The Required *Ansatz* to Construct Lie Point Transformations and the Symmetries of a First-Order Stochastic Differential Equation [1]

Kakanyo Knowledge Masike

Supervision: Dr Ebrahim Fredericks

Co-Supervisor: Dr Francois Ebobisse Bille

Department of Mathematics and Applied Mathematics

University of the Cape Town

Private Bag x3

Rondebosch

7701

South Africa

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Declaration

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Kakanyo Knowledge Masike

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Contents

Abstract

In this thesis we demonstrate how to obtain the required *ansatz* to determine Lie point transformations of evolution-type equations from the contact transformation approach. We indicate that the Lie point transformations of the Fokker-Planck equation (FPE), which is a second-order linear parabolic partial differential equation (PDE), are projectable by using the *ansatz*. We further obtain the symmetries of a stochastic ordinary differential equation (SODE) which corresponds to those of the FPE. This is possible because there exists a relationship between an SODE and the associated (deterministic) FPE. The study of SODEs is an interesting and applicable concept in the real world and one of the building factors to this study is an Itô integral. These Itô integrals are of much use, for instance, in the field of mathematical finance whereby its use has shown the relationship between call options and their non-deterministic underlying stock prices.

Wiener processes must be considered in finding an approximation of these integrals. Acclimatization of Sophus Lie's work to SODEs has been done by (Gaeta and Quintero [2]; Wafo Soh and Mahomed [41]; Ünal [42]; Fredericks and Mahomed [43]). The determining equations for the first-order SODEs are derived in an Itô calculus context and are non-stochastic. Consequently, symmetries of an SODE are obtained without the consultation of its corresponding FPE.

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Chapter 1

Introduction

The world that we reside in is complex in nature and as scientists our elementary role is to strive to understand it better, whether it be from a chemical, biological, physical or economical point of view. Generally, the best language available to us is that of mathematics. Most often, this representation is accomplished by means of differential equations. Linear differential equations are well understood and there are many techniques available to us for solving them, such as The Method of Separation of Variables, The Integration Factor Method (Zill and Cullen [40]), etc. However, the real world is not that simple and most processes are described by nonlinear differential equations which are far more complex to solve and comprehend. It is this complexity, among other things, that inspired Sophus Lie and Felix Klein to study mathematical systems from the perspective of those transformation groups, which left the systems invariant. Klein, in his famous "Erlanger" program, pursued the role of finite groups in the studies of regular bodies and the theory of algebraic equations [50], while Lie developed his notion of continuous transformation groups and applied them to the solution of differential equations [36]. Recently, the theory of continuous groups has become a fundamental tool in such diverse areas as analysis, differential geometry, number theory, atomic structure and high-energy physics. In this thesis, Lie's theorems and their extensions are applied to differential equations.

Background on Developments made in Mathematical Sciences

One of the important developments in applied mathematics over the recent years is that many nonlinear equations can be treated as they are, without approximations, and be solved by essentially linear techniques.

One of the standard techniques for solving linear PDEs is the Fourier transformation. It was shown in former times that a class of physically interesting nonlinear PDEs can be solved by a nonlinear extension of the Fourier technique, namely the inverse scattering transform (Ablowitz and Clarkson [21]). This inverse scattering transform reduces the nonlinear equation that one is dealing with to a sequence of linear steps. This method, originally applied to the Korteweg-de Vries equation, is now known to be applicable to a large class of nonlinear evolution equations in the one space and one time variable and to some equations in higher dimensions.

Continuous group theory, Lie algebras and differential geometry play a critical role in the understanding of the structure of nonlinear PDEs, in particular for generating integrable equations, finding Lax pairs, recursion operators, Bäcklund transformations and finding exact solutions. Lax pairs are pairs of time-dependent matrices that describe solutions of differential equations. The inverse scattering transform uses the Lax equations to solve exactly solvable models. Bäcklund transformations are very useful as they are transformations in the solution space of the nonlinear equations. Thus once one has a solution to a nonlinear equation, the Bäcklund transformations generate new solutions.

Most nonlinear equations are not integrable and cannot be treated *via* the inverse scattering transform, nor its generalizations. They can be treated by numerical methods but significant features are, however, often missed in this manner. It is of great value to be able to obtain at least particular exact analytic solutions of non-integrable equations. Here, group theory and Lie algebras play an important role. Indeed, Lie group theory was originally created as a tool for solving ordinary and partial differential equations, be they linear or nonlinear.

New developments have also occurred in Lie's theory. Recently, there are numerous algebraic computing languages such as SymbolicC++, REDUCE, MACSYMA, AXIOM, MAPLE, MATHEMATICA, MuPAD etc, that have made it possible to write computer programs which construct the Lie algebra of the symmetry group of a differential equation.

Evolution Equation

Evolution equations are also known as diffusion equations. They are PDEs which describe density fluctuations in a material undergoing diffusion. These phenomena range in diversity from heat conduction presented by Bluman and Cole [3] and Bluman [4], diffusion of particle within a media presented by Crank [5], Fife [6] and Murray [7], stock option pricing on financial exchanges by Black and Scholes [8] and the study of waves in quantum mechanics by Kalmins and Miller [9] and Boyer, Sharp and Winternitz [10].

The equation is usually written as

$$\frac{\partial \phi(\vec{r},t)}{\partial t} = \nabla \cdot (D(\phi,\vec{r})\nabla \phi(\vec{r},t)), \tag{1.1}$$

where $\phi(\vec{r},t)$ is the density of diffusion material at location $\vec{r}$ and time t and $D(\phi,\vec{r})$ is the collective diffusion coefficient for density at location $\vec{r}$. The nabla symbol ∇ represents the vector differential operator *Del* acting on the space coordinates. If the diffusion coefficient depends on the density then the equation is nonlinear, otherwise it is linear. If D is constant, then (1.1) reduces to the following linear equation

$$\frac{\partial \phi(\vec{r},t)}{\partial t} = D\nabla^2 \phi(\vec{r},t)), \tag{1.2}$$

which is commonly known as the heat equation. It is a difficult task to find some exact solutions of these evolution equations, especially if they are nonlinear or have a dependence on arbitrary functions. Fortunately, Lie group theory provides a useful methodology to obtain some solutions of these PDEs. This approach calculates the transformations (symmetries) that leave the equation under consideration unchanged, and uses these transformations to reveal exact solutions. Therefore, it is necessary that a Lie point transformation generator must be properly determined, as presented in Anderson and Ibragimov [11], Ibragimov [12], Olver [14], Bluman and Kumei [15] and Stephan [16]. Once the Lie point transformation generators are determined, they can be used to obtain special solutions (group-invariant solutions) of the differential equation under consideration. A reduction in the number of variables and transformations to other simplified equations, which may be easier to solve, are also possible.

As mentioned above, a plethora of literature has been written on this topic of Lie's theory. In Ibragimov [17], Lie point transformation generators and their applications to some evolution equations are listed.

In Bluman and Cole [3], a general similarity solution for the heat equation is determined. The demonstration to use the method of infinitesimal transformations for the construction of similarity solutions of PDEs is revealed.

The arbitrary initial value problem for the Black-Scholes model in finance is considered in Ibragimov and Wafo Soh [18]. It is then acknowledged that Lie's theory has provided insight to many physical phenomena, which may otherwise not have been possible.

Contact transformations

In their paper, Momoniat and Mahomed [1] investigated the existence of contact transformations for evolution-type equations. This is one of the articles that is entirely reviewed in this thesis. The results found were applied in Momoniat [22] to determine Lie point transformations of nonlinear evolution equations and to perform a group classification on a fourth-order nonlinear evolution equation describing the effects of non-uniform surface tension on the spreading of a thin liquid drop. It has been proven in Ibragimov [12] that systems of equations do not admit contact transformation and in that regard only a one-dependent variable case was studied. In Anderson and Ibragimov [11] and Olver [14] an interesting property of evolution equations is obtained, which is that they do admit Lie point and non-trivial Lie-Bäcklund transformations. In Momoniat and Mahomed [1], it is found that they do not admit non-trivial contact transformations.

The main aim of this thesis is to obtain the *ansatz* required to construct the Lie point transformation for n-th order evolution equations with m independent variables. This *ansatz* provides a means for studying properties of transformations of Lie's theory, as it is a well-known and widely studied theory. This application is extended to study transformations of the Fokker-Planck equations (FPEs) whereby by inspection, the symmetries of corresponding stochastic ordinary differential equations (SODEs) can be obtained, a method initially introduced by Gaeta and Quintero [2]. Furthermore, an alternative methodology for constructing the symmetries of a stochastic differential equation (SDE) is studied, which was initially introduced by Wafo Soh and Mahomed [41], modified by Ünal [42] and Fredericks and Mahomed [43]. A crucial question, which will be formally asked in the relevant chapter is, will the symmetries of a SODE obtained in the former technique by Gaeta and Quintero [2] be equivalent to those obtained in the latter technique by Fredericks and Mahomed [43]? Or will extra symmetries be obtained from the one technique as compared to the other? These techniques are discussed in more detail in their relevant chapters.

A brief outline of this thesis

In every chapter, the relevant concepts and definitions are introduced. Chapter 1 introduces evolution equations, discusses contact transformations and provides developments in Lie's theory.

In Chapter 2, an introduction to the preliminary results that accentuate this research is given, whereby both Lie's theory and Itô calculus are discussed.

Chapter 3 presents the methodology of obtaining the required *ansatz* to construct the Lie point transformation for n-th order evolution equations with m independent variables.

Bluman and Kumei [15] presents a theorem which states that the symmetries admitted by any linear PDE have the temporal infinitesimal being a function of both time and space variables, the spatial infinitesimal being a function of both time and space variables and the remaining dependent infinitesimal being linear in the dependent variable. Parts of this theorem are proved in Ovsionnikov [53] and in Bluman [54]. We will be using contact transformation for the special case of Bluman and Kumei theorem, i.e. the evolution equation. For this special case, the admitted symmetries have the temporal infinitesimal being a function of time only, the spatial infinitesimal being a function of both the time and space variables and the remaining dependent infinitesimal being linear in the dependent variable.

The *ansatz* and the determining equations of the FPE is discussed in Chapter 4. The relationship between the point symmetries of FPE and its associated SODE is analyzed in detail, whereby the symmetries of a SODE are obtained from the calculation of those of the FPE. The calculations of Gaeta and Quintero [2], are done in more detail, whereby a projectable transformation is used to forecast the determining equations of the FPE.

Chapter 5 presents an alternative method of obtaining the symmetries of a SODE without the consultation of the FPE. The points of improvement in the technique will be explained and the observations of Wafo Soh and Mahomed [41], Ünal[42] and Fredericks and Mahomed [43] are highlighted.

Random time change is necessary for the transformation of one stochastic differential equation into another. Øksendal [31] presents the derivation for random time change formula for Itô integrals. We will derive this outcome using form invariance.

We conclude our thesis with Chapter 6, where all our findings are placed in perspective.

Chapter 2

Preliminaries

In this chapter, we introduce the key features of both the Lie group analysis of differential equations and Itô calculus. The material is necessary to provide a clear picture of the concepts and results needed in the subsequent chapters. For a broader view on Lie group theory, one can consult the following list of sources: Bluman and Kumei [15], Ovsiannikov [53], Lie and Scheffers [52], Stephan [16] and Olver [14]. A plethora of literature is available on Itô calculus, Shreve [35], Oksendal [47], Ross [51], etc.

2.1 Symmetry analysis of differential equations

2.1.1 Mathematical idea of symmetry

Symmetry includes all features of a physical system that exhibit the property of symmetry. That is, under certain transformations, aspects of these systems are "unchanged", according to a particular observation.

The transformations may be *continuous* (such as rotation of a circle) or *discrete* (such as a rotation of a regular polygon). Continuous and discrete transformations give rise to corresponding types of symmetries. Continuous symmetries can be described by Lie groups while discrete symmetries are described by finite groups.

Symmetry as invariance

Invariance is specified mathematically by transformations that leave some quantity unchanged. For example, temperature may be constant throughout a room. Since the temperature is independent of position within the room, the temperature is *invariant* under a shift in the measurer's position.

Continuous symmetries

Mathematically, *continuous symmetries* are described by continuous or smooth functions. An important subclass of continuous symmetries in physics are space-time symmetries. These are symmetries involving transformations of space and time. These may be further classified as spatial symmetries, involving only the spatial geometry associated with a physical system; temporal symmetries, involving only changes in time; or spatio-temporal symmetries, involving changes in both space and time.

Discrete symmetries

A *discrete symmetry* is a symmetry that describes non-continuous changes in a system. For example, a square possesses discrete rotational symmetry, as only rotations by multiples of right angles will preserve the square's original appearance.

2.1.2 Local one-parameter group of transformations

An illustration of the concept of a group of transformations of the equations which involve a single parameter is given below. Each such group is generated by an infinitesimal operator. The latter is a fundamental concept in application. By a transformation it is understood to mean an invertible transformation, i.e. a one-to-one and onto map. Also, functions are assumed to be smooth and summation over repeated indices is adopted when needed.

Consider a set of invertible transformations T_a in $\mathbb{R}^n$ defined by:

$$\bar{z}^i = f^i(z, a), \quad i = 1, ..., n, \tag{2.1}$$

where $z = (z^1, ..., z^n) \in \mathbb{R}^n$, $\bar{z} = (\bar{z}^1, ..., \bar{z}^n) \in \mathbb{R}^n$ and a is a real parameter from a neighborhood $U \subset \mathbb{R}$ of $a = 0$. We impose the condition that T_a is the identity if and only if $a = 0$, i.e.

$$f^i(z, a) = z^i \quad \text{for all } z \in \mathbb{R}^n \text{ if and only if } a = 0. \tag{2.2}$$

This indicates that $f^i(z, 0) = z^i$.

Definition 2.1 *The set G of transformations T_a given by (2.1) and satisfying condition (2.2) is called a (local) continuous one-parameter group of transformations in $\mathbb{R}^n$ if the function $f = (f^1, ..., f^n)$ satisfies the following property: for all $a, b \in U \subset U' \subset \mathbb{R}$, there exists $c \in U'$ such that*

$$f(f(z, a)b) = f(z, c) \quad \textit{for all } z \in \mathbb{R}^n, \tag{2.3}$$

where

$$c = \varphi(a, b), \tag{2.4}$$

is a smooth function φ of its arguments such that for all $a \in U$, the equation

$$\varphi(a, b) = 0, \tag{2.5}$$

admits a unique solution $b \in U$.

Condition (2.3) is sometimes referred to as the *group property* and φ is termed the *group composition law*. According to Definition 2.1, a continuous one-parameter group G of transformations T_a contains the (unique) identity transformation $Id_{\mathbb{R}^n} = T_0$. Furthermore, the group property (2.3) means that any two transformations $T_a, T_b \in G$ carried out one after the other results in a transformation which belongs again to G for any $a, b \in U$. The solvability of $\varphi(a, b) = 0$ for all $a \in U$, together with the group property (2.3), provide the inverse transformation $T_a^{-1} \in G$

to $T_a \in G$. Thus G is a group with the parameter a varies continuously in $U \subset \mathbb{R}$ and any transformation T_a is continuously connected to the identity T_o within G.

From now on, by a one-parameter group we mean a continuous one-parameter group.

Definition 2.2 *The group parameter a is said to be canonical if the composition law (2.4) is $\varphi(a, b) = a + b$, i.e. if the group property (2.3) has the form*

$$f(f(z, a), b) = f(z, a + b). \tag{2.6}$$

The next theorem shows that we can always assume that the parameter of an arbitrary one-parameter group is canonical.

Theorem 2.3 *(Mahomed [36])*

Given an arbitrary composition law (2.4), the canonical parameter $\tilde{a}$ is defined by the formula

$$\tilde{a} = \int_0^a \frac{da}{Z(a')}, \tag{2.7}$$

where

$$Z(a) = \left.\frac{\partial \varphi(a, b)}{\partial b}\right|_{b=0}. \tag{2.8}$$

Henceforth, we adopt the canonical parameter when referring to one-parameter groups.

Consider the first-order Taylor expansion of T_a about $a = 0$:

$$\bar{z}^i \approx z^i + a\xi^i(z), \tag{2.9}$$

where we have used the fact that $f(z, 0) = z$ and

$$\xi^i(z) = \left.\frac{\partial f^i(z, a)}{\partial a}\right|_{a=0} \quad i = 1, ..., n. \tag{2.10}$$

Note that $\xi = (\xi^1, ..., \xi^n)$ is the vector tangent at z to the orbit defined by (2.1).

Definition 2.4 *The first-order linear differential operator*

$$H = \xi^i(z)\frac{\partial}{\partial z^i}, \tag{2.11}$$

is known as the generator of the group G.

Example

Consider the set S of transformations

$$\bar{z} = e^a z \equiv f(z, a), \tag{2.12}$$

where $a \in U = \mathbb{R}$

- $f(z, a) = z$ for all $z \in \mathbb{R}^n$ if and only if $a = 0$.
- For $a, b \in U$ and $z \in \mathbb{R}^n$,

  $$f(f(z, a), b) = f(z, a + b) = f(z, c), \tag{2.13}$$

  where $c = \varphi(a, b) = a + b$.
- For $a \in U$, $\varphi(a, b) = 0$ if and only if $b = -a \in U$.

Hence S is a one-parameter group called the *scaling group*. The group parameter a is canonical. The symbol of S is then given by

$$H = z^i\frac{\partial}{\partial z^i}. \tag{2.14}$$

The symbol of a one-parameter group defines completely the finite transformations of the group as shown by Lie's theorem:

Theorem 2.5 *(Lie and Scheffers [52])*

The transformations (2.1) of any one-parameter group, with the identity transformation of which has parameter value $a = 0$, are put into one-to-one correspondence, *by the introduction of the canonical parameter (2.7), with the transformation (2.1) represented with respect to the canonical parameter $\bar{a}$. Let (2.1) be represented in the canonical parameter a with infinitesimal transformations (2.9), where (2.10) holds. The functions f^i are solutions of the initial value problem*

$$\frac{d\bar{z}}{da} = \xi(\bar{z}), \quad \bar{z}\Big|_{a=0} = z, \tag{2.15}$$

The system (2.15) is known as Lie's equations.

Remark

The initial value problem (2.15) has the formal solution (*Lie′s series*)

$$\bar{z} = e^{aH}z \equiv \sum_{k=0}^{\infty} \frac{a^k}{k!} H^k z, \tag{2.16}$$

where H is given by (2.11).

Theorem (2.5) provides a simplification in the investigation of continuous one-parameter groups: it reduces the study of continuous one-parameter groups to that of infinitesimal transformations (2.9). Thus it offers a 'linearization' to continuous one-parameter groups. This fact is crucial in the calculation of symmetries of differential equations.

Definition 2.6 *A function F(z) is invariant under the group of transformations (2.1) if*

$$F(z) = F(\bar{z}) \quad \text{for all } z \in \mathbb{R}^n \text{ and } a \in U. \tag{2.17}$$

The following theorem provides an infinitesimal criterion for invariance.

Theorem 2.7 *(Mahomed [36])*

A function F(z) is invariant under the group of transformations (2.1) if and only if

$$HF = 0, \tag{2.18}$$

where H is the generator of the group.

Theorem 2.8 *Every continuous one-parameter group of transformations given by (2.1), reduces to the group of translations*

$$\bar{y}^1 = y^1 + a, \quad \bar{y}^i = y^i, \quad i = 2, ..., n, \tag{2.19}$$

under a suitable change of variables

$$y^i = y^i(z), \quad i = 1, ..., n. \tag{2.20}$$

The new variables y^i are called canonical variables. They are singular for values of z such that $\xi(z) = 0$ (invariant points).

Proof

Using the chain rule, we find that under the change of variables (2.20), the differential operator (2.11) transforms according to the formula

$$H = H(y^i)\frac{\partial}{\partial y^i}. \tag{2.21}$$

Therefore, canonical variables are found from the system of linear partial differential equations

$$H(y^1) = 1, H(y^2) = 0, ..., H(y^n) = 0. \tag{2.22}$$

■

2.1.3 Extension of the one-parameter groups: Prolongation formulas

(Mahomed [36])

Consider a one-parameter group G of point transformations

$$\bar{x} = f^i(x, u, a), \quad f^i(x, u, 0) = x, \tag{2.23}$$

$$\bar{u}^\alpha = \varphi^\alpha(x, u, a), \quad \varphi^\alpha(x, u, 0) = u, \tag{2.24}$$

given that $i = 1, ..., n, \alpha = 1, ..., m$, where x is the independent variable and u the dependent variable (for a system of differential equations). Let the generator of G be

$$H = \xi^i(x,u)\frac{\partial}{\partial x_i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{2.25}$$

We extend the action of the group G to the space of extended variables $(x, u, u_1, ..., u_k)$, $k \geq 1$, where $(u_1, u_2, ..., u_k)$ denotes the collection of all first-, second- up to kth-order partial derivatives. Let us further denote the operator of total differentiation with respect to x_i by D_i, i.e.

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + ..., \tag{2.26}$$

and by $\bar{D}_i$ the operator of total differentiation with respect to $\bar{x}_i$. An application of the chain rule gives

$$D_i = D_i(f^j)\bar{D}_j. \tag{2.27}$$

The transformations (2.23)-(2.24) together with the first derivatives u_1 define the first extension of the group G denoted by $G^{[1]}$. This is the *first prolongation* of the group G which acts in the space (x, u, u_1). Similarly, one can also obtain the prolonged groups $G^{[2]}$ until $G^{[k]}$ by using the total derivative transforms.

The infinitesimal transformations of the prolonged groups are:

$$\begin{aligned} \bar{u}_i^\alpha &\approx u_i^\alpha + a\zeta_i^\alpha(x, u, u_1), \\ \bar{u}_{ij}^\alpha &\approx u_{ij}^\alpha + a\zeta_{ij}^\alpha(x, u, u_1, u_2), \\ &\vdots \\ \bar{u}_{i_1 \dots i_k}^\alpha &\approx u_{i_1 \dots i_k}^\alpha + a\zeta_{i_1 \dots i_k}^\alpha(x, u, u_1, ..., u_k). \end{aligned} \tag{2.28}$$

The functions $\zeta_i^\alpha(x, u, u_1)$, $\zeta_{ij}^\alpha(x, u, u_1, u_2)$ and $\zeta_{i_1 \dots i_k}^\alpha(x, u, u_1, ..., u_k)$ are given, recursively by the prolongation formulas

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j) \\ \zeta_{ij}^\alpha &= D_j(\zeta_i^\alpha) - u_{il}^\alpha D_j(\xi^l), \\ &\vdots \\ \zeta_{i_1 \dots i_k}^\alpha &= D_{i_k}(\zeta_{i_1 \dots i_{k-1}}^\alpha) - u_{i_1 \dots i_{k-1}^\alpha l} D_{ik}(\xi^l). \end{aligned} \tag{2.29}$$

Lie characteristic function

If one introduces the Lie characteristic function defined by

$$Q^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \tag{2.30}$$

then

$$\begin{aligned} \zeta_i^\alpha &= D_i(Q^\alpha) + \xi^j u_{ji}^\alpha \\ \zeta_{ij}^\alpha &= D_i D_j(Q^\alpha) + \xi^k u_{kij}^\alpha \\ &\vdots \\ \zeta_{i_1 \dots i_k}^\alpha &= D_{i_1} \dots D_{i_k}(Q^\alpha) + \xi^j u_{j i_1 \dots i_k}^\alpha. \end{aligned} \tag{2.31}$$

The generators of the prolonged groups are thus

$$\begin{aligned} H^{[1]} &= \xi^i(x,u)\frac{\partial}{\partial x_i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u_1)\frac{\partial}{\partial u_i^\alpha}, \\ &\vdots \\ H^{[k]} &= \xi^i(x,u)\frac{\partial}{\partial x_i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u_1)\frac{\partial}{\partial u_i^\alpha} + \cdots \\ &\quad + \zeta_{i_1 \dots i_k}^\alpha(x,u,\dots,u_k)\frac{\partial}{\partial u_{i_1 \dots i_k}^\alpha}. \end{aligned} \tag{2.32}$$

Definition 2.9 *A function $F(x,u,...,u_k)$ with $k \geq 1$, is a differential invariant of order k of a group G of transformations (2.23) and (2.24) if*

$$F(x,u,...,u_k) = F(\bar{x},\bar{u},...,\bar{u}_k), \tag{2.33}$$

for all $(x,u) \in \mathbb{R}^{n+m}$ and $a \in U \subset \mathbb{R}$, i.e. if F is invariant under the prolonged group $G^{[k]}$.

Theorem 2.10 *A function $F(x,u,...,u_k)$ is invariant under the prolonged group $G^{[k]}$, $k \geq 1$, if and only if*

$$H^{[k]}F = 0, \tag{2.34}$$

where $H^{[k]}$ is the generator of $G^{[k]}$.

2.1.4 Derivation of Lie's algorithm for the calculation of infinitesimal symmetries of partial differential equations

Loosely, a *point symmetry* of a differential equation is an invertible transformation (of independent and dependent variables) which leaves it unchanged or invariant. By considering only symmetries which form continuous one-parameter groups of transformations, Lie was able to derive an algorithm for the calculation of symmetries. These kinds of symmetries are known as *continuous symmetries*. Henceforth, by symmetry, it should be understood as continuous symmetry.

Consider a kth-order ($k \geq 1$) system of differential equations

$$E^{\sigma}(x, u, u_1, ..., u_k) = 0, \quad \sigma = 1, ..., s. \tag{2.35}$$

Assume that a symmetry of equations (2.35) is a one-parameter group of transformations of which the infinitesimal transformations are

$$\bar{x}^i \approx x^i + a\xi^i(x, u), \tag{2.36}$$

$$\bar{u}^{\alpha} \approx u^{\alpha} + a\eta^{\alpha}(x, u), \tag{2.37}$$

or equivalently the generator of which is

$$H = \xi^i(x, u)\frac{\partial}{\partial x_i} + \eta^{\alpha}(x, u)\frac{\partial}{\partial u^{\alpha}}. \tag{2.38}$$

Theorem 2.11 *(Infinitesimal criteria of invariance) The operator H defines a symmetry of (2.35) if and only if*

$$H^{[k]}E^{\sigma}(x, u, u_1, ..., u_k)\Big|_{(2.35)} = 0, \quad \sigma = 1, ..., s, \tag{2.39}$$

where $\Big|_{(2.35)}$ means evaluated on the manifold defined by (2.35).

Remark

Equations (2.39) are known as the *determining equations* for the symmetries of (2.35) as their solutions lead to the full set of symmetries. For $k \geq 2$, the determining equations are in general an overdetermined system of linear partial differential equations which are much easier to solve than that for the finite form of invariance of (2.35).

Definition 2.12 *A Lie algebra consists of a vector space L over a field $\mathbb{F}$ together with a binary operation $[,]$, called a Lie bracket or commutator, defined on L such that the following axioms are satisfied:*

- *Bilinearity: If X, Y, $Z \in L$ and $a, b \in \mathbb{F}$,*

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[X, aY + bZ] = a[X, Y] + b[X, Z],$$

- *Skew-symmetry: for any $X, Y \in L$,*

$$[X, Y] = -[Y, X],$$

- *Jacobi identity: for any $X, Y, Z \in L$,*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

2.2 Itô calculus

2.2.1 Brief overview of probability theory

A basic notion in probability theory is *random experiment*: an experiment whose outcome cannot be determined in advance.

Below are the basic concepts from probability theory.

1. Experiment (E): An experiment is any well-defined action that may result in a number of outcomes.

2. Outcome (O): An outcome is defined as any possible result of an experiment.

3. Sample space (S): The sample space is defined as the set of all possible outcomes of an experiment.

4. Event: An event is a collection of outcomes.

5. Union of two events A and B, ($A \cup B$): The union of two events A and B is the set of outcomes that belong to A or B or both.

6. Intersection of two events A and B, ($A \cap B$): The intersection of two events A and B is the set of outcomes that belong to both A and B.

7. Complement of event A, ($\bar{A}$): A complement of an event A contains all outcomes of the sample space, S, that do not belong to A.

8. Null event, ($\emptyset$): A null event is an empty set which has no outcomes.

9. Probability: Probability is a numerical measure of the likelihood of an event relative to a set of alternative events.

Probability Properties

The probability of an event A is expressed as $\mathbb{P}(A)$, and has the following properties:

1. $0 \leq \mathbb{P}(A) \leq 1$.

2. $\mathbb{P}(A) = 1 - \mathbb{P}(\bar{A})$.

3. $\mathbb{P}(\emptyset) = 0$.

4. $\mathbb{P}(S) = 1$.

In other words, when an event is certain to occur, it has a probability equal to 1; when it is impossible for an event to occur, it has a probability equal to 0.

It can also be shown that the probability of the union of two events A and B is:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$$

Mutually Exclusive Events

Two events A and B are said to be mutually exclusive if it is impossible for them to occur simultaneously. In such cases, the expression for the union of these two events reduces to the following,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B),$$

since the probability of the intersection of these events is defined as zero.

Conditional Probability

The conditional probability of two events A and B is defined as the probability of one of the events occurring, knowing that the other event has already occurred. The expression below denotes the probability of A occurring given that B has already occurred.

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}. \tag{2.40}$$

Independent Events

If knowing B gives no information about A, then the events are said to be independent and the conditional probability expression reduces to

$$\mathbb{P}(A \mid B) = \mathbb{P}(A). \tag{2.41}$$

From the definition of conditional probability, (2.40) can be written as:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\mathbb{P}(B) \tag{2.42}$$

Since events A and B are independent, the expression reduces to:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \tag{2.43}$$

If a group of n events A_i are independent, then:

$$\mathbb{P}\left[\cap_{i=1}^{n} A_i\right] = \Pi_{i=1}^{n}\mathbb{P}(A_i) \tag{2.44}$$

2.2.2 Brownian motion (construction and properties)

Probability theory plays a substantial role in the understanding of the unpremeditated results obtained in the SDEs, also known as the Wiener process.

One needs a probability space which will enable one to discuss the randomness which drives the SDE, which is the Wiener process. Fredericks and Mahomed [43] provides details of how to derive this space in question, and it is as follows.

In order to research SDEs, one needs to accustom oneself to how events ω belonging to a sample space Ω with probability measure $\mathbb{P}$ are associated. One must apply the probability measure specifically to a system of subsets of Ω , which is denoted by $\mathcal{F}$. This σ-algebra $\mathcal{F}$ contains the complement and countable union of any of its arbitrary members, which is known as open sets. A *natural filtration* is then established by forming an indexed family of σ-algebras $\mathcal{F}_t$, where t is a time index, to which the sample paths of the processes are adapted, [44]. By following this approach one obtains the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that was required. The Wiener process is a family of random variables indexed by time t, which belong to the interval I, which can be taken to be the positive real line. This process is a mathematical strategy used for formalizing the physical phenomena of the Wiener process, its sample paths, which will be obtained by focusing on a fixed realization of a particular event $\omega \in \Omega$ and following their families of random variables through time, which are almost surely continuous, and are almost surely nowhere differentiable in the usual sense. Literature is available that explains this concept (Freidlin [45]; Brzeźniak [46]). One represents it as a function $\mathbf{W}(t, \omega)$ which performs the following:

$$(t, \omega) \in I \times \Omega \longrightarrow \mathbf{W}(t, \omega) \in \mathbb{R}. \tag{2.45}$$

The ω in the argument of the function is an arbitrary event and thus suppressed throughout the discussion. The Wiener process $\mathbf{W}(t)$ also has the following characteristics:

Firstly, at time zero with probability one, the Wiener process is equal to zero, i.e. $W(0) = 0$. Secondly, the Wiener process is continuous in time variable t. Thirdly, the Wiener process $\{W(t)\}_{t\geq 0}$ has *stationary*, *independent increments*. The term *independent increments* means that for every choice of non-negative real numbers

$$0 \leq s_1 < t_1 \leq s_2 < t_2 \leq ... \leq s_n < t_n < \infty,$$

the increment random variables

$$W(t_1) - W(s_1), W(t_2) - W(s_2), ..., W(t_n) - W(s_n),$$

are jointly independent. The term *stationary* means that for any two time indexes $s, t \in \mathbb{R}^+$

$$0 < s, t < \infty,$$

the distribution of the increment

$$W(t + s) - W(s),$$

has the same distribution as

$$W(t) - W(0) = W(t).$$

Finally, the increments $W(t + s)$ - $W(s)$ are normally distributed with a zero mean and a variance of t.

2.2.3 Itô's equation

An *Itô* equation is given as

$$dx^i = f^i(t;\mathbf{x})dt + G^i_k(t;\mathbf{x})dW^k, \text{ for } k = 1, ...M, \text{ and } i = 1, ...N, \tag{2.46}$$

where $\mathbf{f}(t;\mathbf{x})$ is the instantaneous drift coefficient, $\mathbf{G}(t;\mathbf{x})$ is the instantaneous diffusion coefficient and the W^k are independent homogeneous standard Wiener process, where

$$\mathbf{W}(t) = \{W^1(t), ..., W^m(t)\}, \tag{2.47}$$

and

$$\mathbf{x} = \{x^1(t), ..., x^n(t)\}. \tag{2.48}$$

2.2.4 Itô's formula

Below, a discussion on the Itô formula which is used instead of the basic Newtonian calculus methods for solving stochastic differential equations (SDEs) is given.

Theorem 2.13 *(Itô's Formula, Øksendal [31]).*

If $\boldsymbol{X}(t)$, an N-dimensional vector, is an Itô process

$$d\mathbf{X}(t) = \mathbf{f}dt + \mathbf{G}d\mathbf{W}(t), \tag{2.49}$$

where $\boldsymbol{f} = \boldsymbol{f}(t, \boldsymbol{X}(t))$ and $\boldsymbol{G} = \boldsymbol{G}(t, \boldsymbol{X}(t))$ are an N-dimensional drift vector coefficient and diffusion matrix coefficient of dimension $N \times M$, respectively; then for an arbitrary application $\boldsymbol{F}$: $I \times \mathbb{R}^N \to \mathbb{R}^M$, which is twice differentiable in the spatial coordinates, $\boldsymbol{F}(t,.) \in \mathcal{C}^2(\mathbb{R}^N, \mathbb{R}^M)$ and only differentiable with respect to time once, $\boldsymbol{F}(.,x) \in \mathcal{C}^1(I, \mathbb{R}^M)$ for all $(s, \boldsymbol{y}) \in I \times \mathbb{R}^N$, an Itô process $\boldsymbol{F}(t, \boldsymbol{X}(t))$ exists and is written in component form as

$$dF_j(t, \mathbf{X}(t)) = \frac{\partial F_j(t, \mathbf{x})}{\partial t}\bigg|_{(t,\mathbf{X}(t))} dt + \frac{\partial F_j(t, \mathbf{x})}{\partial x_i}\bigg|_{(t,X(t))} dX_i(t) + \frac{1}{2}\frac{\partial^2 F_j(t, \mathbf{x})}{\partial x_i \partial x_m}\bigg|_{(t,\mathbf{X}(t))} dX_i(t)dX_m(t) \textit{ for } j = 1, ..., N. \quad (2.50)$$

*The evaluation of each of the partial derivatives on the right-hand side is made at (t, **X**(t)), which we give as*

$$dF_j(t, \mathbf{X}(t)) = \frac{\partial F_j}{\partial t}dt + \frac{\partial F_j}{\partial x_i}dX_i(t) + \frac{1}{2}\frac{\partial^2 F_j}{\partial x_i \partial x_m}dX_i(t)dX_m(t). \quad (2.51)$$

*It is kept in mind that though **X**(t) is indexed by time, it is by its random nature independent of time. The repeated index summation convention is assumed throughout this work. The terms $dX_i(t)$ and $dX_i(t)dX_m(t)$ are evaluated using (2.49) and the Itô multiplication table to obtain*

$$dF_j(t, \mathbf{X}(t)) = \Gamma(F_j)(t, \mathbf{X}(t))dt + \Upsilon(F_j)^l(t, \mathbf{X}(t))dW_l(t), \quad (2.52)$$

where

$$\Gamma(F_j)(t, \mathbf{X}(t)) = \frac{\partial F_j}{\partial t} + f_i\frac{\partial F_j}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_i^k G_m^k \frac{\partial^2 F_j}{\partial x_i \partial x_m}, \quad (2.53)$$

$$\Upsilon(F_j)^l(t, \mathbf{X}(t)) = \frac{\partial F_j}{\partial x_i}G_i^l \qquad \textit{for each } l = 1, ..., M. \quad (2.54)$$

Below is the derivation to demonstrate how (2.52) is obtained

To show the derivation from

$$dF_j(t, \mathbf{X}(t)) = \frac{\partial F_j}{\partial t}dt + \frac{\partial F_j}{\partial x_i}dX_i(t) + \frac{1}{2}\frac{\partial^2 F_j}{\partial x_i \partial x_m}dX_i(t)dX_m(t), \quad (2.55)$$

to

$$dF_j(t, \mathbf{X}(t)) = \Gamma(F_j)(t, \mathbf{X}(t))dt + \Upsilon(F_j)^l(t, \mathbf{X}(t))dW_l(t), \quad (2.56)$$

where

$$\Gamma(F_j)(t, \mathbf{X}(t)) = \frac{\partial F_j}{\partial t} + f_i \frac{\partial F_j}{\partial x_i} + \frac{1}{2} \sum_{k=1}^{M} G_i^k G_m^k \frac{\partial^2 F_j}{\partial x_i \partial x_m}, \tag{2.57}$$

$$\Upsilon(F_j)^l(t, \mathbf{X}(t)) = \frac{\partial F_j}{\partial x_i} G_i^l \qquad \text{for each } l = 1, ..., M. \tag{2.58}$$

Solution

The terms $dX_i(t)$ and $dX_i(t)\, dX_m(t)$ in (2.55) are evaluated using (2.49) and the Itô multiplication table.

Therefore (2.55) becomes:

$$\begin{aligned} dF_j(t, \mathbf{X}(t)) = & \frac{\partial F_j}{\partial t} dt + \frac{\partial F_j}{\partial x_i} \left(f_i dt + G_i^k dW_i(t) \right) \\ & + \frac{1}{2} \frac{\partial^2 F_j}{\partial x_i \partial x_m} \left(f_i dt + G_i^k dW_i(t) \right) \left(f_m dt + G_m^k dW_m(t) \right) = 0. \end{aligned} \tag{2.59}$$

From the Itô multiplication table [31] one obtains that:

$$dt\, dt = 0, \quad dt\, dW_i(t) = 0, \quad dt\, dW_m(t) = 0, \quad \text{and} \quad dW_i(t)\, dW_m(t) = 0, \tag{2.60}$$

where $dW_i(t)$ and $dW_m(t)$ are two independent Wiener processes, further investigation on these Wiener processes is done in Chapter 5. Therefore (2.59) results to be

$$dF_j(t, \mathbf{X}(t)) = \frac{\partial F_j}{\partial t} dt + \frac{\partial F_j}{\partial x_i} \left(f_i dt + G_i^k dW_i(t) \right) + \frac{1}{2} \frac{\partial^2 F_j}{\partial x_i \partial x_m} (G_i^k G_m^k). \tag{2.61}$$

Hence

$$dF_j(t, \mathbf{X}(t)) = \Gamma(F_j)(t, X(t)) dt + \Upsilon(F_j)^l(t, X(t)) dW_l(t). \tag{2.62}$$

2.2.5 Itô integral ($\mathcal{L}^2$-theory)

Let $W(t)$ be the Wiener process (Brownian motion), starting at 0, defined on a probability space Ω, and T be a fixed positive number.

Definition 2.14 *A simple stochastic process is a function of the form*

$$f(t) = f(t, \omega) = \sum_{j=0}^{n-1} \alpha_j(\omega)_{[t_j, t_{j+1}]}(t), \tag{2.63}$$

where $0 = t_0 < t_1 < ... < t_n = T$ *and* $\alpha_j = \alpha_j(\omega)$ *are random variables.*

Definition 2.15 *The Itô integral of a simple stochastic process* $f(t)$ *expressed like in (2.63) is defined by*

$$\int_0^T f(t)dW(t) = \sum_{j=0}^{n-1} \alpha_j(\omega)(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)). \tag{2.64}$$

Theorem 2.16 *(Cosimano and Himonas [49]) If* f *is a simple stochastic process then*

- $\mathbb{E}\Big(\int_0^T f(t)dW(t) \Big) = 0,$
- *The Isometry:* $\mathbb{E}\Big[\Big(\int_0^T f(t)dW(t) \Big)^2 \Big] = \mathbb{E}\Big(\int_0^T f^2(t)dt \Big).$

Proof:

The first property follows from the fact that $\mathbb{E}(W(t)) = 0$. To prove the Isometry, let $\triangle W_j = W_{t_{j+1}} - W_{t_j}$. Then

$$\mathbb{E}\Big[\Big(\int_0^T f(t)dW(t) \Big)^2 \Big] = \sum_{j,k=0}^{n-1} \mathbb{E}[\alpha_j \alpha_k \triangle W_j \triangle W_k]. \tag{2.65}$$

If $j < k$ then $\alpha_j \alpha_k \triangle W_j$ and $\triangle W_k$ are independent. Since $\mathbb{E}[\triangle W_k] = 0$, the expectation of such products is 0. If $j = k$ then

$$\begin{aligned} \mathbb{E}[\alpha_j^2 \triangle W_j^2] &= \mathbb{E}[\alpha_j^2]\mathbb{E}[\triangle W_j^2] \\ &= \mathbb{E}[\alpha_j^2](t_{j+1} - t_j). \end{aligned} \tag{2.66}$$

Therefore

$$\begin{aligned} \mathbb{E}\Big[\Big(\int_0^T f(t)dW(t) \Big)^2 \Big] &= \mathbb{E}\Big(\sum_{j=0}^{n-1} \alpha_j^2 (t_{j+1} - t_j) \Big) \\ &= \mathbb{E}\Big[\int_0^T f^2(t)dt \Big]. \end{aligned} \tag{2.67}$$

■

The key to defining the Itô integral for general integrands f is the following result from measure theory:

Proposition 2.17 ,

If $f(t,\omega)$ is a general stochastic process such that

$$\mathbb{E}\Big[\int_0^T f^2(t,\omega)dt\Big] < \infty, \tag{2.68}$$

then there exists a sequence of simple stochastic processes $f_n(t,\omega)$ such that

$$\mathbb{E}\Big[\int_0^T \big(f(t,\omega) - f_n(t,\omega)\big)^2 dt\Big] \longrightarrow 0 \quad \text{as } n \to \infty. \tag{2.69}$$

Then, it can be shown that the sequence of the Itô integrals of the simple processes f_n

$$I_n(\omega) = \int_0^T f_n(t)dW(t), \tag{2.70}$$

which are defined according to (2.64), converges in $\mathcal{L}^2(\Omega)$ to a random variable $I(\Omega)$. That is

$$\mathbb{E}\Big(\big[I_n(\omega) - I(\omega)\big]\Big) \longrightarrow 0 \text{ as } n \to \infty. \tag{2.71}$$

This random variable $I(\omega)$ is defined to the Itô integral of f. Thus we have the following:

Definition 2.18 *The Itô integral for a general process satisfying the integrability condition (2.68) is given by*

$$\int_0^T f(t,\omega)dW(t)(\omega) \overset{L^2(\Omega)}{=} \lim_{n\to\infty} \int_0^T f_n(t,\omega)dW(t)(\omega), \tag{2.72}$$

where $f_n(t,\omega)$ is a sequence of simple processes approximating $f(t,\omega)$ in $\mathcal{L}^2$ sense, as described in (2.69).

Proof of (2.71) Let f and f_n as in Proposition (2.17). By the Itô Isometry applied to $f_n - f_m$ we obtain

$$\mathbb{E}\bigg[\bigg(\int_0^T (f_n(t) - f_m(t))dW(t)\bigg)^2\bigg] = \mathbb{E}\bigg[\int_0^T (f_n(t) - f_m(t))^2\bigg]dt. \tag{2.73}$$

Since

$$(f_n(t) - f_m(t))^2 \leq 2(f_n(t) - f_t)^2 + 2(f(t) - f_m(t))^2, \tag{2.74}$$

we have that

$$\mathbb{E}\bigg[\bigg(\int_0^T (f_n(t) - f_m(t))dW(t)\bigg)^2\bigg] \leq 2\mathbb{E}\bigg[\int_0^T (f_n(t) - f(t))^2\bigg]dt + 2\mathbb{E}\bigg[\int_0^T (f(t) - f_m(t))^2\bigg]dt \longrightarrow 0, \tag{2.75}$$

as $n, m \longrightarrow \infty$. Thus, the sequence $I_n(\omega) = \int_0^T f_n(t)dW(t)$ is Cauchy in $\mathcal{L}^2(\Omega)$. By the completeness of $\mathcal{L}^2(\Omega)$ (a result of measure theory), this sequence has a limit $I(\omega)$ in $\mathcal{L}^2(\Omega)$, which is called the Itô integral of f. ■

Properties of Itô integral

- $\int_R^T f(t)dW(t) = \int_R^S f(t)dW(t) + \int_S^T f(t)dW(t)$, $0 \leq R < S < T$
- $\int_S^T \Big(cf(t) + g(t)\Big)dW(t) = c\int_S^T f(t)dW(t) + \int_S^T g(t)dW(t)$.
- $\mathbb{E}\Big[\int_S^T f(t)dW(t)\Big] = 0$.

2.2.6 Random time change

Theorem 2.19 *(Random time change for Itô integrals, Øksendal [47])*

Let $c(t, \omega)$ be the measurable time change rate, which is related to our time scalar stochastic process $\beta(t, \omega)$, by the following equation:

$$\beta(t, \omega) = \int^t c(s, \omega)ds, \tag{2.76}$$

and $\alpha(t, \omega)$ be a scalar stochastic process satisfying

- *$\alpha(0, \omega) = 0$.*
- *$d\alpha(t, \omega)/dt = 1/c(\alpha(t), \omega) \geq 0$ for almost all positive time and almost all $\omega \in \Omega$.*
- *$\beta(t, \omega)$ and $\alpha(t, \omega)$ are left and right inverses of each other, respectively, $\alpha(\beta(t, \omega), \omega) = \beta(\alpha(t, \omega), \omega) = t$ for all $(t, \omega) \in I \times \Omega$.*

Then under the random time change $\bar{t} = \beta(t, \omega)$, the Wiener process $\mathbf{W}(\alpha(t), \omega)$ is mapped to another Wiener process $\bar{\mathbf{W}}(t, \omega)$ according to the following relationship

$$\sqrt{\frac{d\alpha(t)}{dt}}\, d\bar{\mathbf{W}}(t) = d\mathbf{W}(\alpha(t)), \tag{2.77}$$

where ω has been suppressed in the above expression. This can be expressed as

$$d\bar{\mathbf{W}}(\beta(t)) = \sqrt{c(t)}\, d\mathbf{W}(t), \tag{2.78}$$

by using the inverse relation between $\alpha(t)$ and $\beta(t)$ in conjunction with (2.76).

2.2.7 Derivation of the Fokker-Planck equation

We will now outline the derivation of the FPE, a partial differential equation for the time evolution of the transition probability density function. There are two approaches that we will consider, first the heuristic approach to deriving the FPE and lastly the Feynmann-Kac Theorem. There is an abundance of literature on this topic including Risken [32] and Shreve [35].

The heuristic approach to deriving the FPE

The SDE and its Transition Density (Rouah [39])

Start with the SDE defined by

$$d\mathbf{X}(t) = \mathbf{f}(\mathbf{X}(t))dt + \mathbf{G}(\mathbf{X}(t))d\mathbf{W}(t). \tag{2.79}$$

The transition density $\rho(x,t \mid y,s)$ is defined by

$$\begin{aligned}\int_A \rho(x,t \mid y,s)dx &= \mathbb{P}[X(t+s) \in A \mid X(s) = y] \\ &= \mathbb{P}[X(t) \in A \mid X(0) = y]. \qquad (2.80)\end{aligned}$$

The density $\rho(x,t \mid y,s)$ is time-invariant since $\mathbf{f}(\mathbf{X}(t))$ and $\mathbf{G}(\mathbf{X}(t))$ are assumed to be time invariant, and consequently, that $\mathbf{X}(t)$ is assumed to be stationary.

Derivation of the Equation

Consider a differentiable function $V(\mathbf{X}(t),t) = V(x,t)$ with $V(\mathbf{X}(t),t) = 0$ for $t \notin (0,T)$. Then by Itô's lemma

$$dV = \left[\frac{\partial V}{\partial t} + \mathbf{f}\frac{\partial V}{\partial x} + \frac{1}{2}\mathbf{G}^2\frac{\partial^2 V}{\partial x^2}\right]dt + \left[\mathbf{G}\frac{\partial V}{\partial x}\right]d\mathbf{W}(t), \qquad (2.81)$$

so that

$$\begin{aligned}V(\mathbf{X}(t),T) - V(\mathbf{X}(0),0) &= \int_0^T \left[\frac{\partial V}{\partial t} + \mathbf{f}\frac{\partial V}{\partial x} + \frac{1}{2}\mathbf{G}^2\frac{\partial^2 V}{\partial x^2}\right]dt \\ &\quad + \int_0^T \left[\mathbf{G}\frac{\partial V}{\partial x}\right]d\mathbf{W}(t), \qquad (2.82)\end{aligned}$$

where $\mathbf{f} = \mathbf{f}(X(t))$ and $\mathbf{G} = \mathbf{G}(X(t))$ for notational convenience. Take the conditional expectation of both sides of (2.82) given $X(0)$

$$\begin{aligned}\mathbb{E}[V(\mathbf{X}(t),T) - V(\mathbf{X}(0),0)] &= \mathbb{E}\int_0^T \left[\frac{\partial V}{\partial t} + \mathbf{f}\frac{\partial V}{\partial x} + \frac{1}{2}\mathbf{G}^2\frac{\partial^2 V}{\partial x^2}\right]dt \\ &\quad + \mathbb{E}\int_0^T \left[\mathbf{G}\frac{\partial V}{\partial x}\right]d\mathbf{W}(t) \\ &= \int_{\mathbb{R}}\left\{\int_0^T \left[\frac{\partial V}{\partial t} + \mathbf{f}\frac{\partial V}{\partial x}\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\mathbf{G}^2\frac{\partial^2 V}{\partial x^2}\right]dt\right\}\rho(x,t \mid y,s)dx. \qquad (2.83)\end{aligned}$$

Note, all expectations are conditional on $\mathbf{X}(0)$, so that $\mathbb{E}[\cdot] = \mathbb{E}[\cdot \mid \mathbf{X}(0) = y]$. Since $\mathbb{E}[d\mathbf{W}(t)] = 0$, we rewrite (2.83) as follows:

$$\int_{\mathbb{R}}\int_0^T \rho\frac{\partial V}{\partial t}dtdx + \int_{\mathbb{R}}\int_0^T \rho\mathbf{f}\frac{\partial V}{\partial x}dtdx + \frac{1}{2}\int_{\mathbb{R}}\int_0^T \rho\mathbf{G}^2\frac{\partial^2 V}{\partial x^2}dtdx = d_1 + d_2 + d_3, \quad (2.84)$$

where $\rho = \rho(x,t \mid y,s)$ for notational convenience. The objective of the derivation is to apply integration by parts to get rid of the derivatives of V.

Evaluation of the Integrals

The trick is that d_1 is evaluated using integration by parts on t, while d_2 and d_3 are each evaluated using integration by parts on x.

Evaluation of d_1

Use $u = \rho$, $v' = \frac{\partial V}{\partial t}$ so that $u' = \frac{\partial \rho}{\partial t}$ and $v = V$. Hence for the inside integrand of d_1 we have

$$\int_0^T \rho \frac{\partial V}{\partial t} dt = \rho V \bigg|_0^T - \int_0^T \frac{\partial \rho}{\partial t} V dt = -\int_0^T \frac{\partial \rho}{\partial t} V dt, \tag{2.85}$$

since at the boundaries 0 and T, $V = 0$. Hence

$$d_1 = -\int_{\mathbb{R}} \int_0^T \frac{\partial \rho}{\partial t} V(x,t) dt dx. \tag{2.86}$$

Evaluation of d_2

Change the order of integration in d_2 and write it as

$$d_2 = \int_0^T \int_{\mathbb{R}} \rho \mathbf{f} \frac{\partial V}{\partial x} dx dt. \tag{2.87}$$

Use integration by parts on the integrand, with $u = \rho\mathbf{f}$, $v' = \frac{\partial V}{\partial x}$ so that $u' = \frac{\partial(\rho \mathbf{f})}{\partial x}$,$v = V$

$$\int_{\mathbb{R}} \rho \mathbf{f} \frac{\partial V}{\partial x} dx = \rho \mathbf{f} V \bigg|_{\mathbb{R}} - \int_{\mathbb{R}} \frac{\partial(\rho \mathbf{f})}{\partial x} V dx. \tag{2.88}$$

Hence the integral can be evaluated as

$$\begin{aligned} d_2 &= -\int_0^T \int_{\mathbb{R}} \frac{\partial(\rho \mathbf{f})}{\partial x} V(x,t) dx dt \\ &= -\int_{\mathbb{R}} \int_0^T \frac{\partial(\rho \mathbf{f})}{\partial x} V(x,t) dt dx. \end{aligned} \tag{2.89}$$

Evaluation of d_3

Finally, the evaluation of the integrand of d_3 requires the application of integration by parts on x twice. This is because in the integrand we want to get rid of the $\frac{\partial^2 V}{\partial x^2}$ term and end up with $V(x,t)$ only. Again, change the order of integration and write

d_3 as

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}} \rho \mathbf{G}^2 \frac{\partial^2 V}{\partial x^2} dx dt. \tag{2.90}$$

For the first integration by parts use $u = \rho\mathbf{G}^2$, $v' = \frac{\partial^2 V}{\partial x^2}$ so that $u' = \frac{\partial(\rho\mathbf{G}^2)}{\partial x}$ and $v = \frac{\partial V}{\partial x}$. Hence the integrand can be written

$$\begin{aligned}\int_{\mathbb{R}} \rho\mathbf{G}^2 \frac{\partial^2 V}{\partial x^2} dx &= \rho\mathbf{G}^2 \frac{\partial V}{\partial x}\bigg|_{\mathbb{R}} - \int_{\mathbb{R}} \frac{\partial(\rho\mathbf{G}^2)}{\partial x}\frac{\partial V}{\partial x} dx \\ &= -\int_{\mathbb{R}} \frac{\partial(\rho\mathbf{G}^2)}{\partial x}\frac{\partial V}{\partial x} dx. \end{aligned} \tag{2.91}$$

Apply integration by parts again, with $u = \frac{\partial(\rho\mathbf{G}^2)}{\partial x}$, $v' = \frac{\partial V}{\partial x}$, $u' = \frac{\partial^2(\rho\mathbf{G}^2)}{\partial x^2}$, $v = V$

$$\begin{aligned} -\int_{\mathbb{R}} \frac{\partial(\rho\mathbf{G}^2)}{\partial x}\frac{\partial V}{\partial x} dx &= -\frac{\partial(\rho\mathbf{G}^2)}{\partial x} V\bigg|_{\mathbb{R}} + \int_{\mathbb{R}} \frac{\partial^2(\rho\mathbf{G}^2)}{\partial x^2} V dx \\ &= \int_{\mathbb{R}} \frac{\partial^2(\rho\mathbf{G}^2)}{\partial x^2} V(x,t) dx. \end{aligned} \tag{2.92}$$

This implies that d_3 can be written as

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}} \frac{\partial^2(\rho\mathbf{G}^2)}{\partial x^2} V dx dt = \frac{1}{2}\int_{\mathbb{R}} \int_0^T \frac{\partial^2(\rho\mathbf{G}^2)}{\partial x^2} V(x,t) dt dx. \tag{2.93}$$

Obtaining the Equation

Substitute (2.86), (2.89) and (2.93) into (2.83), we obtain:

$$\begin{aligned} \mathbb{E}[V(\mathbf{X}(t),T) - V(\mathbf{X}(0),0)] &= -\int_{\mathbb{R}}\int_0^T \frac{\partial \rho}{\partial t} V(x,t) dt dx - \int_{\mathbb{R}}\int_0^T \frac{\partial(\rho\mathbf{f})}{\partial x} V(x,t) dt dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}}\int_0^T \frac{\partial^2(\rho\mathbf{G}^2)}{\partial x^2} V(x,t) dt dx \\ &= \int_{\mathbb{R}}\int_0^T V(x,t)\bigg[-\frac{\partial \rho}{\partial t} - \frac{\partial(\rho\mathbf{f})}{\partial x} \\ &\quad + \frac{1}{2}\frac{\partial^2(\rho\mathbf{G}^2)}{\partial x^2}\bigg] dt dx. \end{aligned} \tag{2.94}$$

Since $V(\mathbf{X}(t),t) = 0$ for $t \notin (0,T)$ we have $V(\mathbf{X}(t),T) = V(\mathbf{X}(0),0) = 0$ so that $\mathbb{E}[V(\mathbf{X}(t),T)] - V(\mathbf{X}(0),0) = 0$. This implies that the portion of the integrand in the brackets is zero

$$-\frac{\partial \rho}{\partial t} - \frac{\partial(\rho\mathbf{f})}{\partial x} + \frac{1}{2}\frac{\partial^2(\rho\mathbf{G}^2)}{\partial x^2} = 0, \tag{2.95}$$

from which the Fokker-Planck equation can be obtained

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho \mathbf{f})}{\partial x} + \frac{1}{2}\frac{\partial^2(\rho \mathbf{G}^2)}{\partial x^2} = 0. \tag{2.96}$$

In general, the FPE may be written as follows:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(f^i \rho)}{\partial x_i} + \frac{1}{2}\frac{\partial^2[(GG^T)^{ij}\rho]}{\partial x_i \partial x_j}, \text{ where } i = 1, ..., m \text{ and } j = 1, ..., n. \tag{2.97}$$

Feynman-Kac Theorem

Stochastic differential equations are mostly encountered in the finance and economic fields, to name a few. Stock prices, exchange rates, interest rates, or other economical variables evolve in time according to a stochastic differential equation. The relationship between a SDE and a PDE was also established by Richard Feynmann and Mark Kac, Shreve [35]. This correspondence between PDEs and stochastic processes is called the Feynmann-Kac formula. It offered a method of solving PDEs by simulating random paths of a stochastic process, conversely, stochastic PDEs can be solved by deterministic methods. The presentation of the $Feynmann - Kac\ Theorem$ is given below.

Theorem 2.20 *Shreve [35]*

Consider the stochastic differential equation

$$d\mathbf{X}(t) = \mathbf{f}(t, \mathbf{X}(t))dt + \mathbf{G}(t, \mathbf{X}(t))d\mathbf{W}(t). \tag{2.98}$$

Let $h(y)$ be a Borel-measurable function. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

$$z(t, x) = \mathbb{E}^{t,x} h(\mathbf{X}(T)). \tag{2.99}$$

(Assume that $E^{t,x}\Big|h(\boldsymbol{X}(T))\Big| < \infty$ for all t and x). Then $z(t, x)$ satisfies the partial differential equation

$$\frac{\partial z}{\partial t}(t, x) + \mathbf{f}(t, x)\frac{\partial z}{\partial x}(t, x) + \frac{1}{2}\mathbf{G}^2(t, x)\frac{\partial^2 z}{\partial x^2}(t, x) = 0, \tag{2.100}$$

and the terminal condition

$$z(T,x) = h(x) \text{ for all } x\ . \tag{2.101}$$

A brief discussion on the operation of this theorem:

Let $\mathbf{X}(t)$ be the solution to the stochastic differential equation SDE (2.98) starting at time zero. The first step is to apply *Itô's lemma* to the stochastic process $z(t,\mathbf{X}(t))$, this results in

$$\begin{aligned} dz(t,\mathbf{X}(t)) &= \frac{\partial z}{\partial t}dt + \frac{\partial z}{\partial x}dX + \frac{1}{2}\frac{\partial^2 z}{\partial x^2}dXdX \\ &= \frac{\partial z}{\partial t}dt + \mathbf{f}\frac{\partial z}{\partial x}dt + \mathbf{G}\frac{\partial z}{\partial x}d\mathbf{W} + \frac{1}{2}\mathbf{G}^2\frac{\partial^2 z}{\partial x^2}dt \\ &= \left[\frac{\partial z}{\partial t} + \mathbf{f}\frac{\partial z}{\partial x} + \frac{1}{2}\mathbf{G}^2\frac{\partial^2 z}{\partial x^2}\right]dt + \mathbf{G}\frac{\partial z}{\partial x}d\mathbf{W}. \end{aligned} \tag{2.102}$$

It was proven in Shreve [35] that $z(t,\mathbf{X}(t))$ is a martingale, and therefore one can integrate both sides and express this equation in terms of expectations:

$$\int_t^T dz = \int_t^T \left(\frac{\partial z}{\partial t} + \mathbf{f}\frac{\partial z}{\partial x} + \frac{1}{2}\mathbf{G}^2\frac{\partial^2 z}{\partial x^2}\right)dt + \int_t^T \mathbf{G}\frac{\partial z}{\partial x}d\mathbf{W}. \tag{2.103}$$

By the Fundamental Theorem of Calculus, one obtains

$$\begin{aligned} z(\mathbf{X}_T,T) - z(\mathbf{x},t) = &\left(\int_t^T \left(\frac{\partial z}{\partial t} + \mathbf{f}\frac{\partial z}{\partial x} + \frac{1}{2}\mathbf{G}^2\frac{\partial^2 z}{\partial x^2}\right)dt\right) \\ &+ \left(\int_t^T \mathbf{G}\frac{\partial z}{\partial x}d\mathbf{W}\right). \end{aligned} \tag{2.104}$$

$$\begin{aligned} \mathbb{E}[z(\mathbf{X}_T,T)] - z(\mathbf{x},t) = &\mathbb{E}\left(\int_t^T \left(\frac{\partial z}{\partial t} + \mathbf{f}\frac{\partial z}{\partial x} + \frac{1}{2}\mathbf{G}^2\frac{\partial^2 z}{\partial x^2}\right)dt\right) \\ &+ \mathbb{E}\left(\int_t^T \mathbf{G}\frac{\partial z}{\partial x}d\mathbf{W}\right). \end{aligned} \tag{2.105}$$

Since the expectation of a *Wiener process* is zero, i.e.

$$\mathbb{E}\left(\int_t^T \mathbf{G}\frac{\partial z}{\partial x}d\mathbf{W}\right) = 0, \tag{2.106}$$

(2.105) becomes

$$\begin{aligned} \mathbb{E}[z(\mathbf{X}_T,T)] - z(\mathbf{x},t) = &\mathbb{E}\left(\int_t^T \left(\frac{\partial z}{\partial t} + \mathbf{f}\frac{\partial z}{\partial x}\right.\right. \\ &\left.\left. + \frac{1}{2}\mathbf{G}^2\frac{\partial^2 z}{\partial x^2}\right)dt\right), \end{aligned} \tag{2.107}$$

and since $z(t, \mathbf{x})$ is a martingale, it is known that the expectation of a martingale function at time T given all the previous values, is equivalent to the martingale function at time t, hence

$$\mathbb{E}[z(\mathbf{X}_T, T)] - z(\mathbf{x}, t) = 0, \tag{2.108}$$

thus making (2.107) to become

$$\mathbb{E}\biggl(\int_t^t \left(\frac{\partial z}{\partial t} + \mathbf{f}\frac{\partial z}{\partial x} + \frac{1}{2}\mathbf{G}^2 \frac{\partial^2 z}{\partial x^2} \right) dt \bigg| X(t) = x \biggr) = 0. \tag{2.109}$$

Probability theory explains that the expectation of a deterministic function is itself. This property forces

$$\frac{\partial z}{\partial t} + \mathbf{f}\frac{\partial z}{\partial x} + \frac{1}{2}\mathbf{G}^2 \frac{\partial^2 z}{\partial x^2} = 0. \tag{2.110}$$

Now the solution can be written in terms of expectations as follows:

$$g(\mathbf{x}, t) = \mathbb{E}[g(\mathbf{X}_T, T)] = \mathbb{E}[h(\mathbf{X}(T))] = \mathbb{E}[h(\mathbf{X}(T)) \bigg| \mathbf{X}(t) = \mathbf{x}]. \tag{2.111}$$

The linear PDE (2.110) in its general form is known as the FPE, which is usually written as

$$L_{FPE}(\rho) = \frac{\partial \rho(t, \mathbf{x})}{\partial t} + A_{ij}\frac{\partial^2 \rho(t, \mathbf{x})}{\partial x_i \partial x_j} + B_i \frac{\partial \rho(t, \mathbf{x})}{\partial x_i} + C\rho(t, \mathbf{x}) = 0 \tag{2.112}$$

where $p(t, \mathbf{x})$ is the density function, satisfying

$$\int_{-\infty}^{+\infty} p(t, \mathbf{x}) dx_1 ... dx_N = 1 \tag{2.113}$$

Chapter 3

Classification of evolution equations

3.1 Introduction

In this chapter, an investigation on the existence of a one-parameter group of contact transformations for evolution-type equations

$$u_t = F(t, x, u, u_x, u_{xx}, ..., u_n), \tag{3.1}$$

is done. Subscripts denote differentiation, where u_n is the n-th derivative of the dependent variable u with respect to the space variable x. The arbitrary function F is expandable as a power series in terms of all derivatives of order higher than one.

This result is extended to the case with m-independent space variables. As a consequence, the *ansatz* for determining Lie point transformations for n-th order evolution equations with m-independent space variables are obtained. Examples are given to show that the Lie point transformations of these evolution-type partial differential equations (PDEs) can be calculated from the *ansatz*.

3.2 Objectives and Relevance of the study

A proof made by Momoniat and Mahomed [1], that the contact transformation of evolution equations are just Lie point transformations is reviewed. Most importantly, the objective is to verify their results in showing the nature of the required *ansatz* to determine Lie point transformations of evolution equations from the contact transformation approach. This will be useful in the construction of the Lie point transformations of the Fokker-Planck equation (FPE) in Chapter 4. We acknowledge Theorem (3.2), a theorem given by Bluman and Kumei [15], which gives the symmetry form admitted by any linear partial differential equation. Indeed, we can state the symmetry form admitted by an evolution equation from Theorem 3.2, since it is a second-order linear parabolic partial differential equation. However, we are going to provide the symmetry form admitted by an evolution equation using contact transformation approach, a method discussed in Momoniat and Mahomed [1]. The theorem by Momoniat and Mahomed [1] states that contact transformation generators of nth-order evolution equation-type partial differential equation are just Lie point transformation generators with the temporal infinitesimal being a function of time only, the spatial infinitesimal being a function of both the time and the spatial variables and the remaining dependent infinitesimal being linear in the dependent variable. This means that the Lie point symmetry transformations are projectable for evolution equations, which is a special case of the more generalized theorem by Bluman and Kumei [15].

3.3 Generating the *ansatz* for evolution-type equations

Theorem 3.1 *(Momoniat and Mahomed [1])*

Contact transformation generators of nth-order evolution-type partial differential equa-

tions of the form

$$u_t = F(t,x,u,u_x,u_{xx},...,u_n), \qquad F_{u_n} \neq 0, \tag{3.2}$$

where $u_n = \frac{\partial^n u}{\partial x^n}$ *and the function* F *can be written as a power series in terms of the derivatives* $u_n, u_{n-1},...,u_{xx}$, *are just Lie point transformation generators given by*

$$H = \alpha(t)\frac{\partial}{\partial t} + \beta(t,x,u)\frac{\partial}{\partial x} + \gamma(t,x,u)\frac{\partial}{\partial u}. \tag{3.3}$$

Proof:

We solve the determining equation given by

$$\tilde{H}(u_t - F(t,x,u,u_x,u_{xx},...,u_n))\Big|_{(3.2)} = 0, \tag{3.4}$$

where $\tilde{H}$ is the prolongation of the generator

$$\begin{aligned} Y = {} & \xi^1(t,x,u,u_t,u_x)\frac{\partial}{\partial t} + \xi^2(t,x,u,u_t,u_x)\frac{\partial}{\partial x} + \eta(t,x,u,u_t,u_x)\frac{\partial}{\partial u} \\ & + \zeta_t(t,x,u,u_t,u_x)\frac{\partial}{\partial u_t} + \zeta_x(t,x,u,u_t,u_x)\frac{\partial}{\partial u_x}, \end{aligned} \tag{3.5}$$

given in terms of the Lie characteristic function Q. Expanding (3.4) and separating by the mixed derivatives $\frac{\partial^{n-2}u_{xt}^2}{\partial x^{n-2}}$ and $\frac{\partial^{n-2}u_{xt}}{\partial x^{n-2}}$ gives:

$$\frac{\partial^{n-2}u_{xt}^2}{\partial x^{n-2}} : Q_{u_t u_t}F_{u_n} = 0, \tag{3.6}$$

$$\frac{\partial^{n-2}u_{xt}}{\partial x^{n-2}} : (u_{xx}Q_{u_x u_t} + u_x Q_{u u_t} + Q_{x u_t})F_{u_n} = 0. \tag{3.7}$$

Since $F_{u_n} \neq 0$, from (3.6) we have that the Lie characteristic function Q is linear in u_t, denoted as $Q_{u_t u_t} = 0$, hence

$$Q = u_t C_1(t,x,u,u_x) + C_2(t,x,u,u_x), \tag{3.8}$$

where C_1 and C_2 are as yet arbitrary functions of t,x,u, and u_x. Substituting (3.8) into (3.7) we again obtain $C_1 = C_1(t)$ and therefore (3.8) can be written as

$$Q = u_t C_1(t) + C_2(t,x,u,u_x). \tag{3.9}$$

If the series expansion of F in (3.2) is infinite, then only a truncation of this series needs to be considered. Substituting F into the determining equation and separate firstly by the highest power of the term u_n. The resulting equation is then separated by the highest power of the term u_{n-1}, ..., until finally we separate by the highest power of the term u_{xx} to obtain

$$C_2 = u_x C_3(t,x,u) + C_4(t,x,u), \tag{3.10}$$

where C_3 and C_4 are arbitrary functions of t, x, and u. Equation (3.9) is now

$$Q = u_t C_1(t) + u_x C_3(t,x,u) + C_4(t,x,u), \tag{3.11}$$

from (2.30) we deduce that

$$\xi^1 = -C_1(t), \qquad \xi^2 = -C_3(t,x,u), \qquad \eta = C_4(t,x,u), \tag{3.12}$$

and hence the transformation generator corresponding to (3.11) is given by

$$H = -C_1(t)\frac{\partial}{\partial t} - C_3(t,x,u)\frac{\partial}{\partial x} + C_4(t,x,u). \tag{3.13}$$

Thus contact transformation generators of (3.2) have the form (3.3). Hence (3.12) is the necessary *ansatz* to use to determine Lie transformations of (3.2). ■

Below is the Theorem extracted from Bluman and Kumei [15] which gives the symmetry form of any linear PDE.

Theorem 3.2 *(Bluman and Kumei [15])*

Suppose PDE

$$u_{i_1 i_2 \dots i_l} = f(x,u,u_1,u_2,\dots u_k), \tag{3.14}$$

$(k \geq 2, l \leq k)$ *is a linear PDE which admits infinitesimal generator*

$$H = \xi^i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}. \tag{3.15}$$

Then

$$\frac{\partial \xi^i}{\partial u} = 0, \quad i = 1,2,\dots,n, \tag{3.16}$$

$$\frac{\partial^2 \eta}{\partial u^2} = 0. \tag{3.17}$$

For $n = 2$, if $\frac{\partial \xi^1}{\partial u} = \frac{\partial \xi^2}{\partial u} = 0$, $\frac{\partial^2 \eta}{\partial u^2} = 0$, then an admitted infinitesimal generator is of the form

$$H = \xi^1(t,x)\frac{\partial}{\partial t} + \xi^2(t,x)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{3.18}$$

where

$$\eta(t,x,u) = f(t,x)u + g(t,x), \tag{3.19}$$

for a one time and one space variables.

Ovsiannikov [53] gives the proof for $k = 2$ and Bluman [54] provides the proof for $k > 2$.

Remark

This theorem characterizes the symmetries of any linear partial differential equation. Since the most general transformation which maps an evolution equation into another evolution equation of the same class has the form $\bar{t} = T(t)$, $\bar{x} = X(t,x)$ and $\bar{u} = U(t,x,u)$, the required form of the symmetries of the Fokker-Planck equation is as follows:

$$H = \tau(t)\frac{\partial}{\partial t} + \xi^i(t,x)\frac{\partial}{\partial x_i} + \eta^\alpha(t,x,u)\frac{\partial}{\partial u^\alpha} \text{ for } i = 1,...,n, \quad \alpha = 1,...,m. \tag{3.20}$$

The following two examples are done to demonstrate the results of Theorem 3.1, (Burger's equation and Hopf equation), to find the required *ansatz*.

3.3.1 Burger's equation

Burger's equation given as

$$u_t = uu_x + u_{xx}, \tag{3.21}$$

is solved to show that the required *ansatz* for the Lie point characteristic function to determine the Lie point transformations is given by

$$Q = u_t C_1(t) + u_x C_3(t,x,u) + C_4(t,x,u), \tag{3.22}$$

An evaluation of the Burger's equation (3.21) in the second prolongation formula is done as follows:

$$H^{[2]}(u_t - uu_x - u_{xx})\Big|_{(3.21)} = 0. \tag{3.23}$$

The determining equations with $F = uu_x + u_{xx}$ and $u_t = uu_x + u_{xx}$ is given by

$$\begin{aligned} & (Q - (uu_x + u_{xx})Q_{u_t} - u_xQ_{u_x})u_x - Q_t - (uu_x + u_{xx})Q_u \\ + \quad & (Q_x + u_xQ_u)u + (u_{xx}^2 Q_{u_xu_x} + 2u_{xt}u_{xx}Q_{u_tu_x} \\ + \quad & u_{xt}^2 Q_{u_tu_t} + u_{xx}Q_u + 2u_xu_{xx}Q_{uu_x} + 2u_xu_{xt}Q_{uu_t} \\ + \quad & u_x^2 Q_{uu} + 2u_{xt}Q_{xu_x} + 2u_{xt}Q_{xu_t} + 2u_xQ_{xu} + Q_{xx}) = 0. \end{aligned} \tag{3.24}$$

Following the procedure stated in the theorem (3.1), one separates by mixed derivatives u_{xt}^2 and u_{xt}.

$$u_{xt}^2 : \quad Q_{u_tu_t} = 0, \tag{3.25}$$

$$u_{xt} : \quad u_{xx}Q_{u_tu_x} + u_xQ_{uu_t} + Q_{xu_t} = 0, \tag{3.26}$$

equation (3.25) implies that the Lie characteristic function is linear in u_t, hence

$$Q = u_tC_1(t, x, u, u_x) + C_2(t, x, u, u_x), \tag{3.27}$$

where C_1 and C_2 are arbitrary functions of t, x, u and u_x. Substituting (3.27) into (3.26)

$$u_{xx}C_1u_x + u_xC_1u + C_1x = 0, \tag{3.28}$$

and first separating by u_{xx}, u_x and a *constant* as follows:

$$u_{xx} : \quad C_1u_x = 0 \Rightarrow C_1(t, x, u), \tag{3.29}$$

$$u_x : \quad C_1u = 0 \Rightarrow C_1(t, x, u_x), \tag{3.30}$$

$$\textit{constant} : \quad C_1x = 0 \Rightarrow C_1(t, u, u_x), \tag{3.31}$$

by the rule of commonality, one finds that $C_1 = C_1(t)$, since t is the only common variable. Thus (3.27) becomes

$$Q = u_tC_1(t) + C_2(t, x, u, u_x). \tag{3.32}$$

At this point F is substituted into the determining equation and one separates by the highest power of u_{xx},

$$u_{xx}^2: \quad C_2 u_x u_x, \tag{3.33}$$

and therefore

$$C_2 = u_x C_3(t,x,u) + C_4(t,x,u), \tag{3.34}$$

where C_3 and C_4 are arbitrary functions of t, x and u. Equation (3.32) is now

$$Q = u_t C_1(t) + u_x C_3(t,x,u) + C_4(t,x,u), \tag{3.35}$$

giving the necessary *ansatz* for the Burger equation.

3.3.2 Hopf equation

The general Hopf equation given as

$$u_t = -uu_x + (k(u)u_x)_x, \tag{3.36}$$

is solved to show that the required *ansatz* for the Lie point characteristic function to determine the Lie point transformations is given by

$$Q = u_t C_1(t) + u_x C_3(t,x,u) + C_4(t,x,u). \tag{3.37}$$

The Hopf equation (3.36) is evaluated in the second prolongation formula as follows:

$$H^{[2]}(u_t + uu_x - (k(u)u_x)_x)\Big|_{(3.36)} = 0. \tag{3.38}$$

The determining equation with $F = -uu_x + (k(u)u_x)_x$ and $u_t = -uu_x + (k(u)u_x)_x$ is given by

$$\begin{aligned}
&- Q_t - (-uu_x + k'(u)u_x^2 + k(u)u_{xx})Q_u + (Q_x + u_x Q_u)(-u + k'(u)u_x) \\
&- u_x((Q - (uu_x + (k(u)u_x)_x)Q_{u_t} - u_x Q_{u_x})(k''(u)u_x^2 \\
&+ k'(u)u_{xx} + (u_{xx}^2 Q_{u_x u_x} + 2u_{xt}u_{xx}Q_{u_t u_x} + u_{xt}^2 Q_{u_t u_t} + u_{xx}Q_u \\
&+ 2u_x u_{xx} Q_{uu_x} + 2u_x u_{xt} Q_{uu_t} + u_x^2 Q_{uu} + 2u_{xt}Q_{xu_x} + 2u_{xt}Q_{xu_t} \\
&+ 2u_x Q_{xu} + Q_{xx})k(u) = 0.
\end{aligned} \tag{3.39}$$

As the procedure states, one separates by mixed derivatives u_{xt}^2 and u_{xt}.

$$u_{xt}^2 : \quad Q_{u_t u_t} = 0, \tag{3.40}$$

$$u_{xt} : \quad u_{xx}Q_{u_t u_x} + u_x Q_{uu_t} + Q_{xu_t} = 0. \tag{3.41}$$

Equation (3.40) implies that the Lie characteristic function is linear in u_t, hence

$$Q = u_t C_1(t, x, u, u_x) + C_2(t, x, u, u_x), \tag{3.42}$$

where C_1 and C_2 are arbitrary functions of t, x, u and u_x. Substituting (3.42) into (3.41)

$$u_{xx}C_1 u_x + u_x C_1 u + C_1 x = 0, \tag{3.43}$$

and first separating by u_{xx}, u_x and a *constant* as follows:

$$u_{xx} : C_1 u_x = 0 \Rightarrow C_1(t, x, u), \tag{3.44}$$

$$u_x : C_1 u = 0 \Rightarrow C_1(t, x, u_x), \tag{3.45}$$

and

$$\textit{constant} : C_1 x = 0 \Rightarrow C_1(t, u, u_x), \tag{3.46}$$

by the rule of commonality, one finds that $C_1 = C_1(t)$ and thus (3.42) becomes

$$Q = u_t C_1(t) + C_2(t, x, u, u_x). \tag{3.47}$$

At this point F is substituted into the determining equation and one separates by the highest power of u_{xx},

$$u_{xx}^2 : C_2 u_x u_x, \tag{3.48}$$

and therefore

$$C_2 = u_x C_3(t, x, u) + C_4(t, x, u), \tag{3.49}$$

where C_3 and C_4 are arbitrary functions of t, x and u. Equation (3.47) becomes

$$Q = u_t C_1(t) + u_x C_3(t, x, u) + C_4(t, x, u). \tag{3.50}$$

Equation (3.50) is the required *ansatz* for the Hopf equation.

3.4 Application of the *ansatz* to construct the Lie point symmetries of evolution-type equations

In the following two examples, the *ansatz* of evolution equation is used in order to determine their Lie point transformations (symmetries).

3.4.1 One dimensional heat equation

The *ansatz* given as

$$Q = C_1(t)u_t + C_2(t,x,u)u_x + C_3(t,x,u), \tag{3.51}$$

is used to obtain Lie point transformations of the heat equation, mathematically stated as

$$u_t = u_{xx}, \tag{3.52}$$

By evaluating the heat equation in the second prolongation formula,

$$H^{[2]}(u_t - u_{xx})\Big|_{(3.52)} = 0, \tag{3.53}$$

one obtains the following:

$$\zeta_t - \zeta_{xx} = 0. \tag{3.54}$$

It is known that

$$\zeta_t = Q_t + u_t Q_u, \tag{3.55}$$

and

$$\zeta_{xx} = D_x D_x(Q) - Q_{u_j} u_{jxx}, \qquad \text{where } j = x \text{ and } t. \tag{3.56}$$

By replacing $u_t = u_{xx}$ and substituting into (3.54) one obtains the following:

$$\begin{aligned} &- Q_t - u_{xx}Q_u + (u_{xx}^2 Q_{u_x u_x} + 2u_{xt}u_{xx}Q_{u_t u_x} \\ &+ u_{xt}^2 Q_{u_t ut} + u_{xx}Q_u + 2u_x u_{xx} Q_{uu_x} + 2u_x u_{xt} Q_{uu_t} \\ &+ u_x^2 Q_{uu} + 2u_{xx}Q_{xu_x} + 2u_{xt}Q_{xu_t} + 2u_x Q_{xu} + Q_{xx}) = 0. \end{aligned} \tag{3.57}$$

Now the *ansatz* (3.51) is substituted into equation (3.57), which reduces to

$$
\begin{aligned}
&-Q_t + 2u_x u_{xx} Q_{uu_x} + u_x^2 Q_{uu} \\
&+ 2u_{xx} Q_{xu_x} + 2u_x Q_{xu} + Q_{xx} = 0,
\end{aligned}
$$

which is the same as

$$
\begin{aligned}
&-(C_{1_t} u_t + C_{2_t} u_x + C_{3_t}) + 2u_x u_{xx} C_{2_u} + u_x^2 (C_{2_{uu}} u_x + C_{3_{uu}}) \\
&+ 2u_{xx} C_{2_x} + 2u_x (C_{2_{xu}} u_x + C_{3_{xu}}) + C_{2_{xx}} u_x + C_{3_{xx}} = 0. \qquad (3.58)
\end{aligned}
$$

One separates the resulting equation according to the derivatives of u, which results in the following overdetermined system of equations:

$$C_{2_u} = 0, \qquad (3.59)$$

$$C_{1_t} = 0, \qquad (3.60)$$

$$C_{1_t} - 2C_{2_x} = 0, \qquad (3.61)$$

$$C_{2_t} - 2C_{3_{xu}} - C_{2_{xx}} = 0, \qquad (3.62)$$

$$C_{3_t} - C_{3_{xx}} = 0, \qquad (3.63)$$

$$C_{3_{uu}} + 2C_{2_{xu}} = 0. \qquad (3.64)$$

Then solving the system of determining equations above one obtains the following general solution:

$$C_1 = \frac{a_1}{2} t^2 + a_2 t + a_3, \qquad (3.65)$$

$$C_2 = \frac{a_1}{2} xt + \frac{a_2}{2} x + a_4 t + a_5, \qquad (3.66)$$

$$C_3 = \left(-\frac{x^2}{8} a_1 - \frac{x}{2} a_4 - \frac{a_1}{4} t + a_6 \right) u + \alpha(t, x). \qquad (3.67)$$

In order to simplify the results, one can manipulate and eliminate the fractions in our general solution by letting

$$a_1 = 8C_6, \qquad (3.68)$$

$$a_2 = 2C_4, \qquad (3.69)$$

$$a_3 = C_1, \qquad (3.70)$$

$$a_4 = 2C_5, \tag{3.71}$$

$$a_5 = C_2, \tag{3.72}$$

$$a_6 = C_3. \tag{3.73}$$

Then the general solution in its simplest form is as follows:

$$C_1 = C_1 + 2C_4 t + 4C_6 t^2, \tag{3.74}$$

$$C_2 = C_2 + C_4 x + 2C_5 t + 4C_6 tx, \tag{3.75}$$

$$C_3 = (C_3 - C_5 x - 2C_6 t - C_6 x^2)u + \alpha(t, x). \tag{3.76}$$

where the $C_i s$ are the constants and $\alpha(t, x)$ satisfies the heat equation.

One obtains the Lie point transformation generators as follows:

$$H_1 = \frac{\partial}{\partial t}, \tag{3.77}$$

$$H_2 = \frac{\partial}{\partial x}, \tag{3.78}$$

$$H_3 = u\frac{\partial}{\partial u}, \tag{3.79}$$

$$H_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \tag{3.80}$$

$$H_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \tag{3.81}$$

$$H_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - u(x^2 + 2t)\frac{\partial}{\partial u}, \tag{3.82}$$

and

$$H_\alpha = \alpha(t, x)\frac{\partial}{\partial u}. \tag{3.83}$$

3.4.2 Two dimensional heat equation

The *ansatz*

$$Q = C_1(t) + C_2(t, x, y, u)u_x + C_3(t, x, y, u)u_y + C_4(t, x, y, u), \tag{3.84}$$

is used to obtain Lie point transformations of the two-dimensional Heat equation given as:

$$u_t - u_{xx} - u_{yy} = 0. \tag{3.85}$$

By evaluating the heat equation in the second prolongation formula

$$H^{[2]}(u_t - u_{yy} - u_{xx})\Big|_{(3.85)} = 0, \tag{3.86}$$

one obtains

$$\zeta_t - \zeta_{xx} - \zeta_{yy} = 0. \tag{3.87}$$

It is known that

$$\zeta_t = Q_t + u_t Q_u, \tag{3.88}$$

$$\zeta_{xx} = D_x D_x(Q) - Q_{u_j} u_{jxx}, \text{ where } j = x, y \text{ and } t, \tag{3.89}$$

and

$$\zeta_{yy} = D_y D_y(Q) - Q_{u_j} u_{yy}, \text{ where } j = x, y \text{ and } t, \tag{3.90}$$

therefore,

$$\begin{aligned} &- Q_t - u_t Q_u + (u_{xx}^2 Q_{u_x u_x} + 2u_{xt} u_{xx} Q_{u_t u_x} + u_{xt}^2 Q_{u_t u_t} + u_{xx} Q_u \\ &+ 2u_x u_{xx} Q_{u u_x} + 2u_x u_{xt} Q_{u u_t} + u_x^2 Q_{uu} + 2u_{xx} Q_{x u_x} + 2u_{xt} Q_{x u_t} + 2u_x Q_{xu} \\ &+ Q_{xx}) + (u_{yy}^2 Q_{u_y u_y} + 2u_{yt} u_{yy} Q_{u_t u_y} + u_{yt}^2 Q_{u_t u_t} + u_{yy} Q_u + 2u_y u_{yy} Q_{u u_y} \\ &+ 2u_y u_{yt} Q_{u u_t} + u_y^2 Q_{uu} + 2u_{yy} Q_{y u_y} + 2u_{yt} Q_{y u_t} \\ &+ 2u_y Q - yu + Q_{yy}) = 0. \end{aligned} \tag{3.91}$$

By replacing $u_t = u_{xx} + u_{yy}$ and substituting into the (3.91), one obtains the following:

$$\begin{aligned} &- Q_t - u_{xx} Q_u - u_{yy} Q_u + (u_{xx}^2 Q_{u_x u_x} + 2u_{xt} u_{xx} Q_{u_t u_x} + u_{xt}^2 Q_{u_t u_t} \\ &+ u_{xx} Q_u + 2u_x u_{xx} Q_{u u_x} + 2u_x u_{xt} Q_{u u_t} + u_x^2 Q_{uu} + 2u_{xx} Q_{x u_x} + 2u_{xt} Q_{x u_t} + 2u_x Q_{xu} \\ &+ Q_{xx}) + (u_{yy}^2 Q_{u_y u_y} + 2u_{yt} u_{yy} Q_{u_t u_y} + u_{yt}^2 Q_{u_t u_t} + u_{yy} Q_u \\ &+ 2u_y u_{yy} Q_{u u_y} + 2u_y u_{yt} Q_{u u_t} + u_y^2 Q_{uu} + 2u_{yy} Q - yu_y \\ &+ 2u_{yt} Q_{y u_t} + 2u_y Q_{yu} + Q_{yy}) = 0. \end{aligned} \tag{3.92}$$

Now substituting the required *ansatz*, (3.84) into (3.92), then equation (3.92) reduces to

$$\begin{aligned}
&-Q_t + 2u_x u_{xx} Q_{uu_x} + u_x^2 Q_{uu} + 2u_{xx} Q_{xu_x} + 2u_x Q_{xu} + Q_{xx} \\
&+ 2u_y u_{yy} Q_{uu_y} + u_y^2 Q_{uu} + 2u_{yy} Q_{yu_y} + 2u_y Q_{yu} + Q_{yy} = 0,
\end{aligned} \tag{3.93}$$

which is the same as

$$\begin{aligned}
&-(C_{1_t} u_t + C_{2_t} u_x + C_{3_t} u_y + C_{4_t}) + 2u_x u_{xx} C_{2_u} + u_x^2 (C_{2_{uu}} u_x + C_{3_{uu}} + C_{4_{uu}}) \\
&+ 2u_{xx} C_{2_x} + 2u_x (C_{2_{xu}} u_x + C_{3_{xu}} u_y + C_{4_{xu}}) + C_{2_{xx}} u_x + C_{3_{xx}} u_y + C_{4_{xx}} \\
&+ 2u_y u_{yy} C_{3_u} + u_y^2 (C_{2_{uu}} u_x + C_{3_{uu}} u_y + C_{4_{uu}}) + 2u_{yy} C_{3_y} + 2u_y (C_{2_{yu}} u_x \\
&+ C_{3_{yu}} u_y + C_{4_{yu}}) + C_{2_{yy}} u_x + C_{3_{yy}} u_y + C_{4_{yy}} = 0.
\end{aligned} \tag{3.94}$$

Separating the resulting equation, (3.94) according to the derivatives of u, results in the following overdetermined system of equations:

$$u_x u_{xx}: \quad C_{2_u} = 0, \tag{3.95}$$

$$u_{x^3}: \quad C_{2_{uu}} = 0, \tag{3.96}$$

$$u_{x^2}: \quad 2C_{2_{xu}} + C_{4_{uu}} = 0, \tag{3.97}$$

$$u_x: \quad -C_{2_t} + 2C_{4_{xu}} + C_{2_{xx}} + C_{2_{yy}} = 0, \tag{3.98}$$

$$u_{x^2} u_y: \quad C_{3_{uu}} = 0, \tag{3.99}$$

$$u_x u_y: \quad 2C_{3_{xu}} + 2C_{2_{yu}} = 0, \tag{3.100}$$

$$u_{y^2} u_x: \quad C_{2_{uu}} = 0, \tag{3.101}$$

$$u_y u_{yy}: \quad C_{3_u} = 0, \tag{3.102}$$

$$u_{y^3}: \quad C_{3_{uu}} = 0, \tag{3.103}$$

$$u_{y^2}: \quad 2C_{3_{yu}} + C_{4_{uu}} = 0, \tag{3.104}$$

$$u_y: \quad -C_{3_t} + C_{3_{xx}} + C_{3_{yy}} + 2C_{4_{yu}} = 0, \tag{3.105}$$

$$1: \quad -C_{4_t} + C_{4_{xx}} + C_{4_{yy}} = 0, \tag{3.106}$$

$$u_t = u_{xx} + u_{yy}: \quad -C_{1_t} = 2C_{2_x} + 2C_{3_y}. \tag{3.107}$$

Then solving the system of determining equations above one obtains the following:

$$C_1 = \frac{1}{2}t^2\alpha_1 + t\alpha_2 + \alpha_3, \tag{3.108}$$

$$C_2 = \frac{1}{2}xt\alpha_1 + \frac{1}{2}x\alpha_2 + t\alpha_4 + \alpha_5 + y\alpha_9, \tag{3.109}$$

$$C_3 = \frac{1}{2}yt\alpha_1 + \frac{1}{2}\alpha_2 + t\alpha_6 + \alpha_7 - x\alpha_9, \tag{3.110}$$

$$C_4 = \frac{1}{8}\Bigg(4t\alpha_1 + x^2\alpha_1 + y^2\alpha_1 + 4x\alpha_4 + 4y\alpha_6 + 8\alpha_8\Bigg), \tag{3.111}$$

this implies that

$$\begin{aligned} Q = & \left(\frac{1}{2}t^2\alpha_1 + t\alpha_2 + \alpha_3\right)u_t + \left(\frac{1}{2}xt\alpha_1 + \frac{1}{2}x\alpha_2 + t\alpha_4 + \alpha_5 + y\alpha_9\right)u_x \\ & + \beta(t,x,y) + \left(\frac{1}{2}yt\alpha_1 + \frac{1}{2}\alpha_2 + t\alpha_6 + \alpha_7 - x\alpha_9\right)u_y + \frac{1}{8}\Bigg(4t\alpha_1 + x^2\alpha_1 \\ & + y^2\alpha_1 + 4x\alpha_4 + 4y\alpha_6 + 8\alpha_8\Bigg)u, \end{aligned} \tag{3.112}$$

where the $\alpha_i s$ are arbitrary constants and β satisfies the equation

$$\beta_t - \beta_{xx} - \beta_{yy} = 0. \tag{3.113}$$

One obtains the following operators:

$$H_1 = \frac{\partial}{\partial t}, \tag{3.114}$$

$$H_2 = \frac{\partial}{\partial x}, \tag{3.115}$$

$$H_3 = \frac{\partial}{\partial y}, \tag{3.116}$$

$$H_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \tag{3.117}$$

$$H_5 = u\frac{\partial}{\partial u}, \tag{3.118}$$

$$H_6 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \tag{3.119}$$

$$H_7 = 2t\frac{\partial}{\partial y} + yu\frac{\partial}{\partial u}, \tag{3.120}$$

$$H_8 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{3.121}$$

$$H_9 = 4t^2\frac{\partial}{\partial t} + 4xt\frac{\partial}{\partial y} - (4t + x^2y^2)u\frac{\partial}{\partial u}, \tag{3.122}$$

and

$$H_\beta = \beta(t, x, y)\frac{\partial}{\partial u}. \tag{3.123}$$

3.5 Summary

A successful review on the work done by Momoniat and Mahomed [1] is achieved, whereby their results are verified. It was shown in [1] that evolution equations of the type considered do not admit a non-trivial one-parameter group of contact transformations. Consequently, this has been instrumental in showing the required *ansatz* to determine Lie point transformations of evolution-type equations from the contact transformation approach. The required form of the symmetries admitted by an evolution-type equation has the temporal infinitesimal being a function of time only, the spatial infinitesimal being a function of both time and space variable and the remaining dependent infinitesimal being linear in the dependent variable .i.e.

$$H = \tau(t)\frac{\partial}{\partial t} + \xi^i(t, x)\frac{\partial}{\partial x_i} + \eta^\alpha(t, x, u)\frac{\partial}{\partial u^\alpha}. \tag{3.124}$$

It is important to note that the same result is found using the theorem from Bluman and Kumei [15], Theorem (3.2) in this thesis.

Chapter 4

The relationship between the FPE and the SDE

4.1 Introduction

It is shown in Gaeta and Quintero [2], how to computationally obtain symmetries of the Fokker-Planck equation (FPE), which is an equation for probability density from those of a stochastic differential equation (SDE) which is the equation for space variable $x(t)$, depending on a Wiener process. This can be done because an SDE is intimately associated to the corresponding FPE for the evolution of the probability measure. The FPE describes the time evolution of the probability density function of the position of a moving object. Most often its usage is in studying the statistical description of a Wiener process.

It is known that for deterministic differential equations, particularly for ordinary differential equations (ODEs), continuous symmetries can be used to lower the order of the considered equation. Specific classes of solutions can be found by symmetry reductions for partial differential equations (PDEs) and to generate new solutions from known ones. Therefore, since the FPE is a PDE, its symmetries can be used to determine specific solutions *via* symmetry reductions.

An SDE considered as defining a one-point stochastic motion (Arnold [24]) and the associated FPE carry, except at degenerate cases, the same statistical information, enabling one to find the symmetries of the one equation from the other, and vise versa. An SDE defines not only a one-point process, but also a random dynamic system, i.e. simultaneous motion of all points x under the same realization of the (vector) Wiener process. Thus, together with the one-point process, it also defines n-point processes for integer n. These n-point motions (with $n \geq 2$) contain information which is embodied in the SDE but in principle cannot be obtained from the FPE.

An example of an SDE is an Itô equation given as

$$dx^i = f^i(t;\mathbf{x})dt + G^i_k(t;\mathbf{x})dW^k, \text{ for } k = 1, ...M, \text{ and } i = 1, ...N. \tag{4.1}$$

It is discussed above that an Itô equation is associated with the FPE

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x_i}(f^i\rho) + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}[(GG^T)^{ij}\rho], \tag{4.2}$$

describing the evolution of the probability measure $\rho(t;\mathbf{x})$ for the stochastic process described by (4.1). As studied in (Øksendal [31]; Van Kampen [30]; Arnold [24]), equation (4.1) and (4.2) incorporates the same statistical information provided the instantaneous diffusion matrix $\mathbf{G}$ satisfies the *non-degeneracy condition* ($\mathbf{GG}^T \neq 0$).

Remark

Since equation (4.2) describes the time evolution of the probability measure $\rho(t,\mathbf{x})$ under the stochastic process (4.1), the interpretation of $\rho(t,\mathbf{x})$ should be subjected to the condition

$$\int_{-\infty}^{+\infty} \rho(t;\mathbf{x})dx^1...dx^n = 1. \tag{4.3}$$

This is relevant in connection with the allowed transformations of $(\mathbf{x}, t, \rho)$ because only transformations preserving this normalization do represent symmetries of the FPE compatible with its probabilistic interpretation, and one should expect a correspondence between symmetries of the Itô equation as well as these, rather than all the symmetries of the FPE. Therefore, with the additional constraint (4.3), the equivalence between the SDE and an FPE is only statistical and this condition holds only for an Itô equation considered as defining a one point process.

It is important to disclose that for different Itô equations which have the same instantaneous drift function $\mathbf{f}$ and different diffusion matrices $\mathbf{G}$ can produce the same term $\mathbf{G}\mathbf{G}^T$ and thus the same FPE. This means that the probabilistic equivalence between the FPE and Itô equation does not imply uniqueness between the two. An example to illustrate this concept is when the diffusion matrix $\mathbf{G}$ is orthogonal ($\mathbf{G} \subset O(n)$): in this case the product of the diffusion matrices results in an identity matrix i.e. $\mathbf{G}\mathbf{G}^T = I$, so that all the Itô equations with the same instantaneous drift function $\mathbf{f}$ and any orthogonal diffusion matrix $\mathbf{G}$ gives the same FPE equation.

It is substantial that one must construct the symmetries of an FPE accurately in order to construct those needed for the SDE.

In Gaeta and Quintero [2], the discussion is restricted to only projectable transformations with the form:

$$H = \tau(t)\frac{\partial}{\partial t} + \xi^i(t, \mathbf{x})\frac{\partial}{\partial x^i} + \eta^\alpha(t, \mathbf{x}, u)\frac{\partial}{\partial u^\alpha}. \tag{4.4}$$

which is simpler than the known general transformation given as:

$$H = \tau(t, \mathbf{x}, u)\frac{\partial}{\partial t} + \xi^i(t, \mathbf{x}, u)\frac{\partial}{\partial x^i} + \eta^\alpha(t, \mathbf{x}, u)\frac{\partial}{\partial u^\alpha}. \tag{4.5}$$

The reason for this restriction is that they preferred to consider only transformations such that the independent variables are transformed independently of the values of the vector fields, for instance, such that temporal infinitesimal τ and the spatial infinitesimals ξ^i do not depend on the u. The other reason is that since the concentration is on evolution equations, in order to guarantee that the considered transformations take evolution equations into evolution equations (time keeps its distinguished role), temporal infinitesimal τ must not depend on the space dimensions $\mathbf{x}$. This means that some terms will be made equal to zero when a projectable symmetry is being applied to an evolution equation. The terms which will be of no importance are all the derivatives of temporal infinitesimal with respect to all the spatial and dependent variables, and the derivatives of the spatial infinitesimal with respect to dependent variable, i.e.

$$\frac{\partial \tau}{\partial u} = \frac{\partial \xi^i}{\partial u} = \frac{\partial \tau}{\partial x^i} = 0, \tag{4.6}$$

hence the end result being the transformation (4.4), which is to be used to construct the symmetries of the FPE.

4.2 Objectives and Relevance of the study

The aim of this chapter is to examine if the simplified transformation (4.4) proposed in Gaeta and Quintero [2] has any mathematical basis. Gaeta and Quintero [2] only explained theoretically why the projectable transformation of the form (4.4) is to be used to analyze both the FPE and its associated SODE. However, they did not present the mathematical derivations of this, hence our objective is to verify the proposed mathematical assumption. In order to accomplish the above mentioned task, the method learned in Chapter 3 of finding the required *ansatz* to determine the Lie point transformation has to be employed. If the transformation (4.4) is found to be justifiable, the transformation will be used to obtain symmetries of the FPE, which will enable the discovery of the symmetries of a SDE associated with it.

4.3 Determining the *ansatz* of the Fokker-Planck equation

The work done in Momoniat and Mahomed [1] is applied to establish the *ansatz* of the FPE given as

$$\frac{\partial u}{\partial t} + A_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k} + B_i \frac{\partial u}{\partial x_i} + Cu = 0, \tag{4.7}$$

for the operator H that Gaeta and Quintero [2] used in their paper, i.e.

$$H_0 = \tau(t)\frac{\partial}{\partial t} + \xi_j(t, \mathbf{x})\frac{\partial}{\partial x_j}, \quad j = 1, ..., N. \tag{4.8}$$

Lie characteristic function is given as

$$Q = \eta(t, \mathbf{x}, u) - \frac{\partial u}{\partial t}\tau(t, \mathbf{x}, u) - \frac{\partial u}{\partial x_i}\xi_i(t, \mathbf{x}, u), \tag{4.9}$$

where

$$H = H_0 + \eta(t, \mathbf{x}, u)\frac{\partial}{\partial u}, \tag{4.10}$$

is the Lie point transformation generator. The infinitesimals τ, ξ_i and η can be given in terms of the Lie characteristic function Q

$$\tau = -\frac{\partial Q}{\partial u_t}, \tag{4.11}$$

$$\xi_i = -\frac{\partial Q}{\partial u_i}, \tag{4.12}$$

$$\eta = Q - \frac{\partial u}{\partial t}\frac{\partial Q}{\partial u_t} - \frac{\partial u}{\partial x_i}\frac{\partial Q}{\partial u_i}, \tag{4.13}$$

where

$$u_t = \frac{\partial u}{\partial t}, \tag{4.14}$$

$$u_i = \frac{\partial u}{\partial x_i}, \tag{4.15}$$

$$u_{ik} = \frac{\partial^2 u}{\partial x_i \partial x_k}, \tag{4.16}$$

and in general

$$u_{(i_{1\to N})} = \frac{\partial^N u}{\partial x_{i_1} ... \partial x_{i_N}}, \tag{4.17}$$

where

$$i_{1\to N} = i_1 i_2 ... i_N. \tag{4.18}$$

To obtain the *ansatz* for H, an application of the second prolongation of H to the second-order evolution PDE (4.7) is needed. The second prolongation of H is given by,

$$H^{[2]} = H + \zeta_t\frac{\partial}{\partial u_t} + \zeta_i\frac{\partial}{\partial u_i} + \zeta_{it}\frac{\partial}{\partial u_{it}} + \zeta_{ik}\frac{\partial}{\partial u_{ik}}, \tag{4.19}$$

where

$$\zeta_t = \frac{\partial Q}{\partial t} + u_t \frac{\partial Q}{\partial u}, \tag{4.20}$$

$$\zeta_i = \frac{\partial Q}{\partial x_i} + u_i \frac{\partial Q}{\partial u}, \tag{4.21}$$

and since u is the only dependent variable

$$\begin{aligned}
\zeta_{ik} &= D_{(i)} D_{(k)}(Q), \\
&= \left(\frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{il} \frac{\partial}{\partial u_l} + ... \right) \left(\frac{\partial}{\partial x_k} + u_k \frac{\partial}{\partial u} + u_{kj} \frac{\partial}{\partial u_j} + ... \right)(Q) \\
&= \frac{\partial^2 Q}{\partial x_i \partial x_k} + u_i \frac{\partial^2 Q}{\partial u \partial x_k} + u_{il} \frac{\partial^2 Q}{\partial x_k \partial u_l} + u_{it} \frac{\partial^2 Q}{\partial x_k \partial u_t} \\
&+ u_k \frac{\partial^2 Q}{\partial u \partial x_i} + u_i u_k \frac{\partial^2 Q}{\partial u \partial u} + u_{ik} \frac{\partial Q}{\partial u} + u_{il} u_k \frac{\partial^2 Q}{\partial u \partial u_l} \\
&+ u_{it} u_k \frac{\partial^2 Q}{\partial u \partial u_t} + u_{kj} \frac{\partial^2 Q}{\partial x_i \partial u_j} + u_{kt} \frac{\partial^2 Q}{\partial x_i \partial u_t} \\
&+ u_i u_{kt} \frac{\partial^2 Q}{\partial u \partial u_j} + u_i u_{kt} \frac{\partial^2 Q}{\partial u \partial u_t} + u_{il} u_{kj} \frac{\partial^2 Q}{\partial u_l \partial u_j} \\
&+ u_{ij} u_{kj} \frac{\partial^2 Q}{\partial u_j u_l} + u_{ij} u_{kj} \frac{\partial^2 Q}{\partial u_j \partial u_j} + u_{it} u_{kt} \frac{\partial^2 Q}{\partial u_t \partial u_t} \\
&+ u_{il} u_{kt} \frac{\partial^2 Q}{\partial u_l \partial u_t} + u_{it} u_{kj} \frac{\partial^2 Q}{\partial u_t \partial u_j},
\end{aligned} \tag{4.22}$$

since this FPE does not contain u_{it}, there is no need for the calculation of ζ_{it}. Applying $H^{[2]}$ on $(u_t - F)$ at $u_t = F$, i.e.

$$H^{[2]}(u_t - F)\bigg|_{u_t = F} = 0, \tag{4.23}$$

where

$$F = -A_{ik} \frac{\partial^2 u(t, \mathbf{x})}{\partial x_i \partial x_k} - B_i \frac{\partial u}{\partial x_i} - C u(t, \mathbf{x}), \tag{4.24}$$

gives

$$
\begin{aligned}
&-\frac{\partial Q}{\partial t}-u_t\frac{\partial Q}{\partial u}+A_{ik}\bigg(\frac{\partial^2 Q}{\partial x_i\partial x_k}+u_i\frac{\partial^2 Q}{\partial u\partial x_k}\\
&+u_{il}\frac{\partial^2 Q}{\partial x_k\partial u_l}+u_{it}\frac{\partial^2 Q}{\partial x_k\partial u_t}+u_k\frac{\partial^2 Q}{\partial u\partial x_i}+u_iu_k\frac{\partial^2 Q}{\partial u\partial u}\\
&+u_{ik}\frac{\partial Q}{\partial u}+u_{il}u_k\frac{\partial^2 Q}{\partial u\partial u_l}+u_{it}u_k\frac{\partial^2 Q}{\partial u\partial u_t}\\
&+u_{kj}\frac{\partial^2 Q}{\partial x_i\partial u_j}+u_{kt}\frac{\partial^2 Q}{\partial x_i\partial u_t}+u_iu_{kt}\frac{\partial^2 Q}{\partial u\partial u_j}\\
&+u_iu_{kt}\frac{\partial^2 Q}{\partial u\partial u_t}+u_{il}u_{kj}\frac{\partial^2 Q}{\partial u_l\partial u_j}+u_{ij}u_{kj}\frac{\partial^2 Q}{\partial u_ju_l}\\
&+u_{ij}u_{kj}\frac{\partial^2 Q}{\partial u_j\partial u_j}+u_{it}u_{kt}\frac{\partial^2 Q}{\partial u_t\partial u_t}+u_{il}u_{kt}\frac{\partial^2 Q}{\partial u_l\partial u_t}\\
&+u_{it}u_{kj}\frac{\partial^2 Q}{\partial u_t\partial u_j}\bigg)+B_i\bigg(\frac{\partial Q}{\partial x_i}+u_i\frac{\partial Q}{\partial u}\bigg)+C\bigg(Q-u_t\frac{\partial Q}{\partial u_t}\\
&-u_j\frac{\partial Q}{\partial u_j}\bigg)=0.
\end{aligned}
\tag{4.25}
$$

The substitution of u_t for F gives

$$
\begin{aligned}
&-\frac{\partial Q}{\partial t}-(-A_{ik}u_{ik}-B_iu_i-Cu(t,\mathbf{x}))\frac{\partial Q}{\partial u}\\
&+A_{ik}\bigg(\frac{\partial^2 Q}{\partial x_i\partial x_k}+u_i\frac{\partial^2 Q}{\partial u\partial x_k}+u_{il}\frac{\partial^2 Q}{\partial x_k\partial u_l}\\
&+u_{it}\frac{\partial^2 Q}{\partial x_k\partial u_t}+u_k\frac{\partial^2 Q}{\partial u\partial x_i}\\
&+u_iu_k\frac{\partial^2 Q}{\partial u\partial u}+u_{ik}\frac{\partial Q}{\partial u}+u_{il}u_k\frac{\partial^2 Q}{\partial u\partial u_l}\\
&+u_{it}u_k\frac{\partial^2 Q}{\partial u\partial u_t}+u_{kj}\frac{\partial^2 Q}{\partial x_i\partial u_j}+u_{kt}\frac{\partial^2 Q}{\partial x_i\partial u_t}\\
&+u_iu_{kt}\frac{\partial^2 Q}{\partial u\partial u_j}+u_iu_{kt}\frac{\partial^2 Q}{\partial u\partial u_t}+u_{il}u_{kj}\frac{\partial^2 Q}{\partial u_l\partial u_j}\\
&+u_{il}u_{kj}\frac{\partial^2 Q}{\partial u_ju_l}+u_{ij}u_{kj}\frac{\partial^2 Q}{\partial u_j\partial u_j}+u_{it}u_{kt}\frac{\partial^2 Q}{\partial u_t\partial u_t}\\
&+u_{il}u_{kt}\frac{\partial^2 Q}{\partial u_l\partial u_t}+u_{it}u_{kj}\frac{\partial^2 Q}{\partial u_t\partial u_j}\bigg)+B_i\bigg(\frac{\partial Q}{\partial x_i}\\
&+u_i\frac{\partial Q}{\partial u}\bigg)+C\bigg(Q-(-A_{ik}u_{ik}-B_iu_i-Cu(t,\mathbf{x}))\frac{\partial Q}{\partial u_t}\\
&-u_j\frac{\partial Q}{\partial u_j}\bigg)=0.
\end{aligned}
\tag{4.26}
$$

Separating out (4.26) by derivatives of u mixed in time and space gives

$$
\begin{aligned}
&A_{ik}\left(\frac{\partial^2 Q}{\partial x_k \partial u_t} + u_k \frac{\partial^2 Q}{\partial u \partial u_t} + u_{kj}\frac{\partial^2 Q}{\partial u_t \partial u_j}\right)u_{it} \\
&+ A_{ik}\left(\frac{\partial^2 Q}{\partial x_i \partial u_t} + u_i \frac{\partial^2 Q}{\partial u \partial u_t} + u_{il}\frac{\partial^2 Q}{\partial u_l \partial u_t}\right)u_{kt} \\
&+ A_{ik}\frac{\partial^2 Q}{\partial u_t \partial u_t}u_{it}u_{kt} \\
&- \frac{\partial Q}{\partial t} + (A_{ik}u_{ik} + B_i u_i + Cu)\frac{\partial Q}{\partial u} + A_{ik}\left(\frac{\partial^2 Q}{\partial x_i \partial x_k}\right. \\
&+ u_i \frac{\partial^2 Q}{\partial u \partial x_k} + u_{il}\frac{\partial^2 Q}{\partial x_k \partial u_l} + u_k \frac{\partial^2 Q}{\partial u \partial x_i} \\
&+ u_i u_k \frac{\partial^2 Q}{\partial u \partial u} + u_{ik}\frac{\partial Q}{\partial u} + u_{il}u_k \frac{\partial^2 Q}{\partial u \partial u_l} \\
&+ u_{kj}\frac{\partial^2 Q}{\partial x_i \partial u_j} + u_i u_{kj}\frac{\partial^2 Q}{\partial u \partial u_j} + u_{il}u_{kj}\frac{\partial^2 Q}{\partial u_l \partial u_j} \\
&\left.+ u_{il}u_{kj}\frac{\partial^2 Q}{\partial u_j u_l} + u_{ij}u_{kj}\frac{\partial^2 Q}{\partial u_j \partial u_j}\right) + B_i\left(\frac{\partial Q}{\partial x_i} + u_i \frac{\partial Q}{\partial u}\right) \\
&+ C\left(Q + (A_{ik}u_{ik} + B_i u_i + Cu)\frac{\partial Q}{\partial u_t} - u_j \frac{\partial Q}{\partial u_j}\right) = 0. \qquad (4.27)
\end{aligned}
$$

By considering only the first three terms of (4.27) one obtains

$$u_{it}: \quad A_{ik}\left(\frac{\partial^2 Q}{\partial x_k \partial u_t} + u_k \frac{\partial^2 Q}{\partial u \partial u_t} + u_{kj}\frac{\partial^2 Q}{\partial u_t \partial u_j}\right) = 0, \tag{4.28}$$

$$u_{kt}: \quad A_{ik}\left(\frac{\partial^2 Q}{\partial x_i \partial u_t} + u_i \frac{\partial^2 Q}{\partial u \partial u_t} + u_{il}\frac{\partial^2 Q}{\partial u_l \partial u_t}\right) = 0, \tag{4.29}$$

$$u_{it}u_{kt}: \quad A_{ik}\left(\frac{\partial^2 Q}{\partial u_t \partial u_t}\right) = 0. \tag{4.30}$$

From (4.30) one notices that Q is linear in u_t

$$Q = \alpha_1(t, \mathbf{x}, u)u_t + \alpha_2(t, \mathbf{x}, u), \tag{4.31}$$

where $\alpha_1(t, \mathbf{x}, u)$ and $\alpha_2(t, \mathbf{x}, u)$ are arbitrary functions of t and $\mathbf{x}$. By substituting (4.31) into (4.28) and (4.29); separating by u_k, u_i and u_{il} gives

$$Q = \alpha_1(t)u_t + \alpha_2(t, \mathbf{x}, u). \tag{4.32}$$

Separating out (4.26) with respect to $u_{il}u_{kj}$, $u_{ij}u_{kl}$ and $u_{ij}u_{kj}$, gives

$$
\begin{aligned}
&A_{ik}\left(\frac{\partial^2 Q}{\partial u_l \partial u_j}\right)u_{il}u_{kj} + A_{ik}\left(\frac{\partial^2 Q}{\partial u_j \partial u_l}\right)u_{ij}u_{kl} + A_{ik}\left(\frac{\partial^2 Q}{\partial u_j \partial u_j}\right)u_{ij}u_{kj} \\
&- \frac{\partial Q}{\partial t} - (-A_{ik}u_{ik} - B_i u_i - Cu(t,\mathbf{x}))\frac{\partial Q}{\partial u} + A_{ik}\Bigg(\frac{\partial^2 Q}{\partial x_i \partial x_k} \\
&+ u_i \frac{\partial^2 Q}{\partial u \partial x_k} + u_{il}\frac{\partial^2 Q}{\partial x_k \partial u_l} + u_{it}\frac{\partial^2 Q}{\partial x_k \partial u_t} \\
&+ u_k \frac{\partial^2 Q}{\partial u \partial x_i} + u_i u_k \frac{\partial^2 Q}{\partial u \partial u} + u_{ik}\frac{\partial Q}{\partial u} \\
&+ u_{il}u_k \frac{\partial^2 Q}{\partial u \partial u_l} + u_{it}u_k \frac{\partial^2 Q}{\partial u \partial u_t} + u_{kj}\frac{\partial^2 Q}{\partial x_i \partial u_l} \\
&+ u_{kt}\frac{\partial^2 Q}{\partial x_i \partial u_t} + u_i u_{kj}\frac{\partial^2 Q}{\partial u \partial u_j} + u_i u_{kt}\frac{\partial^2 Q}{\partial u \partial u_t} \\
&+ u_{it}u_{kt}\frac{\partial^2 Q}{\partial u_t \partial u_t} + u_{il}u_{kt}\frac{\partial^2 Q}{\partial u_l \partial u_t} + u_{it}u_{kj}\frac{\partial^2 Q}{\partial u_t \partial u_j}\Bigg) \\
&+ B_i\left(\frac{\partial Q}{\partial x_i} + u_i \frac{\partial Q}{\partial u}\right) \\
&+ C\left(Q - (-A_{ik}u_{ik} - B_i u_i - Cu(t,\mathbf{x}))\frac{\partial Q}{\partial u_t} - u_j \frac{\partial Q}{\partial u_j}\right) = 0. \qquad (4.33)
\end{aligned}
$$

Considering only the first three terms gives

$$u_{il}u_{kj}: \quad A_{ik}\left(\frac{\partial^2 Q}{\partial u_l \partial u_j}\right) = 0, \qquad (4.34)$$

$$u_{ij}u_{kl}: \quad A_{ik}\left(\frac{\partial^2 Q}{\partial u_j \partial u_l}\right) = 0, \qquad (4.35)$$

$$u_{ij}u_{kj}: \quad A_{ik}\left(\frac{\partial^2 Q}{\partial u_j \partial u_j}\right) = 0. \qquad (4.36)$$

From (4.36) one observes that $\alpha_2(t, \mathbf{x}, u)$ is linear in u_j, i.e. the *ansatz* becomes

$$Q = \alpha_1(t)u_t + \alpha_j(t, \mathbf{x}, u)u_j + \alpha_4(t, \mathbf{x}, u). \qquad (4.37)$$

Finally separating out (4.26) with respect to $u_{il}u_k$ and u_iu_{kj}, gives

$$\begin{aligned}
&A_{ik}u_iu_{kj}\frac{\partial^2 Q}{\partial u\partial u_j} + A_{ik}u_{il}u_k\frac{\partial^2 Q}{\partial u\partial u_l} - \frac{\partial Q}{\partial t}\\
&-(-A_{ik}u_{ik} - B_iu_i - Cu(t,\mathbf{x}))\frac{\partial Q}{\partial u}\\
&+A_{ik}\bigg(\frac{\partial^2 Q}{\partial x_i\partial x_k} + u_i\frac{\partial^2 Q}{\partial u\partial x_k} + u_{il}\frac{\partial^2 Q}{\partial x_k\partial u_l}\\
&+u_{it}\frac{\partial^2 Q}{\partial x_k\partial u_t} + u_k\frac{\partial^2 Q}{\partial u\partial x_i} + u_iu_k\frac{\partial^2 Q}{\partial u\partial u}\\
&+u_{ik}\frac{\partial Q}{\partial u} + u_{il}u_k\frac{\partial^2 Q}{\partial u\partial u_l} + u_{it}u_k\frac{\partial^2 Q}{\partial u\partial u_t}\\
&+u_{kj}\frac{\partial^2 Q}{\partial x_i\partial u_l} + u_{kt}\frac{\partial^2 Q}{\partial x_i\partial u_t} + u_iu_{kj}\frac{\partial^2 Q}{\partial u\partial u_j}\\
&+u_iu_{kt}\frac{\partial^2 Q}{\partial u\partial u_t} + u_{il}u_{kj}\frac{\partial^2 Q}{\partial u_l\partial u_j} + u_{ij}u_{kl}\frac{\partial^2 Q}{\partial u_j\partial u_l}\\
&+u_{ij}u_{kj}\frac{\partial^2 Q}{\partial u_j\partial u_j} + u_{it}u_{kt}\frac{\partial^2 Q}{\partial u_t\partial u_t} + u_{il}u_{kt}\frac{\partial^2 Q}{\partial u_l\partial u_t}\\
&+u_{it}u_{kj}\frac{\partial^2 Q}{\partial u_t\partial u_j}\bigg) + B_i\left(\frac{\partial Q}{\partial x_i} + u_i\frac{\partial Q}{\partial u}\right)\\
&+C\left(Q - (-A_{ik}u_{ik} - B_iu_i - Cu(t,\mathbf{x}))\frac{\partial Q}{\partial u_t} - u_j\frac{\partial Q}{\partial u_j}\right) = 0.
\end{aligned} \tag{4.38}$$

Looking at only the first two terms gives

$$u_{il}u_k : A_{ik}\frac{\partial^2 Q}{\partial u\partial u_j} = 0, \tag{4.39}$$

$$u_iu_{kj} : A_{ik}\frac{\partial^2 Q}{\partial u\partial u_l} = 0. \tag{4.40}$$

From the above, one notices that $\alpha_j(t,x,u) = \alpha_j(t,x)$, hence

$$Q = \alpha_1(t)u_t + \alpha_j(t,\mathbf{x})u_j + \alpha_4(t,\mathbf{x},u). \tag{4.41}$$

Thus the *ansatz* which Gaeta and Quintero [2] used in determining the symmetries of the FPE, *viz.*

$$H = \tau(t)\frac{\partial}{\partial t} + \xi_j(t,\mathbf{x})\frac{\partial}{\partial x_j} + \eta(t,\mathbf{x},u)\frac{\partial}{\partial u}, \tag{4.42}$$

has been verified.

4.4 Deriving the Determining Equations of the FPE using the *ansatz*

It is fully understood that Lie's approach is used to acquire the vector field in which the function will be exploited, then with sophisticated mathematics, the determining equations which will be used to forecast the symmetries of this particular equation in question are obtained. These calculations are done in Gaeta and Quintero [2] for the FPE; a review of their work is made in the interest of understanding how the determining equations, the general solution and symmetries of the FPE, are obtained. In the previous section, one obtained that the transformations for the FPE have to be projectable, i.e. (4.42).

An FPE given as

$$\frac{\partial u}{\partial t} + A_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k} + B_i\frac{\partial u}{\partial x_i} + Cu = 0, \tag{4.43}$$

where

$$A_{ik}(t, \mathbf{x}) = -\frac{1}{2}(GG^T)^{ik}, \tag{4.44}$$

$$B_i(t, \mathbf{x}) = f^i - \frac{\partial}{\partial x_i}(GG^T)^{ik}, \tag{4.45}$$

$$C(t, \mathbf{x}) = (\frac{\partial}{\partial x_i} \cdot f^i) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_k}(GG^T)^{ik}, \tag{4.46}$$

is solved. In order to find the determining equations of (4.43) the second prolongation of the symmetry operator H is required.

$$H^{[2]} = \tau(t)\frac{\partial}{\partial t} + \xi_i\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial u} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_i\frac{\partial}{\partial u_i} + \zeta_{ik}\frac{\partial}{\partial u_{ik}}, \tag{4.47}$$

where

$$u_t = \frac{\partial}{\partial t}, \tag{4.48}$$

$$u_i = \frac{\partial u}{\partial x_i}, \tag{4.49}$$

$$u_{ik} = \frac{\partial^2 u}{\partial x_i \partial x_k}, \tag{4.50}$$

and thus in general

$$u_{(i_{1\to N})} = \frac{\partial^N u}{\partial x_{i_1 \dots \partial x_{i_N}}}, \tag{4.51}$$

where

$$i_{1\to N} = i_1 i_2 \dots i_N. \tag{4.52}$$

The remaining extended infinitesimals are

$$\zeta_t = D_{(t)}(\eta) - u_t D_{(t)}(\tau) - u_j D_{(t)}(\xi_j), \tag{4.53}$$

$$\zeta_i = D_{(i)}(\eta) - u_t D_{(i)}(\tau) - u_j D_{(i)}(\xi_j), \tag{4.54}$$

$$\zeta_{ik} = D_{(k)}(\zeta_i) - u_{it} D_{(k)}(\tau) - u_{ij} D_{(k)}(\xi_j). \tag{4.55}$$

The D operator is the *total derivative* operator defined as

$$D_{(t)} = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{(tj)} \frac{\partial}{\partial u_j} + \dots + u_{(ti_{1\to N})} \frac{\partial}{\partial u_{(i_1 \to N)}} + \dots, \tag{4.56}$$

$$D_{(i)} = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \dots + u_{(ii_{1\to N})} \frac{\partial}{\partial u_{(i_1 \to N)}} + \dots, \tag{4.57}$$

where

$$i_{1\to N} = i_1 i_2 \dots i_N. \tag{4.58}$$

Since the symmetry operator was chosen to be projectable, i.e. $\tau = \tau(t)$, the following results

$$\zeta_t = \frac{\partial \eta}{\partial t} - \frac{\partial \xi_j}{\partial t}\frac{\partial u}{\partial x_j} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \tau}{\partial t}\right)\frac{\partial u}{\partial t}, \tag{4.59}$$

$$\zeta_i = \frac{\partial \eta}{\partial x_i} + \frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x_i} - \frac{\partial \xi_j}{\partial x_i}\frac{\partial u}{\partial x_j}, \tag{4.60}$$

$$\begin{aligned} \zeta_{ik} = {} & \frac{\partial^2 \eta}{\partial x_i \partial x_k} + \frac{\partial}{\partial x_k}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_k} - \frac{\partial^2 \xi_j}{\partial x_i \partial x_k}\frac{\partial u}{\partial x_j} \\ & + \frac{\partial \eta}{\partial u}\frac{\partial^2 u}{\partial x_i \partial x_k} - \frac{\partial \xi_j}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_k} - \frac{\partial \xi_j}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial^2 \eta}{\partial u^2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}. \end{aligned} \tag{4.61}$$

An application of our second prolongation on the FPE should be zero in order for the infinitesimal generator H to be a symmetry of (4.43). This means

$$\begin{aligned}
&\frac{\partial \eta}{\partial t} - \frac{\partial \xi_j}{\partial t}\frac{\partial u}{\partial x_j} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \tau}{\partial t}\right)\frac{\partial u}{\partial t} + A_{ik}\left(\frac{\partial^2 \eta}{\partial x_i \partial x_k} + \frac{\partial}{\partial x_k}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_i}\right. \\
&+ \frac{\partial}{\partial x_i}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_k} - \frac{\partial^2 \xi_j}{\partial x_i \partial x_k}\frac{\partial u}{\partial x_j} + \frac{\partial \eta}{\partial u}\frac{\partial^2 u}{\partial x_i \partial x_k} \\
&\left.- \frac{\partial \xi_j}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_k} - \frac{\partial \xi_j}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial^2 \eta}{\partial u^2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\right) + B_i\left(\frac{\partial \eta}{\partial x_i} + \frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x_i} - \frac{\partial \xi_j}{\partial x_i}\frac{\partial u}{\partial x_j}\right) \\
&+ C\eta + \left(\xi_j\frac{\partial A_{ik}}{\partial x_j} + \tau\frac{\partial A_{ik}}{\partial t}\right)\frac{\partial^2 u}{\partial x_i \partial x_k} + \left(\xi_j\frac{\partial B_i}{\partial x_j} + \tau\frac{\partial B_i}{\partial t}\right)\frac{\partial u}{\partial x_i} \\
&+ (\xi_j\frac{\partial C}{\partial x_j} + \tau\frac{\partial C}{\partial t})u = 0.
\end{aligned} \tag{4.62}$$

Substitute for $\frac{\partial u}{\partial t}$, i.e.

$$\frac{\partial u}{\partial t} = -A_{ij}\frac{\partial^2 u(t,\mathbf{x})}{\partial x_i \partial x_j} - B_i\frac{\partial u}{\partial x_i} - Cu(t,\mathbf{x}), \tag{4.63}$$

in the above equation. This yields

$$\begin{aligned}
&\frac{\partial \eta}{\partial t} - \frac{\partial \xi_j}{\partial t}\frac{\partial u}{\partial x_j} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \tau}{\partial t}\right)\left(-A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} - B_i\frac{\partial u}{\partial x_i} - Cu\right) \\
&+ A_{ik}\left(\frac{\partial^2 \eta}{\partial x_i \partial x_k} + \frac{\partial}{\partial x_k}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{\partial \eta}{\partial u}\right)\frac{\partial u}{\partial x_k} - \frac{\partial^2 \xi_j}{\partial x_i \partial x_k}\frac{\partial u}{\partial x_j} + \frac{\partial \eta}{\partial u}\frac{\partial^2 u}{\partial x_i \partial x_k}\right. \\
&\left.- \frac{\partial \xi_j}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_k} - \frac{\partial \xi_j}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial^2 \eta}{\partial u^2}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\right) + B_i\left(\frac{\partial \eta}{\partial x_i} + \frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x_i} - \frac{\partial \xi_j}{\partial x_i}\frac{\partial u}{\partial x_j}\right) \\
&+ C\eta + \left(\xi_j\frac{\partial A_{ik}}{\partial x_j} + \tau\frac{\partial A_{ik}}{\partial t}\right)\frac{\partial^2 u}{\partial x_i \partial x_k} + \left(\xi_j\frac{\partial B_i}{\partial x_j} + \tau\frac{\partial B_i}{\partial t}\right)\frac{\partial u}{\partial x_i} \\
&+ \left(\xi_j\frac{\partial C}{\partial x_j} + \tau\frac{\partial C}{\partial t}\right)u = 0.
\end{aligned} \tag{4.64}$$

Now the terms are collected into four groups, which are coefficients of $\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}$, $\frac{\partial^2 u}{\partial x_i \partial x_k}$, $\frac{\partial u}{\partial x_i}$ and 1 respectively, and each of the four groups are equal to zero. An over-determined system of equations which are also known as determining equations are revealed

$$A_{ik}(t,\mathbf{x})\frac{\partial^2 \eta(t,\mathbf{x},u(t,\mathbf{x}))}{\partial u^2} = 0, \tag{4.65}$$

$$\tau(t)\frac{\partial A_{ik}}{\partial t} + \frac{\partial \tau}{\partial t}\tau A_{ik} + \xi(t,\mathbf{x})_r\frac{\partial A_{ik}}{\partial x_r} - A_{ir}\frac{\partial \xi_k}{\partial x_r} - A_{rk}\frac{\partial \xi_i}{\partial x_r} = 0, \tag{4.66}$$

$$\tau\frac{\partial B_i(t,\mathbf{x})}{\partial t}+\frac{\partial\tau}{\partial t}\tau B_i+\xi_r\frac{\partial B_i}{\partial x_r}-B_r\frac{\partial\xi_i}{\partial x_r}-\frac{\partial\xi_i}{\partial t}+A_{ik}\frac{\partial}{\partial x_k}\left(\frac{\partial\eta}{\partial u}\right)$$
$$+A_{ri}\frac{\partial}{\partial x_r}\left(\frac{\partial\eta}{\partial u}\right)-A_{rk}\frac{\partial^2\xi_i}{\partial x_r\partial x_k}=0, \tag{4.67}$$

and

$$\frac{\partial\eta}{\partial t}-\left(\frac{\partial\eta}{\partial u}-\frac{\partial\tau}{\partial t}\right)C(t,x)u+A_{ik}\frac{\partial^2\eta}{\partial x_i\partial x_k}+B_i\frac{\partial\eta}{\partial x_i}+C\eta+\left(\xi_r\frac{\partial C}{\partial x_r}+\tau\frac{\partial C}{\partial t}\right)u=0. \tag{4.68}$$

Since a non-degeneracy case was assumed, this means that the $A'_{ik}s$ components are not all zero, resulting in

$$\frac{\partial^2\eta}{\partial u^2}=0, \tag{4.69}$$

which means that η is linear in u

$$\eta=\alpha_1(t,\mathbf{x})+\alpha_2(t,\mathbf{x})u. \tag{4.70}$$

Substituting for η in (4.67) one obtains

$$\tau\frac{\partial B_i(t,\mathbf{x})}{\partial t}+\frac{\partial\tau}{\partial t}\tau B_i+\xi_r\frac{\partial B_i}{\partial x_r}-B_r\frac{\partial\xi_i}{\partial x_r}-\frac{\partial\xi_i}{\partial t}+A_{ik}\frac{\partial\alpha_2(t,\mathbf{x})}{\partial x_k}$$
$$+A_{ri}\frac{\partial\alpha_2(t,\mathbf{x})}{\partial x_r}-A_{rk}\frac{\partial^2\xi_i}{\partial x_i\partial x_k}=0. \tag{4.71}$$

After substituting for η, equation (4.68) splits into two separate equations: one as a coefficient of u and the other a coefficient of 1:

$$\left(\frac{\partial\alpha_2(t,\mathbf{x})}{\partial t}+\frac{\partial\tau}{\partial t}C+A_{ik}\frac{\partial^2\alpha_2(t,\mathbf{x})}{\partial x_i\partial x_k}+B_i\frac{\partial\alpha_2(t,\mathbf{x})}{\partial x_i}\right.$$
$$\left.+\xi_r\frac{\partial C}{\partial x_r}+\tau\frac{\partial C}{\partial t}\right)u+\left(\frac{\partial\alpha_1(t,\mathbf{x})}{\partial t}\right.$$
$$\left.+A_{ik}\frac{\partial^2\alpha_1(t,\mathbf{x})}{\partial x_i\partial x_k}+B_i\frac{\partial\alpha_1(t,\mathbf{x})}{\partial x_i}+C\alpha_1(t,\mathbf{x})\right)=0. \tag{4.72}$$

The two groups are each equal to zero, this is because u is explicit throughout to get

$$\frac{\partial\alpha_2(t,\mathbf{x})}{\partial t}+\frac{\partial\tau}{\partial t}C+A_{ik}\frac{\partial^2\alpha_2(t,\mathbf{x})}{\partial x_i\partial x_k}+B_i\frac{\partial\alpha_2(t,\mathbf{x})}{\partial x_i}$$
$$+\xi_r\frac{\partial C}{\partial x_r}+\tau\frac{\partial C}{\partial t}=0, \tag{4.73}$$

and

$$\frac{\partial\alpha_1(t,\mathbf{x})}{\partial t}+A_{ik}\frac{\partial^2\alpha_1(t,\mathbf{x})}{\partial x_i\partial x_k}+B_i\frac{\partial\alpha_1(t,\mathbf{x})}{\partial x_i}+C\alpha_1(t,\mathbf{x})=0. \tag{4.74}$$

Equation (4.74) is just the FPE for $\alpha_1(t,\mathbf{x})$. This equation is left alone from here on as it gives the infinite number of solutions symmetries. The relations (4.45) and (4.46) will now be used in (4.66), (4.71) and (4.73) to simplify the determining equations further.

$$\frac{\partial(\tau A_{ik})}{\partial t}+\left(\xi_r\frac{\partial A_{ik}}{\partial x_r}-A_{ir}\frac{\partial \xi_k}{\partial x_r}-A_{rk}\frac{\partial \xi_i}{\partial x_r}\right)=0, \tag{4.75}$$

$$\begin{aligned}&\frac{\partial(\xi_1-\tau f_i)}{\partial t}+f_r\frac{\partial \xi_i}{\partial x_r}-\xi_r\frac{\partial f_i}{\partial x_r}-A_{rk}\frac{\partial^2\xi_i}{\partial x_r\partial x_k}-2\bigg(\frac{\partial}{\partial t}\left(\tau\frac{\partial A_{ik}}{\partial x_k}\right)\\&+A_{ik}\frac{\partial\alpha_2(t,\mathbf{x})}{\partial x_k}-A_{rk}\frac{\partial^2\xi_i}{\partial x_r\partial x_k}-\frac{\partial A_{rk}}{\partial x_k}\frac{\partial\xi_i}{\partial x_r}+\xi_r\frac{\partial^2 A_{ik}}{\partial x_r\partial x_k}\bigg)=0,\end{aligned} \tag{4.76}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\alpha_2(t,\mathbf{x})+\tau\left(\frac{\partial f_i}{\partial x_i}+\frac{\partial^2 A_{ik}}{\partial x_i\partial x_k}\right)\right)f_i\frac{\partial\alpha_2(t,\mathbf{x})}{\partial x_i\partial x_k}\\&+2\frac{\partial A_{ik}}{\partial x_k}\frac{\partial\alpha_2(t,\mathbf{x})}{\partial x_i}+\xi_r\frac{\partial^2 f_i}{\partial x_i\partial x_r}+\xi_r\frac{\partial^3 A_{ik}}{\partial x_i\partial x_k\partial x_r}=0.\end{aligned} \tag{4.77}$$

One can manipulate the determining systems above. Firstly, one multiplies (4.75) by 2

$$2\frac{\partial(\tau A_{ik})}{\partial t}+\left(2\xi_r\frac{\partial A_{ik}}{\partial x_r}-2A_{ir}\frac{\partial \xi_r}{\partial x_r}-2A_{rk}\frac{\partial \xi_i}{\partial x_r}\right)=0, \tag{4.78}$$

differentiate with respect to the k-th spatial variable x_k

$$\begin{aligned}&2\frac{\partial}{\partial t}\left(\tau\frac{\partial A_{ik}}{\partial x_k}\right)+\bigg(2\xi_r\frac{\partial^2 A_{ik}}{\partial x_r\partial x_k}+2\frac{\partial\xi_r}{\partial x_k}\frac{\partial A_{ik}}{\partial x_r}-2\frac{\partial A_{ir}}{\partial x_k}\frac{\partial\xi_k}{\partial x_r}-2A_{ir}\frac{\partial^2\xi_k}{\partial x_r\partial x_k}\\&-2\frac{\partial A_{rk}}{\partial x_k}\frac{\partial \xi_i}{\partial x_r}-2A_{rk}\frac{\partial^2\xi_i}{\partial x_r\partial x_k}\bigg)=0,\end{aligned} \tag{4.79}$$

and using the repeated index summation convention, one can sum over all k; add the resulting equation (4.79) to (4.76), to arrive at

$$\begin{aligned}&2\frac{\partial}{\partial t}\left(\tau\frac{\partial A_{ik}}{\partial x_k}\right)+\bigg(2\xi_r\frac{\partial^2 A_{ik}}{\partial x_r\partial x_k}+2\frac{\partial\xi_r}{\partial x_k}\frac{\partial A_{ik}}{\partial x_r}-2\frac{\partial A_{ir}}{\partial x_k}\frac{\partial\xi_k}{\partial x_r}-2A_{ir}\frac{\partial^2\xi_k}{\partial x_r\partial x_k}\\&-2\frac{\partial A_{rk}}{\partial x_k}\frac{\partial \xi_i}{\partial x_r}-2A_{rk}\frac{\partial^2\xi_i}{\partial x_r\partial x_k}\bigg)+\frac{\partial(\xi_i-\tau f_i)}{\partial t}+f_r\frac{\partial \xi_i}{\partial x_r}-\xi_r\frac{\partial f_i}{\partial x_r}-A_{rk}\frac{\partial^2\xi_i}{\partial x_r\partial x_k}\\&-2\bigg(\frac{\partial}{\partial t}\left(\tau\frac{\partial A_{ik}}{\partial x_k}\right)+A_{ik}\frac{\partial\alpha_2(t,\mathbf{x})}{\partial x_k}-A_{rk}\frac{\partial^2\xi_i}{\partial x_r\partial x_k}-\frac{\partial A_{rk}}{\partial x_k}\frac{\partial\xi_i}{\partial x_r}\\&+\xi_r\frac{\partial^2 A_{ik}}{\partial x_r\partial x_k}\bigg),\end{aligned} \tag{4.80}$$

which then simplifies to

$$
\begin{aligned}
&\frac{\partial(\xi_i - \tau f_i)}{\partial t} + f_r \frac{\partial \xi_i}{\partial x_r} - \xi_r \frac{\partial f_i}{\partial x_r} - A_{rk} \frac{\partial^2 \xi_i}{\partial x_r \partial x_k} \\
&- 2\left(A_{ir} \frac{\partial^2 \xi_k}{\partial x_r \partial x_k} + A_{ik} \frac{\partial \alpha_2(t, \mathbf{x})}{\partial x_k} \right) = 0. \qquad (4.81)
\end{aligned}
$$

Next differentiating (4.75) with respect to k-th spatial variable x_k and the i-th spatial variable x_i

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\tau \frac{\partial^2 A_{ik}}{\partial x_i \partial x_k} \right) \\
&+ \Bigg(\frac{\partial \xi_r}{\partial x_i} \frac{\partial^2 A_{ik}}{\partial x_r \partial x_k} + \xi_r \frac{\partial^3 A_{ik}}{\partial x_i \partial x_r \partial x_k} + \frac{\partial^2 \xi_r}{\partial x_i \partial x_k} \frac{\partial A_{ik}}{\partial x_r} + \frac{\partial \xi_r}{\partial x_k} \frac{\partial^2 A_{ik}}{\partial x_i \partial x_r} \\
&- \frac{\partial^2 A_{ir}}{\partial x_i \partial x_k} \frac{\partial \xi_k}{\partial x_r} - \frac{\partial A_{ir}}{\partial x_k} \frac{\partial^2 \xi_k}{\partial x_i \partial x_r} - \frac{\partial A_{ir}}{\partial x_i} \frac{\partial^2 \xi_k}{\partial x_r \partial x_k} - A_{ir} \frac{\partial^3 \xi_k}{\partial x_i \partial x_r \partial x_k} \\
&- \frac{\partial^2 A_{rk}}{\partial x_i \partial x_k} \frac{\partial \xi_i}{\partial x_r} - \frac{\partial A_{rk}}{\partial x_k} \frac{\partial^2 \xi_i}{\partial x_i \partial x_r} - \frac{\partial A_{rk}}{\partial x_i} \frac{\partial^2 \xi_i}{\partial x_r \partial x_k} \\
&- A_{rk} \frac{\partial^3 \xi_i}{\partial x_i \partial x_r \partial x_k} \Bigg) = 0. \qquad (4.82)
\end{aligned}
$$

Then differentiating equation (4.81) with respect to the i-th spatial variable x_i to obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial \xi_i}{\partial x_i} - \tau \frac{\partial f_i}{\partial x_i} \right) + \frac{\partial f_r}{\partial x_i} \frac{\partial \xi_i}{\partial x_r} + f_r \frac{\partial^2 \xi_i}{\partial x_i \partial x_r} \\
&- \frac{\partial \xi_r}{\partial x_i} \frac{\partial f_i}{\partial x_r} - \xi_r \frac{\partial^2 f_i}{\partial x_i \partial x_r} - \frac{\partial A_{rk}}{\partial x_i} \frac{\partial^2 \xi_i}{\partial x_r \partial x_k} - A_{rk} \frac{\partial^3 \xi_i}{\partial x_i \partial x_r \partial x_k} \\
&- 2\Bigg(\frac{\partial A_{ir}}{\partial x_i} \frac{\partial^2 \xi_k}{\partial x_r \partial x_k} + A_{ir} \frac{\partial^3 \xi_k}{\partial x_i \partial x_r \partial x_k} + \frac{\partial A_{ik}}{\partial x_i} \frac{\partial \alpha_2(t, \mathbf{x})}{\partial x_k} \\
&+ A_{ik} \frac{\partial^2 \alpha_2(t, \mathbf{x})}{\partial x_i \partial x_k} \Bigg) = 0. \qquad (4.83)
\end{aligned}
$$

One can now add and subtract (4.83) and (4.82), respectively from (4.77) to get

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\alpha_2(t,\mathbf{x}) + \tau\left(\frac{\partial f_i}{\partial x_i} + \frac{\partial^2 A_{ik}}{\partial x_i \partial x_k}\right)\right) + f_i\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_i} + A_{ik}\frac{\partial^2 \alpha_2(t,\mathbf{x})}{\partial x_i \partial x_k} \\
&+ 2\frac{\partial A_{ik}}{\partial x_k}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_i} + \xi_r\frac{\partial^2 f_i}{\partial x_i \partial x_r} + \xi_r\frac{\partial^3 A_{ik}}{\partial x_i \partial x_k \partial x_r} \\
&- \frac{\partial}{\partial t}\left(\tau\frac{\partial^2 A_{ik}}{\partial x_i \partial x_k}\right) \\
&- \left(\frac{\partial \xi_r}{\partial x_i}\frac{\partial^2 A_{ik}}{\partial x_r \partial x_k} + \xi_r\frac{\partial^3 A_{ik}}{\partial x_i \partial x_k \partial x_r} + \frac{\partial^2 \xi_r}{\partial x_i \partial x_k}\frac{\partial A_{ik}}{\partial x_r} + \frac{\partial \xi_r}{\partial x_k}\frac{\partial^2 A_{ik}}{\partial x_i \partial x_r}\right. \\
&- \frac{\partial^2 A_{ir}}{\partial x_i \partial x_k}\frac{\partial \xi_k}{\partial x_r} - \frac{\partial A_{ir}}{\partial x_k}\frac{\partial^2 \xi_k}{\partial x_i \partial x_r} - \frac{\partial A_{ir}}{\partial x_i}\frac{\partial^2 \xi_r}{\partial x_k \partial x_r} - A_{ir}\frac{\partial^3 \xi_k}{\partial x_i \partial x_r \partial x_k} \\
&\left.- \frac{\partial^2 A_{rk}}{\partial x_i \partial x_k}\frac{\partial \xi_i}{\partial x_r} - \frac{\partial A_{rk}}{\partial x_k}\frac{\partial^2 \xi_i}{\partial x_i \partial x_r} - \frac{\partial A_{rk}}{\partial x_i}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} - A_{rk}\frac{\partial^3 \xi_i}{\partial x_i \partial x_r \partial x_k}\right) \\
&+ \frac{\partial}{\partial t}\left(\frac{\partial \xi_i}{\partial x_i} - \tau\frac{\partial f_i}{\partial x_i}\right) + \frac{\partial f_r}{\partial x_i}\frac{\partial \xi_i}{\partial x_r} + f_r\frac{\partial^2 \xi_i}{\partial x_i \partial x_r} \\
&- \frac{\partial \xi_r}{\partial x_i}\frac{\partial f_i}{\partial x_r} - \xi_r\frac{\partial^2 f_i}{\partial x_i \partial x_r} - \frac{\partial A_{rk}}{\partial x_i}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} - A_{rk}\frac{\partial^3 \xi_i}{\partial x_i \partial x_r \partial x_k} \\
&- 2\left(\frac{\partial A_{ir}}{\partial x_i}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k} + A_{ir}\frac{\partial^3 \xi_k}{\partial x_i \partial x_r \partial x_k} + \frac{\partial A_{ik}}{\partial x_i}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k}\right. \\
&\left.+ A_{ik}\frac{\partial^2 \alpha_2(t,\mathbf{x})}{\partial x_i \partial x_k}\right) = 0.
\end{aligned} \tag{4.84}
$$

This simplifies to

$$
\begin{aligned}
&\frac{\partial \alpha_2(t,\mathbf{x})}{\partial t} + \frac{\partial}{\partial t}\frac{\partial \xi_i}{\partial x_i} + f_i\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_i} + f_r\frac{\partial^2 \xi_i}{\partial x_i \partial x_r} - A_{rk}\frac{\partial^3 \xi_i}{\partial x_i \partial x_r \partial x_k} \\
&- A_{ik}\frac{\partial^2 \alpha_2(t,\mathbf{x})}{\partial x_i \partial x_k} = 0.
\end{aligned} \tag{4.85}
$$

This can also be written as follows:

$$
\left(\frac{\partial}{\partial t} + f_i\frac{\partial}{\partial x_i} - A_{ik}\frac{\partial^2}{\partial x_i \partial x_k}\right)\left(\alpha_2(t,\mathbf{x}) + \frac{\partial \xi_r}{\partial x_r}\right) = 0. \tag{4.86}
$$

Thus, giving a new system of determining equations

$$
\frac{\partial(\tau A_{ik})}{\partial t} + \left(\xi_r\frac{\partial A_{ik}}{\partial x_r} - A_{ir}\frac{\partial \xi_k}{\partial x_r} - A_{rk}\frac{\partial \xi_i}{\partial x_r}\right) = 0, \tag{4.87}
$$

$$
\begin{aligned}
&\frac{\partial(\xi_i - \tau f_i)}{\partial t} + f_r\frac{\partial \xi_i}{\partial x_r} - \xi_r\frac{\partial f_i}{\partial x_r} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} \\
&- 2\left(A_{ir}\frac{\partial^2 \xi_k}{\partial x_r \partial x_k} + A_{ik}\frac{\partial \alpha_2(t,\mathbf{x})}{\partial x_k}\right) = 0,
\end{aligned} \tag{4.88}
$$

and

$$\left(\frac{\partial}{\partial t} + f_i\frac{\partial}{\partial x_i} - A_{ik}\frac{\partial^2}{\partial x_i \partial x_k}\right)\left(\alpha_2(t,\mathbf{x}) + \frac{\partial \xi_r}{\partial x_r}\right) = 0. \tag{4.89}$$

Thus the same determining equations found by Gaeta and Quintero [2] are discovered.

4.5 Determining Equations of Stochastic differential equations

It is known that there is a partial correspondence between the point symmetries of an Itô equation and those of the associated FPE which preserve the normalization condition (4.3), rather than all symmetries of the FPE (4.2). The condition to be satisfied for a vector field of the form (4.70) to preserve (4.3) is discussed in Gaeta and Quintero [2] in detail, where the integral of $\alpha_1(t,\mathbf{x})$ is equated to 0, i.e.

$$\int_{-\infty}^{\infty} \alpha_1(t,\mathbf{x})dx_1...dx_N = 0, \tag{4.90}$$

and

$$\alpha_2(t,\mathbf{x}) = -div(\xi) = -\frac{\partial \xi^r}{\partial x_r}. \tag{4.91}$$

One observes that the complete probabilistic equivalence between the FPEs and the SDEs is maintained by the conditions discussed above, (4.90) and (4.91). With this observation, one can substitute (4.91) in the determining equations (4.88) and (4.89). The first of the determining equations (4.87) contains no α_2 term, therefore it is left alone. The second equation (4.88) becomes

$$\frac{\partial(\xi_i - \tau f_i)}{\partial t} + f_r\frac{\partial \xi_i}{\partial x_r} - \xi_r\frac{\partial f_i}{\partial x_r} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} = 0, \tag{4.92}$$

and the last equation (4.89) becomes zero. Therefore, the determining equations of an SDE associated with the FPE are given by

$$\frac{\partial(\tau A_{ik})}{\partial t} + \left(\xi_r\frac{\partial A_{ik}}{\partial x_r} - A_{ir}\frac{\partial \xi_k}{\partial x_r} - A_{rk}\frac{\partial \xi_i}{\partial x_r}\right) = 0, \tag{4.93}$$

$$\frac{\partial(\xi_i - \tau f_i)}{\partial t} + f_r\frac{\partial \xi_i}{\partial x_r} - \xi_r\frac{\partial f_i}{\partial x_r} - A_{rk}\frac{\partial^2 \xi_i}{\partial x_r \partial x_k} = 0. \tag{4.94}$$

4.6 Examples

In this section, several exercises are carried out to illustrate the discoveries in this chapter, most importantly, to investigate the relationship between the point symmetries of FPE and those of the associated Itô equation. The observation of the symmetries of the FPE that obey the normalization condition is carried out.

In these examples, $\alpha(t, \mathbf{x})$ will denote an arbitrary solution of the FPE. In all cases the FPE admits the symmetries

$$H_\alpha = \alpha(t, \mathbf{x})\partial_u \tag{4.95}$$

and the scaling symmetry

$$H_0 = u\partial_u \tag{4.96}$$

both implied by the linearity of the equation.

4.6.1 Example 1

As the first one-dimensional example, one considers the case where the instantaneous drift coefficient is given by $f(t, x) = 0$ and the instantaneous diffusion coefficient by $G(t, x) = \sigma_0 = \text{constant} \neq 0$, i.e. the equation

$$dx = \sigma_0 \, dW(t), \tag{4.97}$$

which represents a free particle subject to constant noise. The corresponding FPE is given as

$$u_t - \frac{\sigma_0^2}{2} u_{xx} = 0, \tag{4.98}$$

in which the dependent variable is u and independent variables are t and x. Let

$$H = \tau(t)\frac{\partial}{\partial t} + \xi(t, x)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}, \tag{4.99}$$

be its symmetry generator. The second prolongation of H is given by

$$H^{[2]} = H + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}}. \tag{4.100}$$

The determining equation is

$$H^{[2]}(u_t - \frac{\sigma_0^2}{2}u_{xx})\Big|_{(4.98)} = 0, \tag{4.101}$$

or equivalently

$$\zeta_t - \frac{\sigma_0^2}{2}\zeta_{xx} = 0. \tag{4.102}$$

Only the expansions of ζ_t and ζ_{xx} are needed. The determining equation, using the expansions, becomes

$$\eta_t + u_t(\eta_u - \tau_t) - u_x\xi_t - \frac{\sigma_0^2}{2}(\eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_{x^2}\eta_{uu} - 2u_{xx}\xi_x - u_x\xi_{xx}) = 0. \tag{4.103}$$

Now replace u_t by $\frac{\sigma_0^2}{2}u_{xx}$:

$$\eta_t - \frac{\sigma_0^2}{2}u_{xx}\tau_t - u_x\xi_t - \frac{\sigma_0^2}{2}\eta_{xx} - \sigma_0^2 u_x\eta_{xu} - \frac{\sigma_0^2}{2}u_{x^2}\eta_{uu} + \sigma_0^2 u_{xx}\xi_x + \frac{\sigma_0^2}{2}u_x\xi_{xx} = 0. \tag{4.104}$$

The separation of the coefficients yields:

$$u_{xx} : -\frac{\sigma_0^2}{2}\tau_t + \sigma_0^2\xi_x = 0, \tag{4.105}$$

$$u_{x^2} : -\frac{\sigma_0^2}{2}\eta_{uu} = 0, \tag{4.106}$$

$$u_x : \frac{\sigma_0^2}{2}\xi_{xx} - \xi_t - \sigma_0^2\eta_{xu} = 0, \tag{4.107}$$

and

$$1 : \eta_t - \frac{\sigma_0^2}{2}\eta_{xx} = 0. \tag{4.108}$$

The general solution of the system is:

$$\tau = C_1 + 2C_4 t + 2\sigma_0^2 C_6 t^2, \tag{4.109}$$

$$\xi = C_2 + C_4 x + \sigma_0^2 C_5 t + 2\sigma_0^2 C_6 t^2, \tag{4.110}$$

and

$$\eta = (C_3 - C_5 x - C_6(x^2 + \sigma_0^2 t))u + \alpha(t,x), \tag{4.111}$$

where the $C_i s$ are the constants and $\alpha(t,x)$ satisfies the heat equation. This yields the operators

$$H_1 = \frac{\partial}{\partial t}, \tag{4.112}$$

$$H_2 = \frac{\partial}{\partial x}, \tag{4.113}$$

$$H_3 = u\frac{\partial}{\partial u}, \tag{4.114}$$

$$H_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \tag{4.115}$$

$$H_5 = \sigma_0^2 t\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \tag{4.116}$$

$$H_6 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} - \frac{1}{2}(t + \frac{x^2}{\sigma_0^2})u\frac{\partial}{\partial u}, \tag{4.117}$$

and

$$H_\alpha = \alpha(t,x)\frac{\partial}{\partial u}. \tag{4.118}$$

Since the symmetries of the PDE (4.98) are obtained, one can find the symmetries of the SDE (4.97) associated with the PDE. From the determining equations (4.93) and (4.94) one obtains,

$$\frac{\partial \xi}{\partial t} + \frac{1}{2}\sigma_0^2\frac{\partial^2 \xi}{\partial x^2} = 0, \tag{4.119}$$

and

$$\frac{\partial \xi}{\partial x} - \frac{1}{2}\frac{\partial \tau}{\partial t} = 0. \tag{4.120}$$

Differentiating (4.120) with respect to the spatial variable x, results in the following:

$$\frac{\partial^2 \xi}{\partial x^2} = 0, \tag{4.121}$$

whereby the spatial infinitesimal ξ is linear in the spatial variable x

$$\xi = a(t)x + b(t). \tag{4.122}$$

Substituting (4.122) into (4.119) gives

$$\dot{a}(t)x + \dot{b}(t) = 0. \tag{4.123}$$

Separation by coefficients yields

$$a(t) = c_1, \tag{4.124}$$

and

$$b(t) = c_2, \tag{4.125}$$

where c_1 and c_2 are constants. Therefore the general solution is given as

$$\tau(t) = 2c_1 t + c_3, \tag{4.126}$$

$$\xi(t,x) = c_1 x + c_2. \tag{4.127}$$

Therefore H_1, H_2 and H_4 do indeed span the symmetry algebra of (4.97).

4.6.2 Example 2

The second case considered is when $f(t,x) = x$, $G(t,x) = 1$, i.e. the Itô equation

$$dx = xdt + dW(t), \tag{4.128}$$

the corresponding FPE is

$$u_t - \frac{1}{2}u_{xx} + xu_x + u = 0, \tag{4.129}$$

in which the dependent variable is u and independent variables are time variable t and spatial variable x. Let

$$H = \tau(t)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{4.130}$$

be its generator of symmetry. The second prolongation of H is

$$H^{[2]} = H + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}}. \tag{4.131}$$

The determining equation is

$$H^{[2]}(u_t - \frac{1}{2}u_{xx} + xu_x + u)\Big|_{(4.129)} = 0, \tag{4.132}$$

or equivalently

$$\zeta_t - \frac{1}{2}\zeta_{xx} + \xi u_x + x\zeta_x + \eta = 0. \tag{4.133}$$

Only the expansions of ζ_t, ζ_x and ζ_{xx} are needed. The determining equation, using the expansions, becomes

$$\begin{aligned} & \eta_t + u_t(\eta_u - \tau_t) - u_x\xi_t - \frac{1}{2}\eta_{xx} - u_x\eta_{xu} - \frac{1}{2}u_{xx}\eta_u \\ - \; & \frac{1}{2}u_{x^2}\eta_{uu} + u_{xx}\xi_x + \frac{1}{2}u_x\xi_{xx} \\ + \; & \xi u_x + x\eta_x + xu_x\eta_u - xu_x\xi_x + \eta = 0. \end{aligned} \tag{4.134}$$

Now replace u_t by $\frac{1}{2}u_{xx} - xu_x - u$:

$$\begin{aligned} & \eta_t + (\frac{1}{2}u_{xx} - xu_x - u)(\eta_u - \tau_t) - u_x\xi_t \\ - \quad & \frac{1}{2}\eta_{xx} - u_x\eta_{xu} - \frac{1}{2}u_{xx}\eta_u - \frac{1}{2}u_{x^2}\eta_{uu} \\ + \quad & +u_{xx}\xi_x + \frac{1}{2}u_x\xi_{xx} + \xi u_x + x\eta_x + xu_x\eta_u - xu_x\xi_x + \eta = 0. \end{aligned} \tag{4.135}$$

The separation of the coefficients yields

$$u_{xx} : \xi_x - \frac{1}{2}\tau_t = 0, \tag{4.136}$$

$$u_{x^2} : -\frac{1}{2}\eta_{uu} = 0, \tag{4.137}$$

$$u_x : \xi - \eta_{xu} - \xi_t - x\xi_x + \frac{1}{2}\xi_{xx} + x\tau_t = 0, \tag{4.138}$$

and

$$1 : \eta_t - \frac{1}{2}\eta_{xx} + x\eta_x + \eta + u\tau_t - u\eta_u = 0. \tag{4.139}$$

The general solution of the system is

$$\tau = C_1 + e^{2t}C_5 - e^{-2t}C_6, \tag{4.140}$$

$$\xi = e^tC_3 + e^{-t}C_4 + e^{2t}xC_5 + xe^{-2t}C_6, \tag{4.141}$$

and

$$\eta = (C_2 + e^{-t}2xC_4 - e^{2t}C_5 + 2x^2C_6e^{-2t})u + \alpha(t,x), \tag{4.142}$$

where the C_is are the constants and $\alpha(t,x)$ satisfies the heat equation. One obtains the operators

$$H_1 = \frac{\partial}{\partial t}, \tag{4.143}$$

$$H_2 = u\frac{\partial}{\partial u}, \tag{4.144}$$

$$H_3 = e^t\frac{\partial}{\partial x}, \tag{4.145}$$

$$H_4 = e^{-t}\left[\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}\right], \tag{4.146}$$

$$H_5 = e^{2t}\left[\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}\right], \tag{4.147}$$

$$H_6 = e^{-2t}\left[-\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2x^2u\frac{\partial}{\partial u}\right], \tag{4.148}$$

and

$$H_\alpha = \alpha(t,x)\frac{\partial}{\partial u}, \tag{4.149}$$

which generates an infinite dimensional Lie algebra as a result of H_α

One strives to unearth the symmetry generators of (4.128). The determining equations (4.93) and (4.94) gives

$$\frac{\partial \xi}{\partial t} + x\frac{\partial \xi}{\partial x} - \xi - x\frac{\partial \tau}{\partial t} + \frac{1}{2}\frac{\partial^2 \xi}{\partial x2} = 0, \tag{4.150}$$

and

$$\frac{\partial \xi}{\partial t} - \frac{1}{2}\frac{\partial \tau}{\partial t} = 0. \tag{4.151}$$

Differentiating (4.151) with respect to the spatial variable x, one obtains

$$\frac{\partial^2 \xi}{\partial x^2} = 0, \tag{4.152}$$

whereby the spatial infinitesimal ξ is linear in the spatial variable x.

$$\xi(t,x) = a(t)x + b(t). \tag{4.153}$$

Consequently, the derivative of the temporal infinitesimal with respect to time variable t is given as

$$\frac{\partial \tau}{\partial t} = 2a(t). \tag{4.154}$$

By substituting (4.153) and (4.154) into (4.150) we get

$$\dot{a}(t)x + \dot{b}(t) - b(t) - 2a(t)x = 0. \tag{4.155}$$

Separation of coefficients yields

$$a(t) = e^{2t}c_1, \tag{4.156}$$

and

$$b(t) = e^{t}c_2. \tag{4.157}$$

Substituting (4.156) and (4.157) into (4.153) and (4.154), results in the following general solution

$$\xi(t,x) = e^{2t}c_1x + e^{t}c_2, \tag{4.158}$$

and

$$\tau(t) = e^{2t}c_1 + c_3. \tag{4.159}$$

The symmetry generators are

$$H_1 = \frac{\partial}{\partial t}, \tag{4.160}$$

$$H_3 = e^t \frac{\partial}{\partial x}, \tag{4.161}$$

and

$$\tilde{H}_5 = e^{2t}\left(\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right). \tag{4.162}$$

Hence H_1, H_3 and a projection of H_5 to (t, x) space, given by $\tilde{H}_5$ are symmetries of (4.128).

4.6.3 Example 3

As a first example in two space dimensions (with coordinates (x, y), one chooses the instantaneous diffusion coefficient and the instantaneous drift coefficient to be

$$\mathbf{G} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2k^2} \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} y \\ -k^2 y \end{pmatrix} \tag{4.163}$$

respectively. Thus, considering the equations

$$dx = ydt, \qquad dy = -k^2 ydt + \sqrt{2k^2}\, dW(t) \tag{4.164}$$

with k^2 a positive constant. The corresponding FPE is the Kramers equation

$$u_t = k^2 u_{yy} - yu_x + k^2 yu_y + k^2 u, \tag{4.165}$$

in which the dependent variable is u and the independent variables are the time variable t, the first spatial variable x and the second spatial variable y. Let

$$H = \tau(t)\frac{\partial}{\partial t} + \xi^2(t,x,y)\frac{\partial}{\partial x} + \xi^3(t,x,y)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u}, \tag{4.166}$$

be its symmetry generator. The second prolongation of H is

$$H^{[2]} = H + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{ty} \frac{\partial}{\partial u_{ty}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{xy} \frac{\partial}{\partial u_{xy}} + \zeta_{yy} \frac{\partial}{\partial u_{yy}}. \tag{4.167}$$

The determining equation is

$$H^{[2]}(u_t - k^2 u_{yy} + y u_x - k^2 y u_y + k^2 u)\Big|_{(4.165)} = 0, \tag{4.168}$$

or equivalently

$$\zeta_t - k^2 \zeta_{yy} + \xi^3 u_x + y\zeta_x - k^2 \xi^3 u_y - k^2 y \zeta_y + k^2 \eta. \tag{4.169}$$

Only the expansions of ζ_t, ζ_x, ζ_y and ζ_{yy} are needed. The determining equation, using the expansions, becomes

$$\begin{aligned} & \eta_t + u_t(\eta_u - \tau_t) - u_x \xi^2_t - u_y \xi^3_t - k^2(\eta_{yy} + 2u_y \eta_{yu} \\ + \; & u_{yy}\eta_u + u_{y^2}\eta_{uu} - 2u_{yy}\xi^3_y - u_y \xi^3_{yy}) + \xi^3 u_x \\ + \; & y(\eta_x + u_x \eta_u - u_x \xi^3_x) - k^2 \xi^3 u_y - k^2 y(\eta_y + u_y \eta_u \\ - \; & u_y \xi^3_y) - k^2 \eta = 0. \end{aligned} \tag{4.170}$$

Now replace u_t by $k^2 u_{yy} - y u_x + k^2 y u_y + k^2 u$:

$$\begin{aligned} & \eta_t + (k^2 u_{yy} - y u_x + k^2 y u_y + k^2 u)(\eta_u - \tau_t) - u_x \xi^2_t \\ & - u_y \xi^3_t - k^2(\eta_{yy} + 2u_y \eta_{yu} + u_{yy}\eta_u + u_{y^2}\eta_{uu} - 2u_{yy}\xi^3_y - u_y \xi^3_{yy}) \\ & + \xi^3 u_x + y(\eta_x + u_x \eta_u - u_x \xi^3_x) - k^2 \xi^3 u_y - k^2 y(\eta_y \\ & + u_y \eta_u - u_y \xi^3_y) - k^2 \eta = 0. \end{aligned} \tag{4.171}$$

The separation of the coefficients yield

$$u_{y^2} : -k^2 \eta_{uu} = 0, \tag{4.172}$$

$$u_{yy} : -k^2 \tau_t + 2k^2 \xi^3_y = 0, \tag{4.173}$$

$$u_y : k^2 y \xi^3_y - k^2 \xi^3 + k^2 \xi^3_{yy} - 2k^2 \eta_{yu} - \xi^3_t - k^2 y \tau_t = 0, \tag{4.174}$$

$$u_x : y\tau_t - \xi^2_t + \xi^3 - y\xi^2_x = 0, \tag{4.175}$$

and

$$1 : \eta_t - k^2\eta_{yy} + y\eta_x - k^2 y\eta_y - k^2\eta + uk^2\eta_u - uk^2\tau_t = 0. \tag{4.176}$$

The general solution of the system is

$$\tau = C_1, \tag{4.177}$$

$$\xi^2 = C_5 + C_3 e^{-k^2 t} k^{-2} + C_4 t + C_6 e^{k^2 t} k^{-2}, \tag{4.178}$$

$$\xi^3 = C_4 - C_3 e^{-k^2 t} + C_6 e^{k^2 t}, \tag{4.179}$$

and

$$\eta = (C_2 - \frac{1}{2} C_4 (y + k^2 x) - C_6 y e^{k^2 t}) u + \alpha(t, x, y), \tag{4.180}$$

where the $C_i s$ are the constants and $\alpha(t, x)$ satisfies the two space dimensional heat equation. One obtains the operators as

$$H_1 = \frac{\partial}{\partial t}, \tag{4.181}$$

$$H_2 = u\frac{\partial}{\partial u}, \tag{4.182}$$

$$H_3 = e^{-k^2 t}\left[k^{-2}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right], \tag{4.183}$$

$$H_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - \frac{1}{2}(y + k^2 x)u\frac{\partial}{\partial u}, \tag{4.184}$$

$$H_5 = \frac{\partial}{\partial x}, \tag{4.185}$$

$$H_6 = e^{k^2 t}\left[k^{-2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - yu\frac{\partial}{\partial u}\right], \tag{4.186}$$

and

$$H_\alpha = \alpha(t, x, y)\frac{\partial}{\partial u}, \tag{4.187}$$

which generates an infinite dimensional Lie algebra as a result of H_α.

The goal is to unearth the symmetry generators of (4.164) from the determining equations (4.93) and (4.94).

The determining equations are as follows:

$$\frac{\partial \xi^1}{\partial y} = 0, \tag{4.188}$$

$$\frac{\partial \xi^2}{\partial y} = 0, \tag{4.189}$$

$$\frac{\partial \tau}{\partial t} = 0, \tag{4.190}$$

$$\frac{\partial \xi^1}{\partial t} + y\frac{\partial \xi^1}{\partial x} + \xi^2 = 0, \tag{4.191}$$

and

$$\frac{\partial \xi^2}{\partial t} + y\frac{\partial \xi^2}{\partial x} + k^2\xi^2 = 0. \tag{4.192}$$

From (4.190), it is observed that the temporal infinitesimal τ is a *constant*, i.e,

$$\tau(t) = c_1, \tag{4.193}$$

where c_1 is a constant. The determining equations are reduced to

$$\frac{\partial \xi^1}{\partial t} + y\frac{\partial \xi^1}{\partial x} = \xi^2, \tag{4.194}$$

and

$$\frac{\partial \xi^2}{\partial t} + y\frac{\partial \xi^2}{\partial x} = -k^2\xi^2. \tag{4.195}$$

Equating zero to the coefficients of powers of the second spatial variable y and the second spatial variable y independent terms in both equations gives

$$\frac{\partial \xi^1}{\partial t} - \xi^2 = 0, \tag{4.196}$$

$$\frac{\partial \xi^1}{\partial x} = 0, \tag{4.197}$$

$$\frac{\partial \xi^2}{\partial t} + k^2\xi^2 = 0, \tag{4.198}$$

and

$$\frac{\partial \xi^2}{\partial x} = 0. \tag{4.199}$$

The general solution is given as follows:

$$\tau = c_1, \tag{4.200}$$

$$\xi^1 = c_2 k^{-2} e^{-k^2 t} + c_3, \tag{4.201}$$

and

$$\xi^2 = -c_2 e^{-k^2 t}. \tag{4.202}$$

The symmetry generators are H_1, H_3 and H_5.

4.6.4 Example 4

One considers next the case where the instantaneous diffusion and drift are

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} \frac{a_1}{x} \\ a_2 \end{pmatrix} \tag{4.203}$$

respectively.

i.e. the equation

$$dx = \frac{a_1}{x} dt + dW_1(t) \text{ and } \quad dy = a_2 dt + dW_2(t) \tag{4.204}$$

The associated FPE is

$$u_t = \frac{1}{2}(u_{xx} + u_{yy}) + \frac{a_1}{x^2} u - \frac{a_1}{x} u_x - a_2 u_y, \tag{4.205}$$

in which the dependent variable is u and the independent variables are the time variable t, the first spatial variable x and the second spatial variable y. Let

$$H = \tau(t)\frac{\partial}{\partial t} + \xi^2(t,x)\frac{\partial}{\partial x} + \xi^3(t,y)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u}, \tag{4.206}$$

be its generator of symmetry. The second prolongation of H is

$$\begin{aligned} H^{[2]} = H &+ \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} \\ &+ \zeta_{ty} \frac{\partial}{\partial u_{ty}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{xy} \frac{\partial}{\partial u_{xy}} + \zeta_{yy} \frac{\partial}{\partial u_{yy}}. \end{aligned} \tag{4.207}$$

The determining equation is

$$H^{[2]}(u_t - \frac{1}{2}(u_{xx} + u_{yy}) - \frac{a_1}{x^2} u + \frac{a_1}{x} u_x + a_2 u_y \Big|_{(4.205)} = 0, \tag{4.208}$$

or equivalently

$$\zeta_t - \frac{1}{2}(\zeta_{xx} + \zeta_{yy}) - a_1 x^{-2}\eta + 2a_1 x^{-3}u + a_1 x^{-1}\zeta_x - a_1 x^{-2}u_x + a_2\zeta_y. \tag{4.209}$$

Only the expansions of ζ_t, ζ_x, ζ_y, ζ_{xx} and ζ_{yy} are needed. The determining equation, using the expansions, becomes

$$\begin{aligned} & \eta_t + u_t(\eta_u - \tau_t) - u_x\xi^2_t - u_y\xi^3_t - \frac{a_1}{x^2}\eta + 2\frac{a_1}{x^3}u \\ - & \frac{a_1}{x^2}u_x + a_2(\eta_y + u_y\eta_u - u_y\xi^3_y) + \frac{a_1}{x}(\eta_x + u_x\eta_u - u_x\xi^2_x) \\ - & \frac{1}{2}\eta_{xx} - u_x\eta_{xu} - \frac{1}{2}u_{xx}\eta_u - \frac{1}{2}u_{x^2}\eta_{uu} + u_{xx}\xi^2_x \\ + & \frac{1}{2}u_x\xi^2_{xx} - \frac{1}{2}\eta_{yy} - u_y\eta_{yu} - \frac{1}{2}u_{yy}\eta_u - \frac{1}{2}u_{y^2}\eta_{uu} + u_{yy}\xi^3_y \\ + & \frac{1}{2}u_y\xi^3_{yy} = 0. \end{aligned} \tag{4.210}$$

Now replace u_t by $\frac{1}{2}(u_{xx} + u_{yy}) + \frac{a_1}{x^2}u - \frac{a_1}{x}u_x - a_2 u_y$:

$$\begin{aligned} & \eta_t + (\frac{1}{2}(u_{xx} + u_{yy}) + \frac{a_1}{x^2}u - \frac{a_1}{x}u_x - a_2u_y)(\eta_u - \tau_t) - u_x\xi^2_t - u_y\xi^3_t \\ - & \frac{a_1}{x^2}\eta + 2\frac{a_1}{x^3}u - \frac{a_1}{x^2}u_x + a_2(\eta_y + u_y\eta_u - u_y\xi^3_y) \\ + & \frac{a_1}{x}(\eta_x + u_x\eta_u - u_x\xi^2_x) - \frac{1}{2}\eta_{xx} - u_x\eta_{xu} - \frac{1}{2}u_{xx}\eta_u - \frac{1}{2}u_{x^2}\eta_{uu} \\ + & u_{xx}\xi^2_x + \frac{1}{2}u_x\xi^2_{xx} - \frac{1}{2}\eta_{yy} - u_y\eta_{yu} - \frac{1}{2}u_{yy}\eta_u \\ - & \frac{1}{2}u_{y^2}\eta_{uu} + u_{yy}\xi^3_y + \frac{1}{2}u_y\xi^3_{yy} = 0. \end{aligned} \tag{4.211}$$

The separation by the coefficients yields

$$u_{yy} : \xi^3_y - \frac{1}{2}\tau_t = 0, \tag{4.212}$$

$$u_{y^2} : \eta_{uu} = 0, \tag{4.213}$$

$$u_y : \frac{1}{2}\xi^3_{yy} - \eta_{yu} - a_2\xi^3_t + a_2\tau_t = 0, \tag{4.214}$$

$$u_{xx} : \xi^2_x - \frac{1}{2}\tau_t = 0, \tag{4.215}$$

$$u_{x^2} : \eta_{uu} = 0, \tag{4.216}$$

$$u_x : \frac{1}{2}\xi^2_{xx} - \eta_{xu} - \xi^2_t - \frac{a_1}{x^2} - \frac{a_1}{x}\xi^2_x + \frac{a_1}{x}\tau_t = 0, \tag{4.217}$$

and

$$1 : \eta_t - \frac{1}{2}(\eta_{xx} + \eta_{yy}) - \frac{a_1}{x^2}\eta + \frac{a_1}{x}\eta_x + a_2\eta_y + u\left(\frac{a_1}{x^2}\left(\frac{2}{x} + \eta_u - \tau_t\right)\right) = 0. \quad (4.218)$$

The general solution of the system is

$$\tau = C_1 + 2tC_4 - tC_5 + t^2C_6, \quad (4.219)$$

$$\xi^2 = xC_4 + txC_6, \quad (4.220)$$

$$\xi^3 = C_3 + (y + a_2t)C_4 + tyC_6, \quad (4.221)$$

and

$$\eta = (C_2 - 2C_4 + C_5(y - a_2t) + C_6(t(a_1 + a_2y - 1) - \frac{1}{2}(x^2 + y^2a_2^2t^2)))u + \alpha(t, x, y), \quad (4.222)$$

where the C_is are the constants and $\alpha(t, x, y)$ satisfies the heat equation. One obtains the operators as

$$H_1 = \frac{\partial}{\partial t}, \quad (4.223)$$

$$H_2 = u\frac{\partial}{\partial u}, \quad (4.224)$$

$$H_3 = \frac{\partial}{\partial y}, \quad (4.225)$$

$$H_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (y + a_2t)\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}, \quad (4.226)$$

$$H_5 = -t\frac{\partial}{\partial y} + (y - a_2t)u\frac{\partial}{\partial u}, \quad (4.227)$$

$$H_6 = t(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + (t(a_1 + a_2y - 1) - \frac{1}{2}(x^2 + y^2 + a_2^2t^2))u\frac{\partial}{\partial u}, \quad (4.228)$$

and

$$H_\alpha = \alpha(t, x, y)\frac{\partial}{\partial u}. \quad (4.229)$$

One strives to unearth the symmetry generators of (4.204) utilizing (4.93) and (4.94).

The determining equations are as follows

$$\frac{\partial \xi^1}{\partial y} = 0, \quad (4.230)$$

$$\frac{\partial \xi^1}{\partial x} - \frac{1}{2}\frac{\partial \tau}{\partial t} = 0, \tag{4.231}$$

$$\frac{\partial \xi^2}{\partial x} = 0, \tag{4.232}$$

$$\frac{\partial \xi^2}{\partial y} - \frac{1}{2}\frac{\partial \tau}{\partial t} = 0, \tag{4.233}$$

$$\frac{\partial \xi^1}{\partial t} + \frac{a_1}{x}\frac{\partial \xi^1}{\partial x} + \frac{a_1}{x^2}\xi^1 - \frac{a_1}{x}\frac{\partial \tau}{\partial t} + \frac{1}{2}\frac{\partial^2 \xi^1}{\partial x^2} = 0, \tag{4.234}$$

and

$$\frac{\partial \xi^2}{\partial t} + a_2\frac{\partial \xi^2}{\partial y} - a_2\frac{\partial \tau}{\partial t} + \frac{1}{2}\frac{\partial^2 \xi^2}{\partial y^2} = 0. \tag{4.235}$$

One can substitute (4.233) into (4.235) and obtain the following quasi linear partial differential equation

$$\frac{\partial \xi^2}{\partial t} - a_2\frac{\partial \xi^2}{\partial y} = 0. \tag{4.236}$$

Equation (4.236) may be represented as

$$\frac{dt}{1} = \frac{dy}{-a_2} = \frac{d\xi^2}{0}. \tag{4.237}$$

The second spatial infinitesimal is given as a constant i.e.

$$\begin{aligned}\xi^2(t,y) &= c_1(d_1)\\ &= c_1(y + a_2 t),\end{aligned} \tag{4.238}$$

where c_1 is a function of the time and second spatial variables (t, y) and $d_1 = y + a_2 t$. Substituting (4.238) into (4.233) gives that the temporal infinitesimal is linear in time variable, i.e.

$$\tau = c_1 t + c_2, \tag{4.239}$$

where c_2 is a constant. Given the temporal infinitesimal $\tau(t) = c_1 t + c_2$, from (4.231) one notices that the first spatial infinitesimal is given as

$$\xi^1 = \frac{1}{2}c_1 x + F_1(t), \tag{4.240}$$

where F_1 is a function of time and from (4.233) the second spatial infinitesimal is given as

$$\xi^2 = \frac{1}{2}c_1 y + F_2(t), \tag{4.241}$$

where F_2 is a function of time. By substituting the first spatial infinitesimal (4.240) into (4.234) gives

$$\dot{F}_1(t) - \frac{3}{4}\frac{a_1}{x}c_1 + \frac{a_1}{x^2}F_1(t) = 0, \tag{4.242}$$

and separating by coefficient results in $F_1(t) = 0$, therefore the first spatial infinitesimal is given as

$$\xi^1 = \frac{1}{2}c_1 x. \tag{4.243}$$

Substituting the second spatial infinitesimal (4.241) into (4.235) gives

$$\dot{F}_2(t) = \frac{1}{2}a_2 c_1, \tag{4.244}$$

implying that

$$F_2(t) = \frac{1}{2}a_2 c_1 t + c_3. \tag{4.245}$$

Therefore, the second spatial infinitesimal can be written as follows

$$\xi^2 = \frac{1}{2}c_1 y + \frac{1}{2}a_2 c_1 t + c_3. \tag{4.246}$$

Hence, the symmetry generators of (4.204) are spanned by H_1, H_3 and

$$\tilde{H}_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (y + a_2 t)\frac{\partial}{\partial y} \tag{4.247}$$

which is a projection of H_4 to (t, x, y)- space.

4.6.5 Example 5

This last example has the instantaneous drift and diffusion given by

$$\mathbf{f} = \begin{pmatrix} -\frac{1}{2}x \\ -\frac{1}{2}y \end{pmatrix}, \tag{4.248}$$

and

$$\mathbf{G} = \begin{pmatrix} -y \\ x \end{pmatrix}. \tag{4.249}$$

Thus from Gaeta and Quintero's determining equations (4.93) and (4.94), one obtains

$$x\xi^1_y - y\xi^1_x + \xi^2 + \frac{1}{2}y\tau_t = 0, \tag{4.250}$$

$$\xi^1 + 2\xi^1_t - x\xi^1_x - y\xi^1_y + x\tau_t + y^2\xi^1_{xx} + x^2\xi^1_{yy} - 2xy\xi^1_{xy} = 0, \tag{4.251}$$

$$x\xi^2_y - y\xi^2_x - \xi^1 - \frac{1}{2}x\tau_t = 0, \tag{4.252}$$

and

$$\xi^2 + 2\xi^2_t - x\xi^2_x - y\xi^2_y + y\tau_t + y^2\xi^2_{xx} + x^2\xi^2_{yy} - 2xy\xi^2_{xy} = 0. \tag{4.253}$$

Since the temporal infinitesimal τ is a function of time only, one can further simplify these determining equations. Differentiating (4.250) once by the first spatial variable x and also by the second spatial variable y respectively as follows:

$$\xi^2_x = y\xi^1_{xx} - \xi^1_y - x\xi^1_{xy}, \tag{4.254}$$

and

$$\xi^2_y = \xi^1_x + y\xi^1_{xy} - x\xi^1_{yy}, \tag{4.255}$$

then substitute both these equations into (4.252), one obtains the following:

$$\xi^1 = x\xi^1_x + xy\xi^1_{xy} - x^2\xi^1_{yy} - y^2\xi^1_{xx} + y\xi^1_y + xy\xi^1_{xy} - \frac{1}{2}x\tau_t. \tag{4.256}$$

Substituting (4.256) into (4.251) gives

$$x\tau_t + 4\xi^1_t = 0, \tag{4.257}$$

separating by coefficients yields to

$$x : \tau_t = 0, \tag{4.258}$$

$$constant : \xi^1_t = 0, \tag{4.259}$$

one notices from (4.258) that the temporal infinitesimal τ is a *constant*, because the temporal infinitesimal is not a function of the spatial variables x and y

$$\tau = c_0, \tag{4.260}$$

where c_0 is a constant and from (4.259) one notices that the first spatial infinitesimal ξ^1 is a function of the first spatial variable x and the second spatial variable y.

$$\xi^1 = d(x, y). \tag{4.261}$$

The same procedure is done on the second spatial infinitesimal ξ^2 by differentiating the first spatial infinitesimal ξ^1 with respect to the first spatial variable x and the second spatial variable y respectively and substituting those equations found in (4.251). In this case, by separation of coefficients one obtains that the temporal infinitesimal τ is a *constant* since the temporal infinitesimal is not a function of both the spatial variables x and y. The second spatial infinitesimal ξ^2 is a function of both the first spatial variable x and the second spatial variable y.

$$\tau = c_0, \tag{4.262}$$

and

$$\xi^2 = e(x, y). \tag{4.263}$$

The simplified determining equations are as follows:

$$\tau = c_0, \tag{4.264}$$

$$x\xi^1_y - y\xi^1_x + \xi^2 = 0, \tag{4.265}$$

$$\xi^1 - x\xi^1_x - y\xi^1_y + y^2\xi^1_{xx} + x^2\xi^1_{yy} - 2xy\xi^1_{xy} = 0, \tag{4.266}$$

$$x\xi^2_y - y\xi^2_x - \xi^1 = 0, \tag{4.267}$$

and

$$\xi^2 - x\xi^2_x - y\xi^2_y + y^2\xi^2_{xx} + x^2\xi^2_{yy} - 2xy\xi^2_{xy} = 0. \tag{4.268}$$

Equation (4.265) and (4.267) gives the following:

$$\frac{dy}{x} = -\frac{dx}{y} = \frac{dt}{0} = -\frac{d\xi^1}{\xi^2} = \frac{d\xi^2}{\xi^1}. \tag{4.269}$$

From (4.269) one observes that

$$\frac{d\xi^1}{ds} = -\xi^2, \tag{4.270}$$

$$\frac{d\xi^2}{ds} = \xi^1, \tag{4.271}$$

$$\frac{dy}{ds} = x, \tag{4.272}$$

and

$$\frac{dx}{ds} = -y. \tag{4.273}$$

Differentiating (4.270) with respect to s variable, one obtains

$$\frac{d^2\xi^1}{ds^2} + \xi^1 = 0, \tag{4.274}$$

which gives the following solution of ξ^1

$$\begin{aligned} \xi^1 &= Pe^{is} + Ze^{-is} \\ &= P(\cos s + i \sin s) + Z(\cos s - i \sin s) \\ &= (P + Z)\cos s + (Pi - Zi)\sin s. \end{aligned} \tag{4.275}$$

Letting $(P + Z) = A$ and $(Pi - Zi) = B$, one obtains

$$\xi^1 = A\cos s + B\sin s, \tag{4.276}$$

where A and B are arbitrary constants. Consequently, from (4.270)

$$\xi^2 = A\sin s - B\cos s. \tag{4.277}$$

Also from (4.273), one observes that

$$\frac{d^2x}{ds^2} + x = 0, \tag{4.278}$$

which gives the following solution

$$x = a\cos s + b\sin s, \tag{4.279}$$

consequently,

$$y = a\sin s - b\cos s. \tag{4.280}$$

Now a representation of both $\cos s$ and $\sin s$ in terms of both x and y is required. Multiplying (4.279) by a constant a and (4.280) by a constant b, then subtract the two equations as follows:

$$ax - by = (a^2 + b^2)\cos s, \tag{4.281}$$

gives the value for cos s as follows:

$$\cos s = \frac{ax - by}{a^2 + b^2}. \tag{4.282}$$

Multiplying (4.279) by a constant b, then add (4.280) multiplied by a constant a as follows:

$$bx + ay = (a^2 + b^2)\sin s, \tag{4.283}$$

solves for sin s as follows:

$$\sin s = \frac{bx + ay}{a^2 + b^2}. \tag{4.284}$$

Now one can substitute (4.282) and (4.284) into (4.276) and (4.277) to obtain

$$\xi^1 = \frac{Aa + Bb}{a^2 + b^2}x + \frac{Ba - Ab}{a^2 + b^2}y, \tag{4.285}$$

and

$$\xi^2 = \frac{Ab - Ba}{a^2 + b^2}x + \frac{Bb + Aa}{a^2 + b^2}y, \tag{4.286}$$

where A, B, a and b are constants. From (4.269) one notices that

$$\frac{x^2 + y^2}{2} = constant. \tag{4.287}$$

Let

$$\frac{Aa + Bb}{a^2 + b^2} = c_1 z_1\left(\frac{x^2 + y^2}{2}\right), \tag{4.288}$$

and

$$\frac{Ba - Ab}{a^2 + b^2} = c_2 z_2\left(\frac{x^2 + y^2}{2}\right), \tag{4.289}$$

where c_1 and c_2 are constants and z_1 and z_2 are both functions of both x and y. Equation (4.285) and (4.286) becomes

$$\xi^1 = c_1 z_1\left(\frac{x^2 + y^2}{2}\right)x + c_2 z_2\left(\frac{x^2 + y^2}{2}\right)y, \tag{4.290}$$

$$\xi^2 = -c_2 z_2\left(\frac{x^2 + y^2}{2}\right)x + c_1 z_1\left(\frac{x^2 + y^2}{2}\right)y. \tag{4.291}$$

In summary, the general solution is as follows:

$$\tau = c_0, \tag{4.292}$$

$$\xi^1 = c_1 z_1\left(\frac{x^2+y^2}{2}\right)x + c_2 z_2\left(\frac{x^2+y^2}{2}\right)y, \tag{4.293}$$

$$\xi^2 = -c_2 z_2\left(\frac{x^2+y^2}{2}\right)x + c_1 z_1\left(\frac{x^2+y^2}{2}\right)y. \tag{4.294}$$

To demonstrate that a solution of one SODE is transformed to that of another, one can choose a simpler example where

$$z_1\left(\frac{x^2+y^2}{2}\right) = z_2\left(\frac{x^2+y^2}{2}\right) = 1. \tag{4.295}$$

The symmetry generators are as follows:

$$H_0 = \frac{\partial}{\partial t}, \tag{4.296}$$

$$H_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{4.297}$$

and

$$H_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{4.298}$$

4.7 Summary

The vector field (4.4) proposed by Gaeta and Quintero [2] was shown to have a mathematical basis, implying that the symmetry generator for the FPE has to be projectable. This discovery led to the findings of symmetry generators of the SDE associated with it. In the examples done, it is shown that some of the symmetries of the SDE obeyed the normalization condition and some did not; the ones which did are those which presented the probabilistic interpretation. It is important to note that the projectable vector field (4.4) ensures that the symmetries obtained from SODEs are a sub-algebra of the one obtained from its corresponding FPE.

A question that emerges is whether or not the projectable symmetries are the only type of Lie point transformations of the SODE. This question leads to an investigation in the work done by (Wafo Soh and Mahomed [41]; Ünal[42]; Fredericks and Mahomed [43]) where no assumption of the *ansatz* is made.

Chapter 5

Symmetries of Itô equation

5.1 Introduction

It is significant to mention that the efforts to succeed in solving stochastic ordinary differential equations (SODEs) in the context of group analysis are recent. Gaeta and Quintero [2] showed a relational existence between the symmetries of the Fokker-Planck equation (FPE) and those of its corresponding Itô SODE. They also managed to transform an Itô stochastic differential equation (SDE) into a different one by utilizing the FPE dependent symmetries. Later, Wafo Soh and Mahomed [41] showed how a circumvention of obtaining these symmetries can be achieved; that the Lie point symmetries of an Itô equation can be derived without the consultation of its connected FPE; and to transform an Itô stochastic differential equation (SDE) into a different one without using the FPE dependent symmetries as it had been done in Gaeta and Quintero [2]. In Wafo Soh and Mahomed [41], the determining equations of the first-order SDEs were linear stochastic partial differential equations, meaning that the determining equations incorporated a white noise term. The reason for the existence of the white noise term is because they abandoned the application of the invariance principle to the Wiener process, the property that drives the non-deterministic properties of the Itô SDE, which makes the instantaneous mean of the transformed Wiener process not zero under expectation. The determining equations constructed in Wafo Soh and Mahomed [41] are as follows:

$$\left(H^{[1]}(\dot{x}_j - f_j)\Big|_{\dot{x}=f} + \frac{1}{2}\sum_{k=1}^{M} G_i^k G_p^k \left(\frac{\partial^2 \xi_j}{\partial x_i \partial x_p} - f_j \frac{\partial^2 \tau}{\partial x_i \partial x_p} \right) \right)(t, \mathbf{X}(t)) = 0, \tag{5.1}$$

and

$$HG_i^j - G_i^j \left(\frac{\partial \xi_j}{\partial x_i} \right) + \frac{1}{2} G_i^j \left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i} + \frac{1}{2} G_i^j G_p^j \frac{\partial^2 \tau}{\partial x_i \partial x_p} + G_p^j \frac{\partial \tau}{\partial x_p} \frac{dB_j}{dt} \right) = 0, \tag{5.2}$$

where

$$H^{[1]} = H + \xi_j^{[1]} \frac{\partial}{\partial \dot{x}_j}, \tag{5.3}$$

where

$$\begin{aligned} \dot{x}_j &= \frac{dx_j}{dt} \\ &= D_t x_j, \end{aligned} \tag{5.4}$$

$$\begin{aligned} \xi_j^{[1]} &= D_t(\xi_j) - \dot{x}_j D_t(\tau) \\ &= \frac{\partial \xi_j}{\partial t} + \dot{x}_i \frac{\partial \xi_j}{\partial x_i} + \ddot{x}_i \frac{\partial}{\partial \dot{x}_i} + ... \end{aligned} \tag{5.5}$$

Applying the first prolongation on $(\dot{x}_j - f_j)$ at $\dot{\mathbf{x}} = \mathbf{f}$ can be represented as

$$H^{[1]}(\dot{x}_j - f_j)\Big|_{\dot{\mathbf{x}}=\mathbf{f}} = \xi_j^{[1]} - H(f_j), \tag{5.6}$$

where

$$i_{1\to N} = i_1 i_2 ... i_N, \tag{5.7}$$

and

$$j_{1\to M} = j_1 j_2 ... j_M. \tag{5.8}$$

The noise term $\frac{dB(t)}{dt}$ appears in (5.2), hence the system of (5.1)-(5.2) being classified as linear stochastic partial differential equations.

Contrary to the derivations of Wafo Soh and Mahomed [41], Ünal's [42] determining equations for forecasting the symmetries of the first-order SDE were rather dissimilar; they included no white noise term, so were deterministic determining equations. The reason for this difference was due to an extra condition, that the instantaneous diffusion coefficient of the temporal infinitesimal must be zero, i.e.

$$\Upsilon(\tau)^l(t, \mathbf{X}(t)) = 0 \quad \text{for each } l = 1, ..., M. \tag{5.9}$$

Ünal's [42] approach to solving the determining equations is based on the Itô multiplication table for the transformed variables,

$$d\bar{t}d\bar{t} = 0, \tag{5.10}$$

$$d\bar{W}_i(\bar{t})d\bar{t} = 0, \tag{5.11}$$

$$d\bar{W}_l(\bar{t})d\bar{W}_m(\bar{t}) = \delta_l^m d\bar{t}, \tag{5.12}$$

where δ_l^m is known as the *Kronecker delta*. It is a function of two variables, usually integers, which is 1 if they are equal, and 0 if otherwise.

$$\delta_l^m = \begin{cases} 1 & \text{if } l = m \\ 0 & \text{if } l \neq m. \end{cases} \tag{5.13}$$

The determining equations are as follows:

$$\left(H^{[1]}(\dot{x}_j - f_j)\Big|_{\dot{x}=f} + \frac{1}{2}\sum_{k=1}^{M} G_i^k G_p^k \left(\frac{\partial^2 \xi_j}{\partial x_i \partial x_p} - f_j \frac{\partial^2 \tau}{\partial x_i \partial x_p} \right) \right)(t, \mathbf{X}(t)) = 0, \tag{5.14}$$

and

$$HG_i^j - G_i^j \left(\frac{\partial \xi_j}{\partial x_i} \right) + \frac{1}{2} G_i^j \left(\frac{\partial \tau}{\partial t} + f_i \frac{\partial \tau}{\partial x_i} + \frac{1}{2} G_i^j G_p^j \frac{\partial^2 \tau}{\partial x_i \partial x_p} \right) = 0. \tag{5.15}$$

We notice that the system (5.14)-(5.15) is a non-stochastic differential equations since no noise term, $\frac{dB(t)}{dt}$ is present.

These different forms of determining equations of first-order SDEs led to the endurance of a perplexity mood in this field of study. Clarification on this matter was urgently requested and Fredericks and Mahomed [43] were successful in connecting the works done in both these articles, (Wafo Soh and Mahomed [41]; Ünal [42]). They found the very same extra condition instituted in Ünal's work [42] and in addition to that, they also obtained another condition on the temporal symmetry variable ensuring that the transformed Wiener differential still behaves like a standard Wiener process.

It is also noticed that the construction of the determining equations derived by Wafo Soh and Mahomed [41] is based on the *Itô's multiplication table* - simple mneumonics based on *Itô Isometry*, Øksendal [31], given as

	$dW_l(t)$	$dW_m(t)$	dt
$dW_l(t)$	dt	0	0
$dW_m(t)$	0	dt	0
dt	0	0	0

Here, $dW_l(t)$ and $dW_m(t)$ are two independent standard Wiener processes, where $l, m = 1, ..., N$. It is now evident that an adjustment to the classical Newton-Leibnitz chain rule in differential form, needed in order to apply invariance arguments to the spatial, temporal and Wiener variables, is necessary. The justification for this lies in the quadratic variation of the Wiener process, i.e. $(dW(t))^2$ has the mean value of dt, which is finite.

5.2 Objectives and Relevance of the study

In general, if there exists a technique of solving for the symmetries of SDEs without having to consult their corresponding partial differential equations (PDEs), the FPEs, this will be advantageous to mathematicians. This approach will save time in terms of mathematical computations and may be used to check whether one has obtained the correct symmetries when comparing with the method presented by Gaeta and Quintero [2]. It is worth recalling that Gaeta and Quintero [2] restricted their discussion to only projectable vector fields. In this chapter, a review on the methodology brought forth by Fredericks and Mahomed [43] of solving for the symmetries of the first-order SDEs is done. To be precise, this work focuses on maintaining the properties of the temporal, spatial and the Wiener process variables after the the application of the infinitesimal transformations. This approach, otherwise known as form invariance, will be instrumental in the understanding of the random time change formula in the context of Lie point symmetries without having any knowledge of the work of Øksendal [31]. The determining equations of the first-order SODEs are derived and are found to be non-stochastic.

A comparison between the determining equations obtained by Wafo Soh and Mahomed [41], Ünal[42] and Fredericks and Mahomed is done.

The study concludes with the same examples, discussed in the previous chapter, to provide evidence that the new methodology introduced by Fredericks and Mahomed [43] gives the same symmetries as the technique discussed in Gaeta and Quintero [2], if not more. The answer to the question asked in the summary of the previous chapter is given, namely, are projectable vector fields the solitary type of point symmetries for Itô's equations?

5.3 Derivation of the determining equations

This section reviews the steps given in Fredericks and Mahomed [43], for deriving the determining equations of an Itô equation.

We consider an Itô process

$$d\mathbf{X}(t) = \mathbf{f}(t, \mathbf{X}(t))dt + \mathbf{G}(t, \mathbf{X}(t))\, d\mathbf{W}(t), \tag{5.16}$$

where $\mathbf{f}$(t, $\mathbf{x}$) is an instantaneous drift coefficient, vector of N dimension and $\mathbf{G}$(t, $\mathbf{x}$) is an $N \times M$ instantaneous diffusion coefficient. It is thoroughly comprehended that the Lie Point Theorem symmetry approach for (ODEs) requires spatial and temporal infinitesimals $\xi(t,x)$ and $\tau(t,x)$ in its analysis. In the SODE framework, these entities are functionally based on the spatial stochastic process, $\mathbf{X}$(t) which has the same dimensions as the drift coefficient, and using Itô's formula (2.52), one has the j-th spatial infinitesimal, for $j = 1, ..., N$, and temporal infinitesimal are themselves solutions to Itô process given in component form, respectively, as

$$d\xi_j(t, \mathbf{X}(t)) = \Gamma(\xi_j)(t, \mathbf{X}(t))dt + \Upsilon(\xi_j)^l(t, \mathbf{X}(t))d\,W_l(t), \tag{5.17}$$

and

$$d\tau(t, \mathbf{X}(t)) = \Gamma(\tau)(t, \mathbf{X}(t))dt + \Upsilon(\tau)^l(t, \mathbf{X}(t))d\,W_l(t), \tag{5.18}$$

where $\Gamma(\xi_j)$, $\Upsilon(\xi_j)^l$, $\Gamma(\tau)$ and $\Upsilon(\tau)^l$ are the drift and diffusion coefficients of our spatial and temporal infinitesimals, respectively, and defined using (2.53) and (2.54). The Lie Point Theorem, [41], uses the determining equations to provide symmetries that would enable the transformation of a solution of the equation to another. These determining equations are in fact $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ derived from form invariant (ODE) point transformation analysis. The resultant higher-order equations of this form invariant analysis are functionally dependent on the solution of these equations. The method by Fredericks and Mahomed [43] is now reviewed, where a similar point transformation of (5.16)'s spatial, temporal and the Wiener variables is done.

$$\begin{aligned}\bar{X}_j(t) &= e^{\epsilon H}(X_j(t)) \\ &= \int^t \Gamma(e^{\epsilon H}(X_j(s)))ds + \int^t \Upsilon(e^{\epsilon H}(X_j(s)))d\,W(s),\end{aligned} \tag{5.19}$$

$$\begin{aligned}\bar{t} &= e^{\epsilon H}(t) \\ &= \int^t \Gamma(e^{\epsilon H(s)})ds + \int^t \Upsilon(e^{\epsilon H(s)})d\,W(s),\end{aligned} \tag{5.20}$$

and

$$d\bar{W}_l(\bar{t}) = \sqrt{\frac{d(e^{\epsilon H}(t))}{dt}}d\,W_l(t) \qquad \text{for each } l = 1, ..., M, \tag{5.21}$$

using the random time change formula and Itô's formula, where H is the symmetry generator

$$H = \tau(t, \mathbf{x}) \frac{\partial}{\partial t} + \xi_j(t, \mathbf{x}) \frac{\partial}{\partial x_j}, \tag{5.22}$$

with the spatial and temporal infinitesimal $\xi(t, \mathbf{x})$ and $\tau(t, \mathbf{x})$, respectively. The point transformation of the drift and diffusion coefficients is given by

$$f_j(\bar{t}, \bar{\mathbf{x}}) = e^{\epsilon H}(f_j(t, \mathbf{x})), \tag{5.23}$$

and

$$g_i^k(\bar{t}, \bar{\mathbf{x}}) = e^{\epsilon H}(g_i^k(t, \mathbf{x})), \tag{5.24}$$

for each $i, j = 1, ..., N$ and $k = 1, ..., N$. The transformations (5.19)-(5.21), (5.23) and (5.24) are used in combination with Itô's formula to form an invariant version of the original SODE (5.16).

$$d\bar{\mathbf{X}}(\bar{t}) = \mathbf{f}(\bar{t}, \bar{\mathbf{X}}(\bar{t}))d\bar{t} + \mathbf{G}(\bar{t}, \bar{\mathbf{X}}(\bar{t}))\, d\bar{\mathbf{W}}(\bar{t}). \tag{5.25}$$

5.4 Derivation of the random time change using form invariance

A derivation of the random time change formula for a Itô SDE using form invariance approach is to be constructed. This approach uses the philosophy that the properties of the time index and the Wiener process should be invariant under the Lie group transformations. This means all the properties of the instantaneous drift coefficient, instantaneous diffusion coefficient and the Wiener process must be preserved under form invariant transformation.

Trivially, the time index is deterministic since its expected value is itself, i.e.

$$\mathbb{E}[t] = t, \tag{5.26}$$

which alternatively can be written as

$$t = \int^t ds + \int^t 0\, dW(s), \tag{5.27}$$

where the drift instantaneous coefficient can be seen as the time change rate and that it is deterministic. Therefore, the expectation of the time rate change of t is

$$\mathbb{E}\Big[\int^t ds\Big] = \int^t ds. \tag{5.28}$$

Consequently, the probabilistic nature of the transformed time index should remain form invariant

$$\mathbb{E}\Big[\bar{t}(t,\omega)\Big|\mathbf{W}(t) = \mathbf{w}, \mathbf{X}(t) = \mathbf{x}\Big] = \bar{t}(t,\omega). \tag{5.29}$$

This implies that

$$\mathbb{E}\Big[\int^t \Gamma(e^{\epsilon H(s)})ds + \int^t \Upsilon(e^{\epsilon H(s)})dW(s)\Big] = \int^t \Gamma(e^{\epsilon H(s)})ds + \int^t \Upsilon(e^{\epsilon H(s)})dW(s). \tag{5.30}$$

One of the significant properties from probability theory is that the expectation of a Wiener process is *zero*, i.e.

$$\mathbb{E}\Big[\int^t \Upsilon(e^{\epsilon H(s)})dW(s)\Big] = 0. \tag{5.31}$$

Now applying condition (5.31) into (5.30) gives

$$\mathbb{E}\Big[\int^t \Gamma(e^{\epsilon H(s)})ds\Big] = \int^t \Gamma(e^{\epsilon H(s)})ds + \int^t \Upsilon(e^{\epsilon H(s)})dW(s). \tag{5.32}$$

Since the properties of the time index must be preserved; that the expectation of the time rate change must be deterministic, one notices that from the left-hand side of (5.32), the expectation of the time change rate has to be itself, i.e.

$$\mathbb{E}\Big[\int^t \Gamma(e^{\epsilon H}(s))ds\Big] = \int^t \Gamma(e^{\epsilon H}(s))ds, \tag{5.33}$$

implying that the instantaneous drift coefficient which is the rate of time change, is a function of time.

$$\Gamma(e^{\epsilon H}(s)) = k(s), \tag{5.34}$$

where $k(s)$ is an arbitrary function of time. The temporal infinitesimal drift coefficient, $\Gamma(e^{\epsilon H}(s))$ can be viewed as the time change rate, $c(s)$ in (2.76). From this observation, (5.32) becomes

$$\int^t \Gamma(e^{\epsilon H(s)})ds = \int^t \Gamma(e^{\epsilon H(s)})ds + \int^t \Upsilon(e^{\epsilon H(s)})dW(s). \tag{5.35}$$

For (5.35) to have mathematical basis, the instantaneous diffusion coefficient has to be zero

$$\Upsilon^l(e^{\epsilon H}(t))(t, \mathbf{X}(t)) = 0 \quad \text{for each } l = 1, ..., M. \tag{5.36}$$

Equation (5.36) is general for all the instantaneous diffusion coefficients. Expanding (5.36) by Taylor series gives

$$\Upsilon(1 + \epsilon\tau + \frac{\epsilon^2}{2!}H(\tau) + ...) = 0 \tag{5.37}$$

and this provides the same conclusion that Ünal[42] derived at $\mathcal{O}(\epsilon)$ using a form invariant argument on the Itô multiplication table i.e.

$$\Upsilon(\tau)(t, \mathbf{X}(t)) = 0. \tag{5.38}$$

Similarly, the transformed standard Wiener process, $\bar{\mathbf{W}}(\bar{t})$, which should still agree with the Itô multiplication table should be invariant in terms of the existence of an instantaneous mean of zero and variance $\bar{t}$, which implies that the following should still hold, viz:

the expectation of the transformed Wiener process is equal to zero:

$$\mathbb{E}\left[d\bar{W}_l(\bar{t})\middle|\mathbf{W}(t) = \mathbf{w}, \mathbf{X}(t) = \mathbf{x}\right] = 0. \tag{5.39}$$

The expectation of two independent transformed Wiener processes is zero if both the transformed Wiener processes are not of equal dimensions and $d\bar{t}$ if both the transformed Wiener processes are of equal dimensions:

$$\mathbb{E}\left[d\bar{W}_l(\bar{t})d\bar{W}_m(\bar{t})\middle|\mathbf{W}(t) = \mathbf{w}, \mathbf{X}(t) = \mathbf{x}\right] = d\bar{t}\delta_l^m. \tag{5.40}$$

Below is the demonstration that the expectation of a *brownian* motion is indeed zero, invariance argument (5.39) is substantial.

Expanding (5.39)

$$\mathbb{E}\left[\sqrt{\frac{\Gamma(e^{\epsilon H}(t))dt + \Upsilon^k(e^{\epsilon H}(t))dW_k(t)}{dt}}dW_l(t)\middle|\mathbf{W}(t) = \mathbf{w}, \mathbf{X}(t) = \mathbf{x}\right] = 0, \tag{5.41}$$

substituting the value zero for the instantaneous diffusion coefficient (5.36) into (5.41) gives

$$\mathbb{E}\left[\sqrt{\Gamma(e^{\epsilon H}(t))}dW_l(t)\middle|\mathbf{W}(t)=\mathbf{w},\mathbf{X}(t)=\mathbf{x}\right]=0. \tag{5.42}$$

Thus, the invariance argument (5.39) is satisfied.

As a result of the instantaneous diffusion coefficient being zero, the transformed time change formula (5.20) is given as

$$\begin{aligned}\bar{t} &= \int^t \Gamma(e^{\epsilon H}(s))(s,\mathbf{X}(s))ds \\ &= \int^t k(s)(s,\mathbf{X}(s))ds,\end{aligned} \tag{5.43}$$

and the transformed Wiener process (5.21) is given by

$$\begin{aligned}d\bar{W}_l(\bar{t}) &= \sqrt{\Gamma(e^{\epsilon H}(t))}dW_l(t) \\ &= \sqrt{k(t)}dW_l(t) \qquad \text{for each } l=1,...,M.\end{aligned} \tag{5.44}$$

One notices that under form invariance, the instantaneous drift coefficient for the random time change formula has to be a function of time and that the instantaneous diffusion coefficient has to be zero. However, the temporal infinitesimal can be a function of both the time and spatial variables. At this stage, one has shown that the properties of the infinitesimals has to remain the same under the form invariant approach. Now one needs to discuss the condition to be satisfied in order to ensure the recovery of the invariance preserving finite transformations.

5.5 Lie point SODE condition

The derivation of the condition to ensure the recovery of the invariance preserving finite transformations from the infinitesimal transformations is constructed. By expanding the drift term $f(\bar{t},\bar{X}(\bar{t}))d\bar{t}$ on the right-hand side of (5.25) gives

$$\mathbf{f}(t,\mathbf{X}(t))+\frac{\epsilon^k}{k!}\sum_{j=0}^{k}\binom{k}{j}\Gamma(H^{k-j}(t))H^j(\mathbf{f}). \tag{5.45}$$

Alternatively, this equation can be written in such a manner to demonstrate that all terms of order higher than $\mathcal{O}(\epsilon)$ be functionally dependent on terms of order $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$. This is achieved by adjusting the n and the $n-1$ terms of the expansion, as shown below

$$\Big\{\mathbf{f}(t,\mathbf{X}(t))+\epsilon(\Gamma(H(t))+H)\mathbf{f}(t,\mathbf{X}(t)) + \sum_{k=2}^{\infty}\frac{\epsilon^k}{k!}\Big((\Gamma(H(t))+H)^k\mathbf{f}(t,\mathbf{X}(t)) + \sum_{j=0}^{k-2}\binom{k}{k-j}\mathbf{f}(t,\mathbf{X}(t))H^j(\mathbf{f})(\Gamma(H^{k-j}(t))-[\Gamma(H(t))]^{k-j})\Big)\Big\}dt. \tag{5.46}$$

One shows that (5.45) and (5.46) are equivalent by comparing the following:

$$\sum_{j=0}^{k}\binom{k}{j}\Gamma(H^{k-j}(t))H^j(\mathbf{f}) = \Big((\Gamma(H(t))+H)^k\mathbf{f}(t,\mathbf{X}(t)) + \sum_{j=0}^{k-2}\binom{k}{k-j}\mathbf{f}(t,\mathbf{X}(t))H^j(\mathbf{f})(\Gamma(H^{k-j}(t))-[\Gamma(H(t))]^{k-j})\Big). \tag{5.47}$$

For this condition to be satisfied, one must show that both sides have the same terms when expanded. Expanding the left-hand side gives

$$\Big(\Gamma(H^k(t))+\frac{k!}{(k-1)!}\Gamma(H^{k-1}(t))H(t)+\frac{k!}{(k-2)!2!}\Gamma(H^{k-2}(t))H^2(t)+...+ \frac{k!}{(k-2)!2!}\Gamma(H^2(t))H^{k-2}(t)+\frac{k!}{(k-1)!}\Gamma(H(t))H^{k-1}(t)+H^k(t)\Big)\mathbf{f}. \tag{5.48}$$

Now one can expand the right-hand side. The *binomial theorem* proves that

$$(\Gamma(H(t))+H)^k(\mathbf{f}) = \sum_{j=0}^{k}\binom{k}{j}[\Gamma(H(t))]^j H^{k-j}(\mathbf{f}). \tag{5.49}$$

Therefore we expand

$$\sum_{j=0}^{k}\binom{k}{j}[\Gamma(H(t))]^j H^{k-j}(\mathbf{f}) + \sum_{j=0}^{k-2}\binom{k}{k-j}\mathbf{f}(t,\mathbf{X}(t))H^j(\mathbf{f})(\Gamma(H^{k-j}(t))-[\Gamma(H(t))]^{k-j})\Big). \tag{5.50}$$

This yields

$$\begin{aligned}&\Big(H^k + \frac{k!}{(k-1)!}[\Gamma(H(t))]H^{k-1} + \frac{k!}{(k-2)!2!}[\Gamma(H(t))]^2 H^{k-2} + ... + \\ &\frac{k!}{(k-2)!2!}[\Gamma(H(t))]^{k-2}H^2 + \frac{k!}{(k-1)!}[\Gamma(H(t))]^{k-1}H + [\Gamma(H(t))]^k\Big)\mathbf{f} \\ &-\Big([\Gamma(H(t))]^k + \frac{k!}{(k-1)!}[\Gamma(H(t))]^{k-1}H + \frac{k!}{(k-2)2!}[\Gamma(H(t))]^{k-2}H^2 + ... + \\ &\frac{k!}{(k-2)!2!}[\Gamma(H(t))]^2H^{k-2}\Big)\mathbf{f} + \Big(\Gamma(H^k(t)) + \frac{k!}{(k-1)!}[\Gamma(H^{k-1}(t))]H \\ &+ \frac{k!}{(k-2)!2!}[\Gamma(H^{k-2}(t))]H^2 + ... + \frac{k!}{(k-2)!2!}[\Gamma(H^2(t))]H^{k-2}\Big)\mathbf{f}. \end{aligned} \tag{5.51}$$

Cancelling out the like terms results in

$$\begin{aligned}&\Big(\Gamma(H^k(t)) + \frac{k!}{(k-1)!}\Gamma(H^{k-1}(t))H(t) + \frac{k!}{(k-2)!2!}\Gamma(H^{k-2}(t))H^2(t) + ... + \\ &\frac{k!}{(k-2)!2!}\Gamma(H^2(t))H^{k-2}(t) + \frac{k!}{(k-1)!}\Gamma(H(t))H^{k-1}(t) + H^k(t)\Big)\mathbf{f}. \end{aligned} \tag{5.52}$$

The result of expanding the right-hand side is equivalent to the results of the left-hand side.

In order to use the Lie Point Theorem in the SODE context, one requires that all terms of order higher than $\mathcal{O}(\epsilon)$ be functionally dependent on terms of $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$. As a result of this dependency, higher-order terms can be ignored completely and justifies the methods of Wafo Soh and Mahomed [41] and Ünal[42]. This dependency, however, imposes the following condition:

$$e^{\epsilon\Gamma(H(t))}(t, \mathbf{X}(t)) = \Gamma(e^{\epsilon H}(t)(t, \mathbf{X}(t))), \tag{5.53}$$

and the resultant relationship, by separation of coefficients of ϵ, between the drift components of the left-hand side and right-hand side of (5.25) can be expressed as

$$\Gamma(H^k(\mathbf{x}))(t, \mathbf{X}(t)) = (\Gamma(H(t)) + H)^k \mathbf{f}(t, \mathbf{X}(t)), \tag{5.54}$$

for $k = 1, 2, 3, ...$. Thus for $k = 1$, one has the first determining equation as

$$\Gamma(H(\mathbf{x})) = (\Gamma(H(t)) + H)\mathbf{f}(t, \mathbf{X}(t)), \tag{5.55}$$

which partially solves for the spatial and temporal infinitesimals. By using the determining equation (5.55) in (5.54) for the remaining higher-order equations, a direct functional dependency between the two is established by the following:

$$\Gamma(H^k(\mathbf{x})) = (\Gamma(H(t)) + H)^{k-1}\Gamma(H(\mathbf{x})) \qquad \text{for } k = 2, 3, 4, ... \tag{5.56}$$

A derivation of the remaining determining equation is required. If one expands the diffusion component $\mathbf{G}(\bar{t}, \bar{\mathbf{X}}(\bar{t}))d\bar{\mathbf{W}}(\bar{t})$ of (5.25) and then compares these components on both sides of (5.25) by separation of coefficients of ϵ, one obtains the following:

$$\Upsilon^l(H(\mathbf{x}))(t, \mathbf{X}(t)) = \left(\frac{\Gamma(H(t))}{2} + H\right) G^l(t, \mathbf{X}(t)), \tag{5.57}$$

$$\Upsilon^l(H^k(\mathbf{x}))(t, \mathbf{X}(t)) = \left(\frac{\Gamma(H(t))}{2} + H\right)^{k-1} \Upsilon^l(H(\mathbf{x})) \qquad \text{for } k = 2, 3, 4, ... \tag{5.58}$$

for each $l = 1, ..., M$, where (5.57) is the last determining equation required to solve the infinitesimals. The functional dependency of higher-ordered equations on zero and first-order ones is satisfied in (5.58). All that remains to be shown is that the determining equations are unique to their SODEs from which they are derived. If given the determining equations (5.55) and (5.57), the cononical symmetry that is immediately applicable is the time scaling symmetry $H = \frac{\partial}{\partial t}$. From this, one observes that the drift and diffusion coefficients have to be functions of the spatial variable only in order to satisfy (5.55) and (5.57). Thus, the SODE associated with its particular symmetry is given by

$$d\mathbf{X}(t) = \mathbf{f}(t, \mathbf{X}(t))dt + \mathbf{G}(t, \mathbf{X}(t))\, d\mathbf{W}(t). \tag{5.59}$$

Thus, the study has shown the following theorem which was partially proved in Wafo Soh and Mahomed [41], and the theorem modified by Fredericks and Mahomed [43].

Theorem 5.1 *(Lie Point Theorem for SODE, Fredericks and Mahomed [43])*

The Itô SODE

$$d\mathbf{X}(t) = \mathbf{f}(t, \mathbf{X}(t))dt + \mathbf{G}(t, \mathbf{X}(t))d\mathbf{W}(t), \tag{5.60}$$

has the following determining equations and conditions that have to hold in order to transform a solution of (5.60) to that of another solution using Lie point symmetry methods:

$$\Gamma(H(x))(t,\mathbf{X}(t)) = (\Gamma(H(t)) + H)f(t,\mathbf{X}(t)), \tag{5.61}$$

$$\Upsilon^l(H(x))(t,\mathbf{X}(t)) = \left(\frac{\Gamma H(t)}{2} + H\right)G^l(t,\mathbf{X}(t)), \tag{5.62}$$

$$e^{\epsilon\Gamma(H(t))}(t,\mathbf{X}(t)) = \Gamma(e^{\epsilon H}(t))(t,\mathbf{X}(t)), \tag{5.63}$$

and

$$\Upsilon(\tau)^l(t,\mathbf{X}(t)) = 0 \quad \textit{for each } l = 1,...,M. \tag{5.64}$$

To establish the comparison between these results and those of Wafo Soh and Mahomed [41], one has to resort to the definition of the prolongation of an infinitesimal generator for non-stochastic (ODEs):

$$H^{[1]} = H + \xi_j^{[1]}\frac{\partial}{\partial \dot{x}_j}, \tag{5.65}$$

where

$$\begin{aligned} \dot{x}_j &= \frac{dx_j}{dt} \\ &= D_t x_j, \end{aligned} \tag{5.66}$$

$$\begin{aligned} \xi_j^{[1]} &= D_t(\xi_j) - \dot{x}_j D_t(\tau) \\ &= \frac{\partial \xi_j}{\partial t} + \dot{x}_i\frac{\partial \xi_j}{\partial x_i} + \ddot{x}_i\frac{\partial}{\partial \dot{x}_i} + ... \end{aligned} \tag{5.67}$$

Applying the first prolongation on $(\dot{x}_j - f_j)$ at $\dot{\mathbf{x}} = \mathbf{f}$ can be represented as

$$H^{[1]}(\dot{x}_j - f_j)\Big|_{\dot{\mathbf{x}}=\mathbf{f}} = \xi_j^{[1]} - H(f_j). \tag{5.68}$$

Using (5.67) we find that (5.68) in combination with the second-order derivative terms of the instantaneous spatial and temporal drifts constitutes the whole of (5.55) and can be expressed as

$$\left(H^{[1]}(\dot{x}_j - f_j)\Big|_{\dot{\mathbf{x}}=\mathbf{f}} + \frac{1}{2}\sum_{k=1}^{M} G_i^k G_p^k\left(\frac{\partial^2 \xi_j}{\partial x_i \partial x_p} - f_j\frac{\partial^2 \tau}{\partial x_i \partial x_p}\right)\right)(t,\mathbf{X}(t)) = 0. \tag{5.69}$$

If one now considers (5.57), there is no *white noise* term, $\frac{dW_l(t)}{dt}$, as was found by Wafo Soh and Mahomed since $\Upsilon(\tau)^l = 0$. The remaining determining equation can be expressed as

$$HG_{ij} - G_i^j\left(\frac{\partial \xi_j}{\partial x_i}\right) + \frac{1}{2}G_i^j\left(\frac{\partial \tau}{\partial t} + f_i\frac{\partial \tau}{\partial x_i} + \frac{1}{2}G_i^jG_p^j\frac{\partial^2\tau}{\partial x_i \partial x_p}\right) = 0. \tag{5.70}$$

5.6 Examples

5.6.1 Example 1 revisited

Let $\mathbf{X}$(t) be the Itô process

$$d\mathbf{X}(t) = \mathbf{f}dt + \mathbf{G}\, d\mathbf{W}(t), \tag{5.71}$$

where the instantaneous drift coefficient is $f(t,x) = 0$ and the instantaneous diffusion coefficient is $G(t,x) = \sigma_0 = \text{constant} \neq 0$. The Itô process reads as follows:

$$dx = \sigma_0\, dW(t). \tag{5.72}$$

Thus from Wafo Soh and Mahomed's [41] corrected version of the determining equations (5.69) and (5.70), one has

$$H^{[1]}(\dot{x} - 0)\Big|_{\dot{x}=f} + \frac{1}{2}\sigma_0^2\left(\frac{\partial^2\xi}{\partial x^2}\right) = 0, \tag{5.73}$$

$$\sigma_0\frac{\partial \xi}{\partial x} + \frac{\sigma_0}{2}\left(\frac{\partial \tau}{\partial t}\right) = 0. \tag{5.74}$$

Then the determining equations are as follows:

$$\frac{\partial \xi}{\partial t} + \frac{1}{2}\sigma_0^2\left(\frac{\partial^2\xi}{\partial x^2}\right) = 0, \tag{5.75}$$

$$\frac{\partial \xi}{\partial x} - \frac{1}{2}\frac{\partial \tau}{\partial t} = 0. \tag{5.76}$$

The final determining equation now required is the extra condition (5.64)

$$\sigma_0 \frac{\partial \tau}{\partial x} = 0, \tag{5.77}$$

where the evaluation at $(t, \mathbf{X}(t))$ has not taken place. Solving these determining equations gives

$$\tau = C_1 t + C_2, \tag{5.78}$$

$$\xi = \frac{1}{2} C_1 x + C_3. \tag{5.79}$$

Thus one has the following resulting symmetry generators:

$$H_1 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \tag{5.80}$$

$$H_2 = \frac{\partial}{\partial t}, \tag{5.81}$$

and

$$H_3 = \frac{\partial}{\partial x}. \tag{5.82}$$

5.6.2 Example 2 revisited

Let $\mathbf{X}$(t) be the Itô process

$$d\mathbf{X}(t) = \mathbf{f}dt + \mathbf{G}\, d\mathbf{W}(t), \tag{5.83}$$

where $f(t, x) = x$ and $G(t, x) = \sigma_0 = 1$. The Itô process reads as follows:

$$dx = xdt + dW(t). \tag{5.84}$$

Thus from Wafo Soh and Mahomed's [41] corrected version of the determining equations (5.69) and (5.70), one has

$$H^{[1]}(\dot{x} - x)\Big|_{\dot{x}=f} + \frac{1}{2}\frac{\partial^2 \xi}{\partial x^2} = 0, \tag{5.85}$$

$$\frac{\partial \xi}{\partial x} + \frac{1}{2}\frac{\partial \tau}{\partial t} = 0. \tag{5.86}$$

The determining equations are as follows:

$$\frac{\partial \xi}{\partial x} - \frac{1}{2}\frac{\partial \tau}{\partial t} = 0, \tag{5.87}$$

$$\frac{\partial \xi}{\partial t} + x\frac{\partial \xi}{\partial x} - x\frac{\partial \tau}{\partial t} - \xi + \frac{1}{2}\frac{\partial^2 \xi}{\partial x^2} = 0. \tag{5.88}$$

The final determining equation now required is the extra condition (5.64)

$$\frac{\partial \tau}{\partial x} = 0, \tag{5.89}$$

where the evaluation at $(t, \mathbf{X}(t))$ has not taken place. Solving these determining equations gives:

$$\tau = C_1 + C_2 e^{2t}, \tag{5.90}$$

$$\xi = C_2 e^{2t} x + C_3 e^t. \tag{5.91}$$

Thus, one has the following resulting symmetry generators:

$$H_1 = \frac{\partial}{\partial t}, \tag{5.92}$$

$$H_2 = e^{2t}\left(\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right), \tag{5.93}$$

and

$$H_3 = e^t \frac{\partial}{\partial x}. \tag{5.94}$$

5.6.3 Example 3 revisited

As a first example in two space dimensions (with coordinates (x, y)), one chooses the instantaneous diffusion and drift coefficients as

$$\mathbf{G} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2k^2} \end{pmatrix}, \tag{5.95}$$

and

$$\mathbf{f} = \begin{pmatrix} y \\ -k^2 y \end{pmatrix}, \tag{5.96}$$

respectively. Thus, one is considering the equations

$$dx = ydt, \tag{5.97}$$

and

$$dy = -k^2 ydt + \sqrt{2k^2}\, dW(t), \tag{5.98}$$

with k^2 a positive constant.

The determining equations are

$$\frac{\partial \tau}{\partial y} = 0, \tag{5.99}$$

$$\frac{\partial \xi^1}{\partial y} - \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + y\frac{\partial \tau}{\partial x} - k^2 y\frac{\partial \tau}{\partial y} + k^2\frac{\partial^2 \tau}{\partial y^2}\right) = 0, \tag{5.100}$$

$$\frac{\partial \xi^2}{\partial y} - \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + y\frac{\partial \tau}{\partial x} - k^2 y\frac{\partial \tau}{\partial y} + k^2\frac{\partial^2 \tau}{\partial y^2}\right) = 0, \tag{5.101}$$

$$\frac{\partial \xi^1}{\partial t} + y\frac{\partial \xi^1}{\partial x} - k^2 y\frac{\partial \xi^1}{\partial y} + k^2\frac{\partial^2 \xi^1}{\partial y^2} - y\left(\frac{\partial \tau}{\partial t} - y\frac{\partial \tau}{\partial x} + k^2 y\frac{\partial \tau}{\partial y} + k^2\frac{\partial^2 \tau}{\partial y^2}\right) - \xi^2 = 0, \tag{5.102}$$

and

$$\frac{\partial \xi^2}{\partial t} + y\frac{\partial \xi^2}{\partial x} - k^2 y\frac{\partial \xi^2}{\partial y} + k^2\frac{\partial^2 \xi^2}{\partial y^2} + k^2 y\left(\frac{\partial \tau}{\partial t} - y\frac{\partial \tau}{\partial x} + k^2 y\frac{\partial \tau}{\partial y} + k^2\frac{\partial^2 \tau}{\partial y^2}\right) + k^2\xi^2 = 0. \tag{5.103}$$

Equation (5.99) gives

$$\tau(t, x, y) = F(t, x), \tag{5.104}$$

where F is an arbitrary function of time and the first space variables. To obtain $F(t, x)$, one applies the condition

$$e^{\epsilon\Gamma(H(t))} = \Gamma(e^{\epsilon H}(t)), \tag{5.105}$$

to the temporal infinitesimal, (5.104). The left-hand side gives

$$1 + \epsilon\left(\frac{\partial G}{\partial t} + y\frac{\partial F}{\partial x}\right) + \frac{\epsilon^2}{2!}\left(\frac{\partial^2 F}{\partial t^2} + 2y\frac{\partial^2 F}{\partial t \partial x} + y^2\frac{\partial^2 F}{\partial x^2}\right) + ..., \tag{5.106}$$

while the right-hand side gives

$$\begin{aligned} &1 + \epsilon\left(\frac{\partial F}{\partial t} + y\frac{\partial F}{\partial x}\right) \\ &+ \frac{\epsilon^2}{2!}\left(\left(\frac{\partial F}{\partial t}\right)^2 + F\frac{\partial^2 F}{\partial t^2} + \frac{\partial \xi^1}{\partial t}\frac{\partial F}{\partial x} + \xi^1\frac{\partial^2 F}{\partial t \partial x} + y\frac{\partial F}{\partial t}\frac{\partial F}{\partial x}\right. \\ &\left.+ yF\frac{\partial^2 F}{\partial t \partial x} + y\xi^1\frac{\partial^2 F}{\partial x^2} + y\frac{\partial \xi^1}{\partial x}\frac{\partial F}{\partial x}\right) + ... \end{aligned} \tag{5.107}$$

By comparison of coefficients, $F(t,x)$ is forced to be a *constant* in order to satisfy condition (5.105). Hence the temporal infinitesimal is

$$\tau(t,x) = c_1, \tag{5.108}$$

where c_1 is an arbitrary constant. Then the determining equations are as follows:

$$\frac{\partial \xi^1}{\partial y} = 0, \tag{5.109}$$

$$\frac{\partial \xi^2}{\partial y} = 0, \tag{5.110}$$

$$\frac{\partial \xi^1}{\partial t} + y\frac{\partial \xi^1}{\partial x} - k^2 y\frac{\partial \xi^1}{\partial y} + k^2\frac{\partial^2 \xi^1}{\partial y^2} - \xi^2 = 0, \tag{5.111}$$

and

$$\frac{\partial \xi^2}{\partial t} + y\frac{\partial \xi^2}{\partial x} - k^2 y\frac{\partial \xi^2}{\partial y} + k^2\frac{\partial^2 \xi^2}{\partial y^2} + k^2\xi^2 = 0. \tag{5.112}$$

Further simplification of the determining equations (5.111) and (5.112) can be achieved because of (5.109) and (5.110) to give the following results:

$$\frac{\partial \xi^1}{\partial t} + y\frac{\partial \xi^1}{\partial x} - \xi^2 = 0, \tag{5.113}$$

and

$$\frac{\partial \xi^2}{\partial t} + y\frac{\partial \xi^2}{\partial x} + k^2\xi^2 = 0. \tag{5.114}$$

By comparison of coefficients one obtains

$$\frac{\partial \xi^1(t,x)}{\partial t} - \xi^2(t,x) = 0, \tag{5.115}$$

$$\frac{\partial \xi^1(t,x)}{\partial x} = 0, \tag{5.116}$$

$$\frac{\partial \xi^2(t,x)}{\partial x} = 0, \tag{5.117}$$

and

$$\frac{\partial \xi^2(t,x)}{\partial t} + k^2\xi^2(t,x) = 0. \tag{5.118}$$

The general solution then becomes

$$\tau = c_1, \tag{5.119}$$

$$\xi^1 = c_3 - \frac{c_2}{k^2} e^{-k^2 t}, \tag{5.120}$$

and

$$\xi^2 = c_2 e^{-k^2 t}, \tag{5.121}$$

resulting in the following Lie point symmetries

$$H_1 = \frac{\partial}{\partial t}, \tag{5.122}$$

$$H_2 = e^{-k^2 t}\left(\frac{\partial}{\partial y} - k^{-2}\frac{\partial}{\partial x}\right), \tag{5.123}$$

and

$$H_3 = \frac{\partial}{\partial x}. \tag{5.124}$$

5.6.4 Example 4 revisited

As the second example in two space dimensions (with coordinates (x, y)), one chooses the instantaneous diffusion and drift coefficients as

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.125}$$

and

$$\mathbf{f} = \begin{pmatrix} \frac{a_1}{x} \\ a_2 \end{pmatrix}, \tag{5.126}$$

respectively. The determining equations are

$$\frac{\partial \xi^1}{\partial t} + \frac{a_1}{x}\frac{\partial \xi^1}{\partial x} + a_2 \frac{\partial \xi^1}{\partial y} + \frac{1}{2}\frac{\partial^2 \xi^1}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \xi^1}{\partial y^2} = \frac{a_1}{x}\bigg(\frac{\partial \tau}{\partial t} - \frac{a_1}{x}\frac{\partial \tau}{\partial x} - a_2\frac{\partial \tau}{\partial y} + \frac{1}{2}\frac{\partial^2 \tau}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \tau}{\partial y^2}\bigg) + \xi^1\bigg(-\frac{a_1}{x^2}\bigg), \tag{5.127}$$

$$\frac{\partial \xi^2}{\partial t} + \frac{a_1}{x}\frac{\partial \xi^2}{\partial x} + a_2 \frac{\partial \xi^2}{\partial y} + \frac{1}{2}\frac{\partial^2 \xi^2}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \xi^2}{\partial y^2} = a_2\bigg(\frac{\partial \tau}{\partial t} - \frac{a_1}{x}\frac{\partial \tau}{\partial x} - a_2\frac{\partial \tau}{\partial y} + \frac{1}{2}\frac{\partial^2 \tau}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \tau}{\partial y^2}\bigg), \tag{5.128}$$

$$\frac{\partial \xi^1}{\partial x} = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + \frac{a_1}{x}\frac{\partial \tau}{\partial x} + a_2\frac{\partial \tau}{\partial y} + \frac{1}{2}\frac{\partial^2 \tau}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \tau}{\partial y^2}\right), \tag{5.129}$$

$$\frac{\partial \xi^1}{\partial y} = 0, \tag{5.130}$$

$$\frac{\partial \xi^2}{\partial x} = 0, \tag{5.131}$$

$$\frac{\partial \xi^2}{\partial y} = \frac{1}{2}\left(\frac{\partial \tau}{\partial t} + \frac{a_1}{x}\frac{\partial \tau}{\partial x} + a_2\frac{\partial \tau}{\partial y} + \frac{1}{2}\frac{\partial^2 \tau}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \tau}{\partial y^2}\right), \tag{5.132}$$

$$\frac{\partial \tau}{\partial x} = 0, \tag{5.133}$$

and

$$\frac{\partial \tau}{\partial y} = 0. \tag{5.134}$$

By the rule of commonality

$$\tau(t, x, y) = a(t), \tag{5.135}$$

where $a(t)$ is an arbitrary function of time. One applies condition

$$e^{\epsilon\Gamma(H(t))} = \Gamma(e^{\epsilon H}(t)), \tag{5.136}$$

to the temporal infinitesimal (5.135). From the left-hand side one obtains

$$1 + \epsilon\dot{a} + \frac{\epsilon^2}{2!}\dot{a}^2 + \frac{\epsilon^3}{3!}\dot{a}^3 + ..., \tag{5.137}$$

and on the right-hand side one obtains

$$1 + \epsilon\Gamma(\tau) + \frac{\epsilon^2}{2!}\Gamma(H(\tau)) + \frac{\epsilon^3}{3!}\Gamma(H(H(\tau))) + ... \tag{5.138}$$

By comparison of coefficients, one obtains

$$\dot{a}^2 = \dot{a}^2 + a\ddot{a}. \tag{5.139}$$

One acknowledges that condition (5.136) can only be satisfied if

$$\ddot{a} = 0, \tag{5.140}$$

then the temporal infinitesimal becomes

$$a = c_1 t + c_2. \tag{5.141}$$

The determining equations then becomes

$$\frac{\partial \xi^1}{\partial y} = 0, \tag{5.142}$$

$$\frac{\partial \xi^2}{\partial x} = 0, \tag{5.143}$$

$$\frac{\partial \xi^2}{\partial y} = \frac{1}{2}c_1, \tag{5.144}$$

$$\frac{\partial \xi^1}{\partial x} = \frac{1}{2}c_1, \tag{5.145}$$

$$\frac{\partial \xi^1}{\partial t} + \frac{a_1}{x}\frac{\partial \xi^1}{\partial x} + a_2\frac{\partial \xi^1}{\partial y} + \frac{1}{2}\frac{\partial^2 \xi^1}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \xi^1}{\partial y^2} = \frac{a_1}{x}c_1 + \xi^1\bigg(-\frac{a_1}{x^2}\bigg), \tag{5.146}$$

and

$$\frac{\partial \xi^2}{\partial t} + \frac{a_1}{x}\frac{\partial \xi^2}{\partial x} + a_2\frac{\partial \xi^2}{\partial y} + \frac{1}{2}\frac{\partial^2 \xi^2}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \xi^2}{\partial y^2} = a_2 c_1. \tag{5.147}$$

From (5.145) one obtains

$$\xi^1 = \frac{1}{2}c_1 x + F_1(t), \tag{5.148}$$

and from (5.144) one obtains

$$\xi^2 = \frac{1}{2}c_1 y + F_2(t). \tag{5.149}$$

Substituting (5.148) into (5.146) results in

$$\dot{F}_1(t) + \frac{a_1 c_1}{2x} = \frac{a_1 c_1}{x} - \frac{a_1 c_1}{2x} - F_1(t)\frac{a_1}{x^2}. \tag{5.150}$$

By comparing the coefficient one observes that $F_1(t) = 0$ which results in the following solution for the first spatial infinitesimal

$$\xi^1 = \frac{1}{2}c_1 x. \tag{5.151}$$

Also substituting (5.149) into (5.147) gives

$$\dot{F}_2(t) + \frac{1}{2}a_2 c_1 = a_2 c_1, \tag{5.152}$$

which results in the function $F(t)$ to be linear with respect to time

$$F_2(t) = \frac{1}{2}a_2 c_1 t + c_3. \tag{5.153}$$

The general solution is given by

$$\tau(t) = c_1(t) + c_2, \tag{5.154}$$

and

$$\xi^1 = \frac{1}{2}c_1 x, \tag{5.155}$$

$$\xi^2 = \frac{1}{2}c_1 y + \frac{a_2 c_1}{2}t + c_3. \tag{5.156}$$

Thus the Lie point symmetries are as follows:

$$H_1 = \frac{\partial}{\partial t}, \tag{5.157}$$

$$H_2 = \frac{\partial}{\partial y}, \tag{5.158}$$

and

$$H_3 = t\frac{\partial}{\partial t} + \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}(y + a_2 t)\frac{\partial}{\partial y}. \tag{5.159}$$

5.6.5 Example 5 revisited

One chooses the instantaneous drift and instantaneous diffusion coefficients as

$$\mathbf{f} = \begin{pmatrix} -\frac{1}{2}x \\ -\frac{1}{2}y \end{pmatrix} \tag{5.160}$$

and

$$\mathbf{G} = \begin{pmatrix} -y \\ x \end{pmatrix}, \tag{5.161}$$

respectively. Thus from (5.69) and (5.70), one has for $j = 1$:

$$x\xi^1_y - y\xi^1_x + \xi^2 + \frac{1}{2}y\tau_t = 0, \tag{5.162}$$

and

$$\xi^1 + 2\xi^1_t - x\xi^1_x - y\xi^1_y + x\tau_t + y^2\xi^1_{xx} + x^2\xi^1_{yy} - 2xy\xi^1_{xy} = 0, \tag{5.163}$$

and for $j = 2$:

$$x\xi^2_y - y\xi^2_x - \xi^1 - \frac{1}{2}x\tau_t = 0, \tag{5.164}$$

and

$$\xi^2 + 2\xi^2_t - x\xi^2_x - y\xi^2_y + y\tau_t + y^2\xi^2_{xx} + x^2\xi^2_{yy} - 2xy\xi^2_{xy} = 0. \tag{5.165}$$

The final determining equation now required is the extra condition (5.64), viz

$$-x\frac{\partial\tau}{\partial y} + y\frac{\partial\tau}{\partial x} = 0. \tag{5.166}$$

From (5.166), one obtains

$$\frac{dy}{-x} = \frac{dx}{y} = \frac{d\tau}{0}, \tag{5.167}$$

where one obtains

$$d_1 = \left(\frac{x^2 + y^2}{2}\right), \tag{5.168}$$

hence

$$\tau(t, \mathbf{X}(t)) = c_0 z_0(d_1) = c_0 z_0\left(\frac{x^2 + y^2}{2}\right). \tag{5.169}$$

Since one has obtained that τ is not a function of time, but a function of x and y, one can further simplify the determining equations knowing that

$$\frac{\partial\tau}{\partial t} = 0. \tag{5.170}$$

Alternatively, one can demonstrate (5.170) as follows. Differentiating (5.162) once by x and also by y respectively as follows:

$$\xi^2_x = y\xi^1_{xx} - \xi^1_y - x\xi^1_{xy}, \tag{5.171}$$

and

$$\xi^2_y = \xi^1_x + y\xi^1_{xy} - x\xi^1_{yy}, \tag{5.172}$$

then substitute both these equations into (5.164), one obtains the following:

$$\xi^1 = x\xi^1_x + xy\xi^1_{xy} - x^2\xi^1_{yy} - y^2\xi^1_{xx} + y\xi^1_y + xy\xi^1_{xy} - \frac{1}{2}x\tau_t. \tag{5.173}$$

One then substitutes (5.173) into (5.163) and obtains

$$x\tau_t + 4\xi^1_t = 0, \tag{5.174}$$

and separating by coefficients yields to

$$x : \tau_t = 0, \tag{5.175}$$

$$constant : \xi^1_t = 0, \tag{5.176}$$

it is noticed from (4.258) that

$$\tau = c_0 z_0(x, y), \tag{5.177}$$

where $z_0(x, y)$ is a function of both x and y, hence (5.170) and from (5.176) it is noticed that ξ^1 is a function of space variables, x and y.

$$\xi^1 = d(x, y). \tag{5.178}$$

The same procedure is done on ξ^2 by differentiating ξ^1 with respect to x and y respectively, and substituting those equations found in (5.163). In this case, by separation of coefficients one obtains

$$\tau = c_0 z_0(x, y), \tag{5.179}$$

and

$$\xi^2 = e(x, y). \tag{5.180}$$

The simplified determining equations are as follows:

$$\tau = c_0 z_0(x, y), \tag{5.181}$$

$$x\xi^1_y - y\xi^1_x + \xi^2 = 0, \tag{5.182}$$

$$\xi^1 - x\xi^1_x - y\xi^1_y + y^2\xi^1_{xx} + x^2\xi^1_{yy} - 2xy\xi^1_{xy} = 0, \tag{5.183}$$

$$x\xi^2_y - y\xi^2_x - \xi^1 = 0, \tag{5.184}$$

and

$$\xi^2 - x\xi^2_x - y\xi^2_y + y^2\xi^2_{xx} + x^2\xi^2_{yy} - 2xy\xi^2_{xy} = 0. \tag{5.185}$$

From (5.182) and (5.184) one notices that

$$\frac{dy}{x} = \frac{dx}{-y} = \frac{dt}{0} = -\frac{d\xi^1}{\xi^2} = \frac{d\xi^2}{\xi^1}. \tag{5.186}$$

From (5.186) it is observed that

$$\frac{d\xi^1}{ds} = -\xi^2, \tag{5.187}$$

$$\frac{d\xi^2}{ds} = \xi^1, \tag{5.188}$$

$$\frac{dy}{ds} = x, \tag{5.189}$$

and

$$\frac{dx}{ds} = -y. \tag{5.190}$$

Working firstly with (5.187) one notices that

$$\frac{d^2\xi^1}{ds^2} + \xi^1 = 0, \tag{5.191}$$

which gives the following solution of ξ^1

$$\begin{aligned}\xi^1 &= Pe^{is} + Ze^{-is} \\ &= (P+Z)\cos s + (Pi - Zi)\sin s.\end{aligned} \tag{5.192}$$

By letting $P + Z = A$ and $Pi - Zi = B$, one obtains the first spatial infinitesimal as

$$\xi^1 = A\cos s + B\sin s, \tag{5.193}$$

where A and B are arbitrary constants. Consequently from (5.187) one obtains

$$\xi^2 = A\sin s - B\cos s. \tag{5.194}$$

Also from (5.190) one observes that

$$\frac{d^2x}{ds^2} + x = 0, \tag{5.195}$$

which gives the first spatial variable as follows:

$$x = a\cos s + b\sin s, \tag{5.196}$$

consequently, the second spatial variable is

$$y = a\sin s - b\cos s. \tag{5.197}$$

Now a representation of both $\cos s$ and $\sin s$ in terms of both x and y is required. Multiply (5.196) by a constant a and (5.197) by a constant b, then subtract the two equations as follows:

$$ax - by = (a^2 + b^2)\cos s, \tag{5.198}$$

which solves for $\cos s$ as given below

$$\cos s = \frac{ax - by}{a^2 + b^2}. \tag{5.199}$$

Secondly, multiplying (5.196) by a constant b, then adding (5.197) multiplied by a constant a as follows:

$$bx + ay = (a^2 + b^2)\sin s, \tag{5.200}$$

which solves as for $\sin s$ as given below

$$\sin s = \frac{bx + ay}{a^2 + b^2}. \tag{5.201}$$

Now one can substitute (5.200) and (5.201) into (5.192) and (5.194) to give

$$\xi^1 = \frac{Aa + Bb}{a^2 + b^2}x + \frac{Ba - Ab}{a^2 + b^2}y, \tag{5.202}$$

and

$$\xi^2 = \frac{Ab - Ba}{a^2 + b^2}x + \frac{Bb + Aa}{a^2 + b^2}y, \tag{5.203}$$

where A, B, a and b are constants. From (5.186), it is noticed that

$$\frac{x^2 + y^2}{2} = constant. \tag{5.204}$$

Let

$$\frac{Aa + Bb}{a^2 + b^2} = c_1 z_1\left(\frac{x^2 + y^2}{2}\right), \tag{5.205}$$

and let

$$\frac{Ba - Ab}{a^2 + b^2} = c_2 z_2\left(\frac{x^2 + y^2}{2}\right), \tag{5.206}$$

where z_1 and z_2 are both functions of x and y. Equation (5.202) and (5.203) becomes

$$\xi^1 = c_1 z_1\left(\frac{x^2 + y^2}{2}\right)x + c_2 z_2\left(\frac{x^2 + y^2}{2}\right)y, \tag{5.207}$$

and

$$\xi^2 = -c_2 z_2\left(\frac{x^2+y^2}{2}\right)x + c_1 z_1\left(\frac{x^2+y^2}{2}\right)y. \tag{5.208}$$

One can now demonstrate that condition

$$e^{\epsilon\Gamma(H(t))} = \Gamma(e^{\epsilon H}(t)), \tag{5.209}$$

has mathematical basis. We have obtained that $\tau(t,\mathbf{X}(t)) = c_0 z_0\left(\frac{x^2+y^2}{2}\right)$. From the left-hand side one obtains

$$1 + \epsilon\Gamma(\tau) + \frac{\epsilon^2}{2!}\Gamma(\Gamma(\tau)) + \mathcal{O}(\epsilon^3), \tag{5.210}$$

since

$$\begin{aligned}\Gamma(\tau) &= \left(-\frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2\right)z_0\\ &= 0.\end{aligned} \tag{5.211}$$

The left-hand side gives the value 1. The right-hand side gives

$$\Gamma(t + \epsilon\tau + \frac{\epsilon^2}{2!}H(\tau) + \mathcal{O}(\epsilon^3)), \tag{5.212}$$

We know that

$$\Gamma(t) = 1, \tag{5.213}$$

and already one has obtained $\Gamma(\tau) = 0$. $\Gamma(H(\tau))$ is given by

$$\begin{aligned}&\Bigg(-x^2c_1z_0z_1\left(\frac{x^2+y^2}{2}\right) - y^2c_1z_0z_1\left(\frac{x^2+y^2}{2}\right) + x^2c_1z_0z_1\left(\frac{x^2+y^2}{2}\right) + y^2c_1z_0z_1\left(\frac{x^2+y^2}{2}\right)\\ &-\frac{1}{2}x^2z_0z_1c_1(x^2+y^2) - \frac{1}{2}y^2z_1z_0(x^2+y^2) + \frac{1}{2}x^2z_1z_0(x^2+y^2) + \frac{1}{2}y^2z_1z_0(x^2+y^2)\Bigg)\\ &= 0.\end{aligned} \tag{5.214}$$

Therefore one obtains the right-hand side to be 1, thus one has shown that the left-hand side is equal to the right-hand side, implying that condition (5.209) is satisfied.

In summary, the general solution is as follows:

$$\tau = c_0 z_0\left(\frac{x^2+y^2}{2}\right), \tag{5.215}$$

$$\xi^1 = c_1 z_1\left(\frac{x^2+y^2}{2}\right)x + c_2 z_2\left(\frac{x^2+y^2}{2}\right)y, \tag{5.216}$$

and

$$\xi^2 = -c_2 z_2\left(\frac{x^2+y^2}{2}\right)x + c_1 z_1\left(\frac{x^2+y^2}{2}\right)y. \tag{5.217}$$

To demonstrate that a solution of one SODE is transformed to that of another, we choose a simple example where

$$z_1\left(\frac{x^2+y^2}{2}\right) = z_2\left(\frac{x^2+y^2}{2}\right) = 1. \tag{5.218}$$

The symmetry generators are as follows:

$$H_0 = z_0\left(\frac{x^2+y^2}{2}\right)\frac{\partial}{\partial t}, \tag{5.219}$$

$$H_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{5.220}$$

and

$$H_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{5.221}$$

The symmetry generators obtained above are the same symmetries that were obtained by Fredericks and Mahomed [43] and Ünal[42].

5.7 Summary

A successful investigation on the symmetries of SODEs without consulting the FPE as done in chapter 4 is achieved. Using form invariance, we were able to derive the random time change formula without the knowledge of Øksendal [31], the instantaneous diffusion coefficient was obtained to be zero and the instantaneous drift coefficient (time rate change) to be a function of time, although the temporal infinitesimal can be a function of time and spatial variables. It is noticed that with this new technique introduced by Fredericks and Mahomed [43], for scalar SDEs, the same symmetries are revealed as compared to the technique introduced by Gaeta and Quintero [2] of projectable symmetries, but for systems of equations, more symmetries are obtained. Since the study in this chapter was not restricted to projectable symmetries, it is acknowledged that the projectable symmetries of the SDE i.e. $\tau = \tau(t)$ are not the sole type of symmetries for an SDE.

In an article written by Roman Kozlov [48], Lie point group classification of a scalar SDE with a one-dimensional Wiener process is discussed where the SDE is given as

$$d\mathbf{X}(t) = \mathbf{f}(t, \mathbf{x})dt + \mathbf{G}(t, \mathbf{x})d\mathbf{W}(t), \tag{5.222}$$

where $\mathbf{f}(t, \mathbf{x})$ is an instantaneous drift coefficient, $\mathbf{G}(t, \mathbf{x})$ is the instantaneous diffusion coefficient and $\mathbf{W}(t)$ is a standard Wiener process. The admitted symmetry group can be zero, one, two or three-dimensional. The symmetries admitted by scalar SDEs are projectable symmetries as obtained by Gaeta and Quintero [2], Fredericks and Mahomed [43] and Ünal [42]. For a scalar SDE to admit zero symmetries, the drift coefficient $\mathbf{f}(t, \mathbf{x})$ and diffusion coefficient $\mathbf{G}(t, \mathbf{x})$ must be arbitrary which will enforce that the determining equations (5.64), (5.69) and (5.70) have non-trivial solutions, hence no symmetries. Also, a scalar SDE (5.222) admits a one symmetry if the drift coefficient $f(x)$ is arbitrary and a constant diffusion coefficient. An SDE (5.222) admits a two-dimensional symmetry group if and only if the drift coefficient obeys this condition

$$\mathbf{f}(t, \mathbf{x}) = \frac{C}{x + D(t)} + F(t)(x + D(t)) - D'(t), \qquad C \neq 0 \tag{5.223}$$

where $F(t)$, $G(t)$ and $D(t)$ are arbitrary functions and C is a constant. Kozlov [48] also showed that only equations that admit a three-dimensional symmetry group can be transformed into the equation of a Wiener process by a point change of variables with non-random time transformation. The conditions for an SDE to admit a three-dimensional symmetry group is that an SDE must admit a symmetry of the form

$$H_* = \xi(t, x)\frac{\partial}{\partial x}. \tag{5.224}$$

An SDE admits such a symmetry if and only if the drift coefficient $\mathbf{f}(t, \mathbf{x})$ and the diffusion coefficient $\mathbf{G}(t, \mathbf{x})$ satisfy the condition

$$\left(\frac{G_t}{G} - G\left(\frac{f}{G}\right)_x + \frac{1}{2} G G_{xx} \right)_x = 0. \tag{5.225}$$

For the examples 1 and 2 done in both Chapter 4 and Chapter 5, which are scalar SODEs, the symmetries admitted by the SDE (5.222) correspond with the findings in Kozlov [48].

As discussed above, more symmetries are revealed concerning the systems of SDEs with the method discussed in this chapter as compared to the technique introduced in Gaeta and Quintero [2]. Therefore, one notices that projectable symmetries are not the sole form of symmetries for a system of SDEs.

Chapter 6

Conclusion

We discussed how to construct the *ansatz* needed to determine the Lie point transformations and showed that Theorem (3.1) is a special case for the more generalized Theorem (3.2) in Chapter 3. This *ansatz* was then used to show that the symmetry infinitesimals for the FPE has to be projectable and allowed the discovery of its determining equations. Since a relationship between the SODEs and FPEs was established by Gaeta and Quintero [2] and that the normalization condition must be satisfied, this assisted in obtaining the symmetry generators of the SODEs from those of the FPE associated with it. The work of Wafo Soh and Mahomed [41], Ünal [42], and Fredericks and Mahomed [43] was investigated, which assisted in answering the question asked in the Chapter 4, namely, whether or not the Lie point transformations of the SODEs are applicable if and only if the infinitesimals are not projective. The answer is that projectable infinitesimals are not the sole symmetries of SODEs, as discovered in Chapter 5. Furthermore, the methodology in Chapter 5 revealed that for scalar SDE, the symmetry generators are the same as those found with the methodology of projectable infinitesimals, the method in Chapter 4 and in Kozlov [48]. For a system of SDEs, it is noticed that the technique in Chapter 5 revealed more symmetries that are applicable to SDEs but not necessarily to PDEs. The random time change formula is derived using form invariance approach, whereby the instantaneous drift coefficient is found to be a function of time and the instantaneous diffusion coefficient to be zero.

The determining equations needed to forecast the spatial and temporal infinitesimals are:

$$\Gamma(H(x))(t,\mathbf{X}(t)) = (\Gamma(H(t)) + H) f(t,\mathbf{X}(t)), \tag{6.1}$$

$$\Upsilon^l(H(x))(t,\mathbf{X}(t)) = \left(\frac{\Gamma H(t)}{2} + H\right) G^l(t,\mathbf{X}(t)), \tag{6.2}$$

$$e^{\epsilon\Gamma(H(t))}(t,\mathbf{X}(t)) = \Gamma(e^{\epsilon H}(t))(t,\mathbf{X}(t)), \tag{6.3}$$

and

$$\Upsilon(\tau)^l(t,\mathbf{X}(t)) = 0 \quad \text{for each} \quad l = 1, ..., M, \tag{6.4}$$

where

$$\Gamma = \frac{\partial}{\partial t} + f_j \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{M} G_i^k G_j^k \frac{\partial^2}{\partial x_i \partial x_j} \quad \text{for each} \quad j = 1, ..., N, \tag{6.5}$$

and

$$\Upsilon^l = G_j^l \frac{\partial}{\partial x_j}, \tag{6.6}$$

where N is the dimension of spatial process and M is the dimension of the Wiener process.

Rozlov [48] discussed the Lie point group classification of scalar SDEs with the one dimensional Wiener process. The question that one may ask is: can one provide the Lie point group classification of the system of SDEs ?

The overview of the thesis

This dissertation establishes the connection between the point symmetries of Itô's equation and those of the associated FPE. Loosely, Itô equation describes the motion of a particle in a random environment. The associated FPE which is a second-order linear parabolic partial differential equation provides the model of how to find the distribution of the particle at a given point of the space-time. Thus both the Itô equation and the FPE provide a full probabilistic representation of the dynamics in a random medium. The particle used in our case of Itô's equation may stand for an asset price in finance, the population size in biology or any dynamic variable influenced by chance. Finding the general closed-form solution of Itô's equation is possible only in special cases. Besides the methods used for obtaining solutions in these cases vary according to the underlying problem. Keeping in mind the success of the Lie symmetry analysis in the investigation of the integrability of deterministic differential equations, one may wonder if such an analysis can be extended to Itô's equation. The answer is affirmative and is the result of complementary investigation by Gaeta and Quintero [2], Wafo Soh and Mahomed [41], Ünal [42] and more recently Fredericks and Mahomed [43]. Owing to the connection between Itô's equation and the associated (deterministic) FPE, one may suspect a close link between their symmetries. Indeed such a link exists and is characterized by Gaeta and Quintero [2] after projecting the symmetries of the FPE on the space-time and insisting on the invariance of the 'total mass' of the solution of the FPE.

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