x_j^k)) \\ &= C(e_n, \psi(x_i) \circledast (T^*_{\hat{C}(e_n)} T^*_{e^k} e_j)) \\ &= C(e_n, \psi(x_i) \circledast (T^*_{e^k} T^*_{\hat{C}(e_n)} e_j)) \\ &= C(e_n, \psi(x_i) \circledast (T^*_{e^k} \psi(x_j)) \\ &= C(e_n, \psi(x_i) \circledast \psi(T_{e^k} x_j) \end{aligned}$$

From the other side,

$$\psi(x_i) \circledast \psi(T_{e^k} x_j) = \psi(x_i * (e^k * x_j)) = \psi((x_i * e^k) * x_j)) \tag{4.45}$$

and we see that

$$C(T^*_{e^k} e_j, e_i) = C(e_i, T^*_{e^k} e_j) \tag{4.46}$$

and we are done. □

Finally the formula for C_{ij} can be written also in the form;

$$\begin{aligned} C_{ij} &= C(e_n, \psi(e_i) * \psi(e_j)) \\ &= C(e_n, T^*_{G\psi(e_i)} \psi(e_j)) \\ &= C(T^*_{G\psi(e_i)} e_n, \psi(e_j)) \\ &= C(\psi(e_i), \psi(e_j)) \end{aligned}$$

We remark that comparing the form

$$C_{ij} = C(e_n, \psi(e_i) * \psi(e_j)) \tag{4.47}$$

of the coefficients C_{ij} in the case when ψ is degenerate with the form in equation (4.28) when ψ is assumed to be non-degenerate explains the remarkable similarity of these cases mentioned in [1]. Indeed the coordinate form of (4.47) is given by

$$C_{ij} = \sum_{f=0}^{n} A^f_{ij} C_{nt} \tag{4.48}$$

where A^f_{ij} are as in proposition (4.19).

4.4 Conclusion

We have considered the “universal” extensions and the Casimir invariants from the new viewpoint set out in [2], by commutative associative algebras.

We have given a clear algebraic meaning of the quadratic Casimirs via these structures - Proposition (4.22) and (4.23). Consequently, we are able to give much simpler and transparent proofs to the results about the extensions and reductions of the quadratic Casimirs- see section (4.3)

We also managed to give a clear algebraic meaning to the case where ψ was degenerate and explain the results (within the framework set out in [2]) given by [1] where [2] had left off.
We believe that taking into account the important applications of the structures we are considering in the theory of Poisson-Lie structures the results of this work will stimulate further research both in the theory and in the applications.

Appendix A

It is shown that, through a series of lower-triangular coordinate transformations, we can change $W^{(1)}$ into the identity matrix whilst preserving the lower triangular nilpotent form of $W^{(2)}, W^{(3)}, \ldots, W^{(n)}$. ([1]) By [1], if the coordinate transformation M is of the form $M = I + L$ where I is the identity and L is strictly lower triangular, then $\hat{W}^{(1)} = M^{-1}W^{(1)}M$ still has eigenvalue 1 and the matrices

$$\hat{W}^{(i)} = M^{-1}W^{(i)}M \tag{4.49}$$

$(i > 1)$ are still nilpotent. For $i > 1$ we have

$$\bar{W}_i^{11} = \hat{W}_i^{11} + \hat{W}_i^{1j}L_j^1 = \hat{W}_i^{11} + \sum_{k=2}^{i-1} \hat{W}_i^{1k}L_k^1 + L_i^1, \tag{4.50}$$

since $\hat{W}_k^{1k} = 1$. Clearly, we can always solve for L_i^1 so that W_i^{11} vanishes. It follow then that W_j^{11} becomes δ_1^j.

We now proceed inductively on i. For $i = 1$, it is clearly true. Assume now that $W_k^{1j} = \delta_k^j$ for $k < j$. Hence

$$\sum_{t=1}^{n} W_i^{t1}W_t^{1m} = \sum_{t=1}^{n} W_i^{tm}W_t^{11} = \sum_{t=1}^{n} W_i^{tm}\delta_t^1 = W_i^{1m} \tag{4.51}$$

Since $i > 1$ we get that

$$\begin{aligned} \sum_{t=2}^{n} W_i^{t1}W_t^{1m} &= \sum_{t=1}^{n} W_i^{t1}W_t^{1m} - W_i^{11}W_1^{1m} \\ &= W_i^{1m} - \delta_i^1 W_1^{1m} \\ &= W_i^{1m} \end{aligned}$$

and for $k < j$ we get

$$\sum_{t=2}^{i-1} W_i^{t1}W_t^{1m} = 0 \tag{4.52}$$

Using the inductive hypothesis,

$$\sum_{t=2}^{i-1} W_i^{t1}\delta_t^m = W_i^{m1} = 0 \tag{4.53}$$

Hence, $W_i^{m1} = \delta_i^m$ and we are done.

Appendix B

Here we introduce the notion of an inverse of a linear operator in a finite dimensional vector space over $\mathbb{C}$, i.e. we consider the operator to be a matrix.

Suppose V, W are finite dimensional vector spaces over $\mathbb{C}$ and that $A : V \to W$ is a linear map. It follows then that we can assume $V = \mathbb{C}^m$, $W = \mathbb{C}^n$ and choosing the canonical bases in these spaces, A can be regarded as a $m \times n$ matrix. We identify the spaces $\mathbb{C}^t$ and $(\mathbb{C}^t)^*$ through the bilinear form given by

$$(u, v) = \sum_{i=1}^{t} u_i \bar{v}_i \tag{4.54}$$

where the bar is complex conjugation. Then the conjugated operator (matrix) A^* is the Hermitian conjugated matrix. Consider now $Im(A)$, the image of A as a vector space of dimension r (hence A has rank r). We can assume that $A = g \circ f$ where $f : \mathbb{C}^m \to Im(A)$ and g is the inclusion map of $Im(A)$ into $\mathbb{C}^n$, i.e.

$$\mathbb{C}^m \xrightarrow{f} Im(A) \xrightarrow{g} \mathbb{C}^n, \qquad \mathbb{C}^m \xrightarrow{A} \mathbb{C}^n$$

Note that f and g are thus surjective and injective respectively. So, it now follows then that $A^* = f^* \circ g^*$ and f^* is injective, g^* is surjective. It is also obvious that $Im(A^*) = (kerA)^\perp$, $ker(A^*) = (Im(A))^\perp$. Now

Lemma 4.1. *Let $F : \mathbb{C}^r \to \mathbb{C}^n$ be an injective linear map (or equivalently of rank r). Then the operator;*

$$P = F(F^*F)^{-1}F^* \tag{4.55}$$

(where we identify $\mathbb{C}^p$ and $(\mathbb{C}^P)^*$ as before) is the orthogonal projector onto the space $Im(F)$.

Proof. Since $(Im(F))^\perp = ker(F^*)$, the map $F^*F : \mathbb{C}^r \to \mathbb{C}^r$ is invertible. Now, if $x \in Im(F)$. Then $x = F(y)$ for some $y \in \mathbb{C}^n$ and we have

$$P(x) = F(F^*F)^{-1}F^*F(y) = F(y) = x \tag{4.56}$$

If $x \in (Im(F))^\perp = ker(F^*)$ then $P(x) = 0$. It follows that P is the orthogonal projector over $Im(F)$. As every orthogonal projector P satisfies $P^2 = P$, $P^* = P$. □

We can now apply the above lemma to the injective maps g and f^* and we obtain two orthogonal projectors, P_1 onto the space $Im(A)$ and P_2, onto the space $Im(A^*)$ by

$$P_1 = g(g^*g)^{-1}g^* \quad P_2 = f^*(ff^*)^{-1}f \tag{4.57}$$

Let us define the operator $A^G : W \to V$ of A by

$$A^G = f^*(ff^*)^{-1}(g^*g)^{-1}g^* \tag{4.58}$$

Then it is easy (if not tedious) that

$$AA^G = P_1, \quad A^G A = P_2, \quad A^G P_1 = P_2 A^G, \tag{4.59}$$

$$AA^G A = A, \quad A^G AA^G = A^G \tag{4.60}$$

Definition 4.2. *The operator A^G, satisfying equations* (4.59) *and* (4.60)*, is called the generalized Penrose inverse of A.*

Proposition 4.3. *The generalized Penrose inverse exists and is unique.*

Proof. By construction, we see the existence of A^G. We now show that it is unique. Assume that there are two operators, A_1 and A_2 satisfying equations (4.59) and (4.60).

We know that $A_1 A = P_2$, hence $A_1 A A_2 = P_2 A_2$. But $A_2 A = P_2$ and $A_2 A A_2 = A_2$ implies $A_2 = P_2 A_2 = A_1 A A_2$ and we get that $A_1 A A_2 = A_2$.

Now, also since $AA_2 = P_1$ we get that $A_1 A A_2 = A_1 P_1$ and from $AA_1 = P_1$ and $A_1 A A_1 = A_1$ we get $A_1 P_1 = A_1$ which then leads to $A_1 A A_2 = A_1$. And hence from the above we get that $A_1 = A_2$. □

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