UNIVERSITY OF CAPE TOWN, SOUTH AFRICA

Observations in Inhomogeneous Cosmology and the Cosmic Mass

by

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Abstract

The analysis of modern cosmological data is becoming an increasingly important task as the amount of data multiplies. An important goal is to extract information about the geometry of the spacetime, i.e. the metric of our cosmos, from observational data. The aim of this thesis is a development within the framework that enables one to take into account inhomogeneities when analysing astronomical observations. To this end, two investigations look at firstly the derivation of the observer metric from spherically symmetric observations, and secondly the question of mass in the Lemaître mertic. Background and motivation are given in in chapter 1, followed by discussion of the Lemaître-Tolman metric and its observational relations in chapter 2.

The observer metric is adapted to the reality of observations: information received along the past null cone, and matter flowing along timelike lines. It provides potentially a very good candidate for developing a general numerical data reduction program. As a basis for this, the spherically symmetric solution is re-evaluated in chapter 3. With future numerical implementation in mind, a clear derivation of the mathematical solution in terms of 4 arbitrary functions, and an algorithm for exrtacting the solution from observational data on the past null cone are presented.

Chapter 4 presents the spherically symmetric metric with a comoving perfect fluid and non-zero pressure — the Lemaître metric. It is shown how to solve the field equation for this spacetime, and an algorithm is presented for constructing a Lemaître model. Further, the definition of mass is discussed. It is shown that the introduction of pressure and Λ makes it difficult to separate the mass from other physical parameters in an invariant way. Under the usual mass definition, the apparent horizon relation, that relates the maximum in the diameter distance to the cosmic mass within that radius, remains the same as in the Lemaître-Tolman case.

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Abbreviations

DE	**D**ifferential **E**quation
EFE	**E**instein **F**ield **E**quation
FLRW	**F**riedmann **L**emaître **R**oberston **W**alker
GDE	**G**eodesic **D**eviation **E**quation
GR	**G**eneral **R**elativity
LT	**L**emaître **T**olman
OC	**O**bserver **C**oordinates
PDE	**P**artial **D**ifferential **E**quation
PNC	**P**ast **N**ull **C**one

To my Parents and the rest of my Family

Chapter 1

Introduction

Cosmology is the branch of science that studies the dynamics and the evolution of the entire Universe. However, its status is very specific, for it has only observations to study — because of its gigantic length and time scales compared with the speed of light times a human lifetime, is not possible to arrange any kind of experiment. Therefore, all we know about the Universe is derived from observations only. There can be some concerns whether a science which studies only a sample of one, cannot arrange an experiment, can be called a science [1, 2]. Indeed, up to the 20^{th} century we did not have sophisticated observational techniques and advanced mathematical theorems to study the entire Universe with scientific methods. Hence, cosmology used to be treated as an intellectual activity at the borders of philosophy, religion, and science. It is not possible to analyse observations without a theoretical framework, therefore, the main foundation of cosmology must be a theory. Since the publication of Einstein's general relativity, the modern theory of cosmology has been totally based on the homogeneous isotropic solution. However, the Einstein equations are a set of 10 partial differential equations which are all coupled with each other, and are very hard to solve. That is why different approaches to the analysis and application of the Einstein equations have been developed over the last 80-90 years of relativistic cosmology.

In order to solve the Einstein equations, it was assumed that the Universe is homogeneous and isotropic. The first cosmological models based on this assumption were: the Einstein Universe (spatially homogeneous static solution) and the de Sitter model (homogeneous both in time and space), but they did not meet observational tests. The later works of Friedmann, Lemaître, Robertson, and Walker laid the foundation of the hot big bang model that still plays a major role today. (This model will be referred to as the FLRW model [3–7]).

Since then, homogeneity has become a standard, unquestioned assumption in cosmology. Scientists typically assume the FLRW model and they try to derive its parameters based on observations. Currently the most favoured model is the ΛCDM model, in which the equation of the state is that of the normal baryonic matter plus cold dark matter plus a cosmological constant or "dark energy". It is based on observations of the cosmic microwave radiation [8], galaxy redshift surveys [9, 10] and supernova observations [11, 12]. The parameters of this model are $H_0 \approx 72$ km s^{-1} Mpc^{-1}, $\Omega_m \approx 0.26$, $\Omega_\Lambda \approx 0.74$, where Ω_m is composed of $\Omega_b \approx 0.05$ and $\Omega_{cdm} \approx 0.22$.

Although the standard approach seems successful, there are many unanswered questions about the nature of the matter content of the cosmos, as well as the relationship between observations of many discrete sources, which are necessarily averaged in some sense, and general relativity, which assumes the metric and the matter are described by smooth functions. A significant complication arises from the fact that observers are looking down earth's past null cone, so we do not know the state of the universe at any one time or the history of any one worldline. Therefore, since we know that the Universe is not homogeneous, it is more reasonable not to assume homogeneity at the very beginning.

Ever since Kristian and Sachs [13] there has been interest in the possibility of determining the metric of the spacetime we live in directly from cosmological observations. Kristian and Sachs provided a framework based on series expansions of different geometrical quantities which describe the geometry of the Universe. They made three assumptions to construct their solution, these are: the universe is well described by a Riemannian spacetime, the light travels along null geodesics, and lastly the gravitational field is described by Einstein's Equations for dust. Then they defined a corrected luminosity distance for the source that appears in motion relative to an observer. Accordingly they made series expansions near the observer in a general metric for the redshift, area distance, distortion effect, number count density and proper motion in terms of the corrected luminosity distance. Since these observations have been made for a general case, then they considered the special case of "dust" to re-derive these quantities as parameters whose values must be very rough estimated numerically. They estimated values for the series cofficients from available data, and concluded that the homogeneity of the universe was not proven. The series approach has been also studied in [14–16].

However, since cosmological observations provide us with data on the past null cone, it is more convenient to use a coordinate system that is suitable for this. This was done by Ellis, Nel, Maartens, Stoeger, and Whitman [17] who introduce the observational coordinate system. A new system of "observer's coordinates", different from the usual space and time coordinates, were introduced. They are centered on a single observer's past null cone (i.e the observer on his worldline receiving all information via his null cone). Then, using these coordinates, first they considered cosmographic analysis, which is analysis of the cosmological observational data without taking into account the gravitational field equations, and they showed that we cannot determine the dynamics of the universe or the spacetime structure on the past null cone (PNC) from observations only. Then in the second part, they assumed that spacetime and matter obey the theory of general relativity, and were thus able to determine the spacetime structure on the PNC and off the PNC. This was further developed in [18–29]. These coordinates, allow us to solve Einstein's field equation (EFEs) on our past light cone first, and then evolve the solution off our past null cone to the past or to the future. The solution in the first case can be determined from the cosmological data and can be thought of as the initial conditions for the solution of the second one. The observational coordinate concept follows an idea originally introduced by Temple [30], who noted that traditional time and space coordinates are not well adapted to cosmological observations. In the observer coordinate approach the 'fluid-ray tetrad' is introduced in [18], including a set of spin coefficients, and the basic constraint and dynamic equations are derived from this formalism. Although a general form has been given for the observer metric, work has concentrated on the spherically symmetric case, and to a lesser extent its perturbations.

In principle the Kristian and Sachs type approach enables us to derive the properties of the Universe from observations, but it does not allow us to study structure formation. Therefore the idea of an exact model is needed. In addition to that, a lot of studies have been based on exact inhomogeneous solutions of the Einstein equations. Within this approach the most popular models are based on the Lemaître–Tolman metric [31, 32] (sometimes it is also referred to as Lemaˆtre–Tolman–Bondi [33] metric). An algorithm which shows how to construct a Lemaître-Tolman model from isotropic observations, apparent luminosity, angular diameter, number count density in redshift space, together with their corresponding evolution functions, absolute luminosity, actual diameter, and average mass per source, has been presented by Mustapha, Hellaby & Ellis [34], and further developed by Lu & Hellaby and McClure & Hellaby [35, 36]. Lu & Hellaby developed expressions for the Lemaître-Tolman arbitrary functions in terms of differential equations, and wrote computer programs to integrate them numerically. They showed that the apparent horizon creates a difficulty in the numerics, and therefore, in order to avoid this problem, a power series around the maximum was used. They tested their numerical program with fake observational data, and successfully recovered the original Lemaître-Tolman model that gave the data. Hellaby [37] pointed out that the relations that hold at the apparent horizon actually allow one to check the numerical integration. This was implemented numerically by McClure and Hellaby, who showed how systematic errors in the data can be detected and at least partially corrected for. They also adapted the algorithm to handle noisy data, and showed the DEs are stable except for the mass DE beyond the apparent horizon.

Applications of the Lemaître-Tolman models have recently become very popular for showing that inhomogeneity can explain the dimming discovered in the SN Ia data [38–44] without dark energy. However, the Lemaître-Tolman models are not the only inhomogeneous models that have been employed in cosmology. The Stephani [45–48], Lemaître [48–50], Szekeres [51–55] and Szafron metrics [56] have been used as well. For a review on inhomogeneous cosmological models the reader is referred to a monograph by Krasiński [57] which provides more than seven hundred references.

This Thesis will focus on solving the observer metric and generalising the cosmic mass result to the Lemaître model. It will first present the spherical symmetric inhomogeneous space-times, and discuss how one can define the Lemaître–Tolman model based on cosmological observations (Chapter 2). Then the spherical observer metric will be discussed in detail, and a clear algorithm for obtaining solution will be presented (Chapter 3). Finally, in Chapter 4 it will be shown how the position of the apparent horizon can be observed and can put very tight constraints on the properties of the Universe, such as the cosmic mass on gigaparsec scales, independently of any detaild model of inhomogeneity.

Chapter 2

The Lemaître-Tolman Model and Observations

In the standard approach to cosmology it is assumed that the Universe is spatially homogeneous and isotropic, and it can be described by the FLRW model. However, we know from observations that on small scales the Universe is very inhomogeneous. Therefore models that can describe the evolution of inhomogeneities are very important. In this chapter our studies will concentrate on the spherically symmetric, pressure free, inhomogeneous models. Spherically symmetric models were first studied by Lemaître as early as 1933 [31], who considered matter evolution with anisotropic pressure. A year later Tolman [32] studied a simplified version of the Lemaître's model – the case of zero pressure. Interest in this case was further revived by Bondi [33], who only cited Tolman, despite the fact that Tolman had cited Lemaître, and that is why in the literature the spherical symmetric pressure free models are often referred to as the Tolman–Bondi models. Throughout this thesis, geometric units ($G = c = 1$) will be used.

2.1 The Lemaître-Tolman model

The spherically symmetric metric in co-moving, orthogonal coordinates can be written as

$$ds^2 = -A^2 dt^2 + X^2\, dr^2 + R^2 (d\theta^2 + \sin^2\theta\, d\phi^2)\ , \tag{2.1}$$

where $X = X(t,r)$, $A = A(t,r)$ are unknown functions, while $R = R(t,r)$ is the areal radius. On the other hand, the energy momentum tensor describes a zero pressure perfect fluid (dust), and is given by,

$$T^{\mu\nu} = \rho\, u^{\mu}\, u^{\mu}\ ,$$

where $\rho = \rho(t, r)$ is the mass-energy density of the matter which is space and time dependent, and $u^{\mu} = dx^{\mu}/ds$ is the 4−velocity of the matter. In the comoving coordinates, $u^{\mu} = (\frac{1}{A}\,, 0, 0, 0)$. Since the EFEs are the central equations for cosmology, thus the EFEs $G^{\mu\nu} = \kappa T^{\mu\nu} + g^{\mu\nu}\Lambda$ can be written in terms of the metric functions [58] as follows:

$$\begin{aligned} A^2 G^{tt} &= \left(-\frac{2R''}{R\,A^2\,X^2} - \frac{2R'^2}{A^2\,R\,X^2} + \frac{2\dot{R}\dot{X}}{A^4\,R\,X} + \frac{1}{A^2\,R^2} + \frac{2R'\,X'}{A^2\,R\,X^3} + \frac{\dot{R}^2}{A^4\,R^2} \right) \\ &= \kappa\rho + \Lambda. \end{aligned} \tag{2.2}$$

$$A^2 G^{rt} = \left(\frac{2\dot{R}'}{A^2\,R\,X^2} - \frac{2A'\dot{R}}{A^3\,X^2\,R} - \frac{2\dot{X}R'}{A^2\,R\,X^3} \right) = 0\ . \tag{2.3}$$

$$\begin{aligned} X^2 G^{rr} = &\left(\frac{A'R'}{R^3\,X^2\,A} + \frac{A''}{R^2\,X^2\,A} - \frac{\ddot{X}}{R^2\,X\,A^2} - \frac{\dot{X}\dot{R}}{R^3\,X\,A^2} - \frac{A'X'}{R^2\,A\,X^3} + \frac{\dot{A}\dot{X}}{R^2\,A^3\,X} \right. \\ &\left. + \frac{\dot{A}\dot{R}}{R^3\,A^3} - \frac{\ddot{R}}{R^3\,A^2} + \frac{R''}{R^3\,X^2} - \frac{X'R'}{R^3\,X^3} \right) = -\Lambda\ . \end{aligned} \tag{2.4}$$

$$\begin{aligned} R^2 G^{\theta\theta} = &\left(\frac{A'R'}{R^3\,X^2\,A} + \frac{A''}{R^2\,X^2\,A} - \frac{\ddot{X}}{R^2\,X\,A^2} - \frac{\dot{X}\dot{R}}{R^3\,X\,A^2} - \frac{A'X'}{R^2\,X^3\,A} + \frac{\dot{A}\dot{X}}{R^2\,X\,A^3} \right. \\ &\left. - \frac{\dot{A}\dot{X}}{R^3\,A^3} - \frac{\ddot{R}}{R^3\,A^2} + \frac{R''}{R^3\,X^2} - \frac{R'X'}{R^3\,X^3} \right) = -\Lambda. \end{aligned} \tag{2.5}$$

Note that the "dot" and "prime" refer to the partial derivatives with respect to the time t and the radial coordinate r respectively, and $\kappa = 8\pi G/c^4 = 8\pi$. In addition, from the conservation equations $T^{\mu\nu}{}_{;\nu} = 0$, we obtain the continuity equations:

$$T^{t\mu}{}_{;\mu} = 0 \quad \Rightarrow \quad \frac{\dot{X}}{X} + \frac{\dot{R}}{R} = -\frac{\dot{\rho}}{\rho}, \tag{2.6}$$

$$T^{r\mu}{}_{;\mu} = 0 \quad \Rightarrow \quad \frac{A'}{A} = 0\ . \tag{2.7}$$

As seen from (2.7) we get that $A' = 0$, so without loss of generality we can scale A to one, $A = 1$, thus making Eq. (2.1) synchronous.

Then solving Eq.(2.3) yields,

$$G^{rt} = \left(\frac{2\dot{R}'^2}{RX^2} - \frac{2\dot{X}R'}{R\,X^3} \right) = 0 \qquad \rightarrow \qquad \frac{\partial \ln X}{\partial t} = \frac{\partial \ln R'}{\partial t}. \tag{2.8}$$

which can be integrated to produce,

$$X(t,r) = \frac{R'(t,r)}{\sqrt{1+f}}\,, \tag{2.9}$$

where $f = f(r)$ is the first arbitrary function of integration. Inserting Eq. (2.9) into Eq. (2.1) immediately gives the full line element of the Lemaître Tolman model:

$$ds^2 = -dt^2 + \frac{R'^2\,dr^2}{1+f} + R^2\Big(d\theta^2 + \sin^2\theta\,d\phi^2\Big)\,. \tag{2.10}$$

Since the unknown function X has been determined, the remaining EFEs (2.2), (2.4), and (2.5) can be rewritten as

$$G^{tt} = \left(\frac{-f'}{R'R} + 2\frac{\dot{R}\dot{R}'}{R'R} - \frac{f}{R^2} + \frac{\dot{R}^2}{R^2}\right) = \rho + \Lambda\,, \tag{2.11}$$

$$G^{rr} = \Big(f - \dot{R}^2 - 2\,R\,\ddot{R}\Big) = -\Lambda\,, \tag{2.12}$$

$$G^{\theta\theta} = \left(-\frac{f}{2RR^3} - \frac{\dot{R}\dot{R}'}{R'R^3} - \frac{\ddot{R}}{R'R^3} - \frac{\ddot{R}'}{2R'R^2}\right) = -\Lambda\,. \tag{2.13}$$

Solving Eq. (2.12) allows us to find the evolution equation of the LT model. For the non-static case ($\dot{R} \neq 0$), Eq. (2.4) can be multiplied by $-\dot{R}$ and integrated to yield:

$$\dot{R}^2 = \frac{2M}{R} + f + \frac{\Lambda}{3}R^2\,, \tag{2.14}$$

where $M = M(r)$ is the second arbitrary function of the integration. Eq. (2.14) is a generalised Friedmann equation, and it shows how the areal radius R evolves with time.

The physical meaning of these arbitrary functions can be obtained by comparing the evolution equation in (2.14) with the Newtonian energy equation for the radial motion of a particle in a spherical dust cloud.

$$E = -\frac{GMm}{R} + \frac{mv^2}{2}\,. \tag{2.15}$$

Here the first term on the right hand side is the gravitational potential which contains the active gravitational mass M within the shell of constant radius r, that in principle generates the gravitational field. The second term is the kinetic energy and E is the total energy. Thus, there are two interpretation of f/G. Firstly, in (2.10) it plays a geometrical role which specifies the curvature of the spatial part. Secondly, comparing Eq. (2.14) and (2.15) it plays a dynamic role which determines the local energy per unit mass for the dust particle in that shell of constant r. (For more details see [31, 59]).

Now, substituting Eq. (2.14) and its r derivative in Eq. (2.11), gives an expression for the density,

$$\kappa\rho = \frac{2M'}{R'\,R^2} \,. \tag{2.16}$$

It is clear that the density reaches infinity in two cases. The first case is when $R = 0$ and $M, M' \neq 0$ at the time of the "Big Bang" or "big crunch". The second case is when $R' = 0$ and $M' \neq 0$ which is a shell crossing. Shell crossings can be avoided by setting initial conditions appropriately [60]. Hence the total active gravitational mass $M = M(r)$ within constant r shell can be obtained by integrating Eq. (2.16) along hypersurfaces of constant t with

$$M = \frac{\kappa}{2}\int_0^r \rho R'\,R^2 d\tilde{r} \,, \tag{2.17}$$

whereas the total rest mass $\mathcal{M}$ of matter within the same shell is the volume integral of the density on a constant time surface

$$\mathcal{M} = 4\pi \int_0^r \rho\, \frac{R'\,R^2}{\sqrt{1+f}} dr \,. \tag{2.18}$$

Notice that the two differ by a geometric factor in the integral. Where $\Lambda = 0$, the evolution of the diameter distance R can be obtained by solving Eq. (2.14) in terms of the parametric solution $\eta = \eta(r,t)$. These solutions are:

Elliptic case, $f < 0$:

$$R(t,r) = -\frac{M}{f}(1-\cos\eta) \,,$$
$$\eta - \sin\eta = \frac{(-f)^{3/2}}{M}(t - a(r)). \tag{2.19}$$

Parabolic case, $f = 0$:

$$R(t,r) = \left[\frac{9}{2}M(t-a(r))^2\right]^{1/3} . \tag{2.20}$$

Hyperbolic case, $f > 0$:

$$\begin{aligned} R(t,r) &= \frac{M}{f}(\cosh\eta - 1) , \\ \sinh\eta - \eta &= \frac{f^{3/2}}{M}(t - a(r)) . \end{aligned} \tag{2.21}$$

The parameter η is dependent on t and r and stands for the conformal time on each worldline, and $a(r)$ is a third arbitrary function which describes the local time of the "Big Bang". Eqs (2.19), (2.20) and (2.21) show that the evolution of R depends on M, f and a. The arbitrary functions f, M, and a, the energy per unit mass, the active gravitational mass, and the Big Bang time fully determine the LT model.

2.2 Observable quantities

With any distribution of cosmological sources ("galaxies") four fundamental observational quantities are the redshift z, the angular diameter δ, the apparent luminosity ℓ and the number count density n in redshift space. With each of δ, ℓ and n there is associated an intrinsic source property that may be time dependent, such as the true diameter D, the absolute luminosity L and the average mass per source μ. These source properties are a lot more difficult to determine observationally.

In the rest of this chapter we will present the mathematical framework that relates observation to the theory, on the past null cone (PNC) via the luminosity distance, the diameter distance, and the number count density (for more details see [61]). Cosmological observations are detections of light rays that arrive along our PNC. Therefore, since these light rays follow radial null geodesics, $ds^2 = 0 = d\theta^2 = d\phi^2$, Eq. (2.1) implies

$$\frac{d\hat{t}}{dr} = -\frac{\widehat{R'}(t,r)}{\sqrt{1+f}} . \tag{2.22}$$

Eq (2.22) has solution $t = \hat{t}(r)$. Now each quantity evaluated on the observer's past null cone will be denoted by a hat on top or subscript, for example $R(\hat{t},r) \equiv \hat{R}$ or $[R]_\wedge$.

2.2.1 Redshift

To obtain the redshift formula in the Lemaître-Tolman model, let us consider a cosmological source or "galaxy", located at r, that emits two light rays at t_e and $t_e + T_e$, that have a small oscillation period T_e. On the other side let us consider the observer located at $r = 0$ who receives the emitted light rays at t_0 and $t_0 + T_0$ respectively. The light rays follow null geodesics and cross nearby worldlines r and $r + dr$ at the points $A : (t_A, r)$, $B : (t_B, r + dr)$, $C : (t_C, r)$, and $D : (t_D, r + dr)$ respectively (see Figure (2.1)),

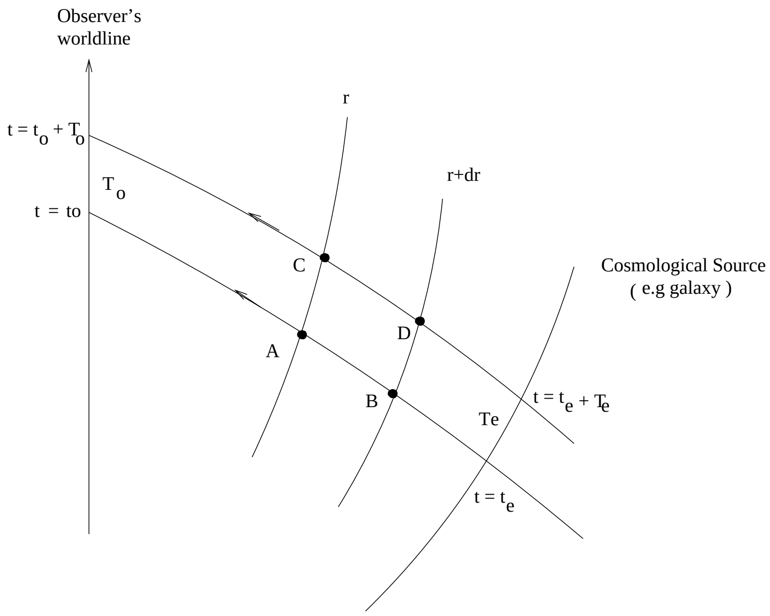

FIGURE 2.1: The path of the light rays from the source to observer

Using Taylor expansions along AB and CD, the change in time $\hat{t}$ between those points is,

$$t_B = t_A + \left.\frac{d\hat{t}}{dr}\right|_A dr + \mathcal{O}^2(r) \;, \tag{2.23}$$

$$t_D = t_C + \left.\frac{d\hat{t}}{dr}\right|_C dr + \mathcal{O}^2(r) \;. \tag{2.24}$$

The change in the interval time (oscillation period) dT between these worldlines can be obtained from Eq (2.23) and (2.24),

$$\begin{aligned} dT = T_{BD} - T_{AC} &= \Big(t_D - t_B\Big) - \Big(t_C - t_A\Big) \\ &= \left(\frac{d\hat{t}}{dr}\bigg|_C - \frac{d\hat{t}}{dr}\bigg|_A \right) dr \ . \end{aligned} \tag{2.25}$$

Again using a Taylor expansion along AC, the worldline of constant r, allows us to write Eq (2.25) as

$$dT = \left[\frac{d\hat{t}}{dr}\bigg|_A + \frac{\partial}{\partial t}\left(\frac{d\hat{t}}{dr}\right) T - \frac{d\hat{t}}{dr}\bigg|_A \right] dr = \frac{\partial}{\partial t}\left(\frac{d\hat{t}}{dr}\right) T \, dr. \tag{2.26}$$

Using Eq (2.22), therefore, the oscillation period in Eq (2.26) becomes

$$d(T) = dT_B - dT_A = \frac{\partial}{\partial t}\left(\frac{-\widehat{R'}}{\sqrt{1+f}} \right) drT \ , \tag{2.27}$$

Integrating this equation, the above expression is equivalent to

$$\int_{T_o}^{T_e} -\frac{dT}{T} = \ln\left(\frac{T_o}{T_e}\right) = \int_0^{r_e} \frac{\widehat{\dot{R}'}}{\sqrt{1+f}}\, dr \,. \tag{2.28}$$

Then using the definition of the redshift,

$$z = \frac{T_o - T_e}{T_e} \ ,$$

we find the redshift in the Lemaître-Tolman model takes the form

$$\ln(1+z) = \int_0^{r_e} \frac{\widehat{\dot{R}'}}{\sqrt{1+f}}\, dr \ . \tag{2.29}$$

Eq.(2.29) determines the redshift for the central observer at $r = 0$, receiving signals from an emitter at $r = r_e$ while the right hand side of Eq. (2.29) implies a numerical integration down the past null cone.

2.2.2 Angular diameter distance

In flat Newtonian spacetime, suppose our cosmological source has true diameter D, and its angular diameter is δ (see Fig (2.2)). Assuming that the measurement of the two ends of the diameter was at the same time, then its angular diameter distance d_D is the ratio between the true diameter D ("proper diameter") and the angular diameter δ of the cosmological source [17, 19],

$$d_D = \frac{D}{\delta}. \tag{2.30}$$

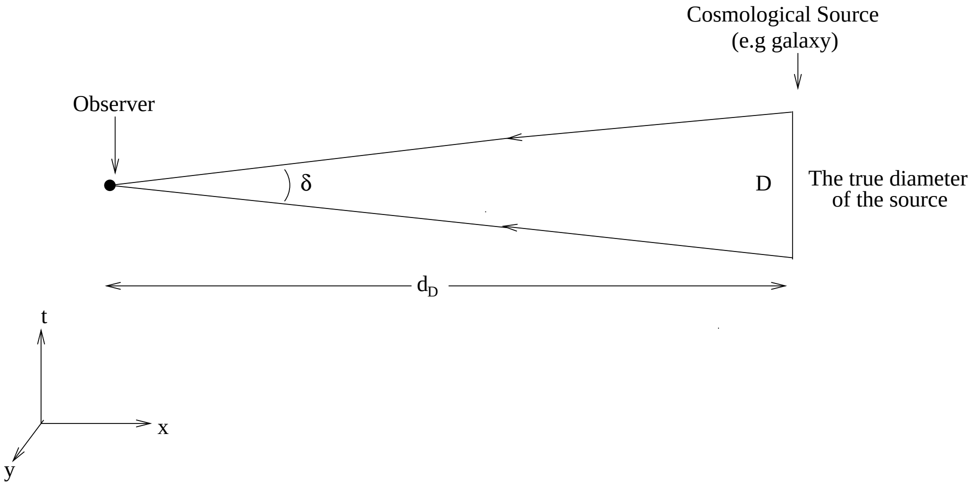

FIGURE 2.2: The Diameter distance between the observer and the Cosmological source (galaxy).

Now consider a curved spacetime having the geometry described by the Lemaître-Tolman metric. Suppose we have two radial null geodesics propagating along constant θ and ϕ directions meeting at the central observer at time t_o with angular separation δ, having been emitted at time t_e from the source of true diameter D at comoving coordinate r_e (see Fig (2.3)). Then the metric Eq. (2.10) relates the angular displacement δ (at constant r_e) to the physical size D, i.e

$$D = R(t_e, r_e)\delta. \tag{2.31}$$

Substituting Eq. (2.31) into (2.30) and noting $t_e = \hat{t}(r_e)$, the angular diameter distance is

$$d_D = \hat{R}(t_e, r_e)\ . \tag{2.32}$$

Eq. (2.32) shows that the angular diameter distance in the Lemaître-Tolman metric corresponds to the areal radius on the PNC.

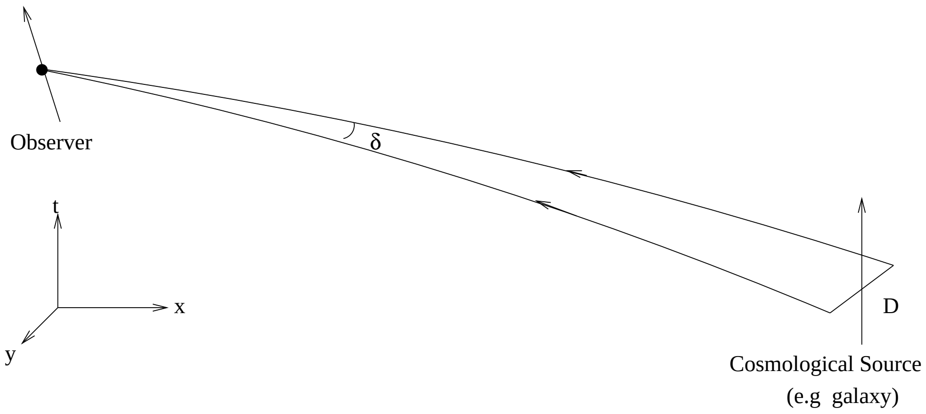

FIGURE 2.3: The Diameter distance between the observer and the Cosmological source (galaxy) in 4-d spacetime.

2.2.3 Luminosity distance

In flat Newtonian spacetime consider a cosmological source located at distance d_L that emits energy J_e per unit time measured in Js^{-1}. The flux received by an observer located at $r = 0$ (See Fig (2.4)) is,

$$\ell = \frac{J_e}{4\pi d_L^2}. \tag{2.33}$$

where ℓ is apparent luminosity of the source measured in $Js^{-1}m^{-2}$. The absolute luminosity L is what would be measured at 10 parsecs,

$$L = \frac{J_e}{4\pi d_{10}^2}. \tag{2.34}$$

where $d_{10} = 10$ pc. Eliminating $J_e/4\pi$ by dividing Eq. (2.33) and (2.34) the luminosity distance d_L for any cosmological source is defined in terms of the ratio between apparent luminosity ℓ, and abslute luminosity L, see Fig (2.4). It is given by

$$d_L = \sqrt{\frac{L}{\ell}} d_{10} \ , \tag{2.35}$$

In the Lemaître-Tolman spacetime, the luminosity distance is very hard to obtain, but the reciprocity theorem gives the relationship between the luminosity distance d_L and the diameter distance d_D in terms of the redshift (see Fig (2.5)), and holds in any 4-d spacetime under very general conditions,

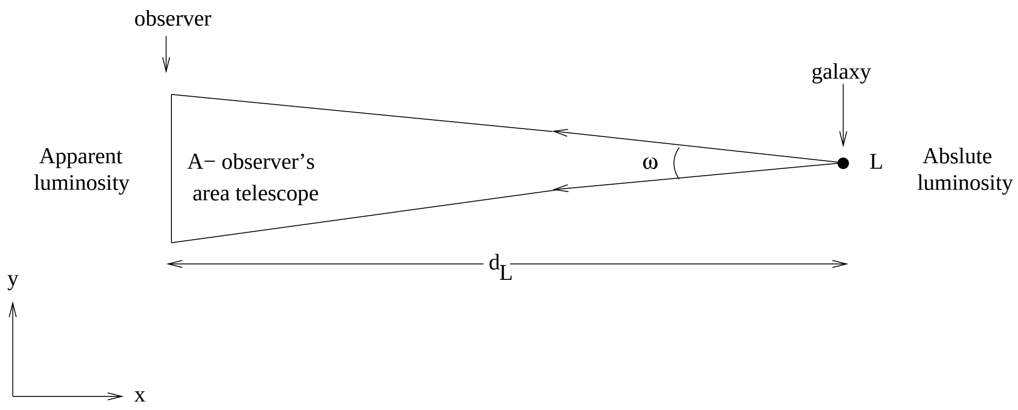

Figure 2.4: The Luminosity distance between the observer and the Cosmological source (galaxy) in flat spacetime.

$$d_L = (1+z)^2 d_D \ . \tag{2.36}$$

This was first introduced by Etherington and then shown more generally by Penrose (for more details see [62–64]).

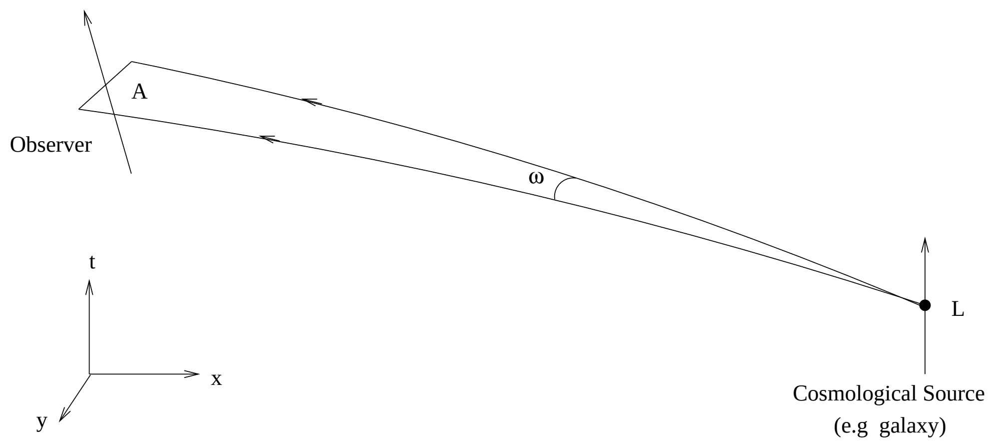

Figure 2.5: The Luminosity distance between the observer and the Cosmological source (galaxy).

2.2.4 Volume and number density

Suppose that we have N galaxies in a small sky area of $d\omega$ steradians which are located between r and $r + dr$ (see Fig (2.6)). Then, in the Lemaître-Tolman model, the proper volume dV of the spatial part, which is measured by a comoving observer at the time of emission, is

$$d\hat{V} = \sqrt{|-{}^3g|}\, dx^1 dx^2 dx^3 = \frac{\widehat{R'}\hat{R}^2}{\sqrt{1+f}} \sin\theta\, d\theta\, d\phi\, dr = \frac{\widehat{R'}\hat{R}^2}{\sqrt{1+f}} d\omega\, dr \ . \tag{2.37}$$

where ${}^3g$ is the determinant of the spatial part of the metric tensor $g_{\mu\nu}$, dx^1, dx^2, dx^3 are comoving coordinates and $d\omega = \sin\theta\, d\theta\, d\phi$. The associated total mass of these sources is

$$dM = \hat{\rho} d\hat{V} = \hat{\rho}\, \frac{\widehat{R'}\hat{R}^2}{\sqrt{1+f}} d\omega\, dr \tag{2.38}$$

when we have evaluated this expression on the PNC. On the other hand, in redshist space (z, θ, ϕ) let $n(z)$ stand for number of the sources per steradian per unit redshift interval, and let r and $r + dr$ correspond to z and $z + dz$ on the PNC, then $N = nd\omega dz$ is the total number of galaxies in solid angle $d\omega$ in this z interval. If μ is the average mass per source, then, the total mass of these galaxies is

$$dM = \mu\, n\, d\omega\, dz \ . \tag{2.39}$$

Eliminating the term dM from Eqs. (2.38) and (2.39), the relation between n and $\hat{\rho}$ is

$$\hat{\mu} n(z) = \hat{\rho}\, \frac{\widehat{R'}\hat{R}^2}{\sqrt{1+f}} \frac{dr}{dz} \, . \tag{2.40}$$

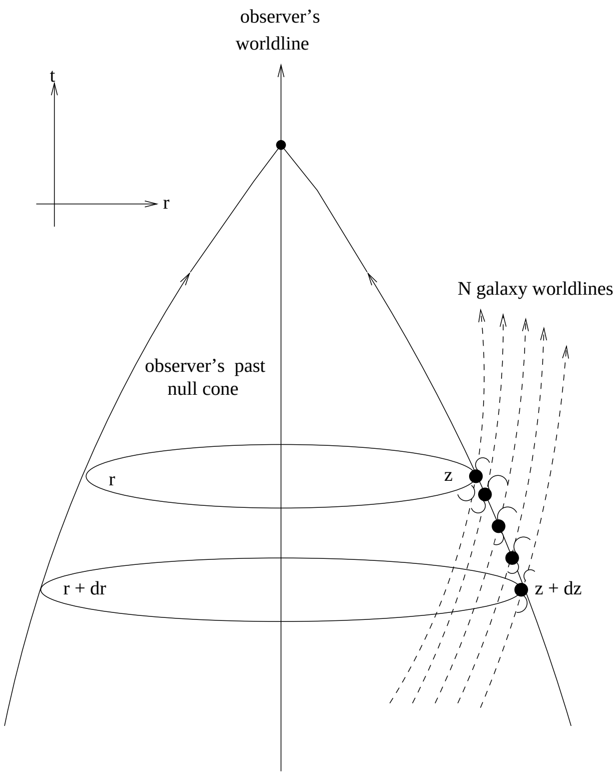

FIGURE 2.6: The number of galaxies counted within an interval r and $r + dr$ and its corresponding redshift z and $z + dz$ on the observer's PNC

2.3 Finding the metric from the PNC

We have shown that the Lemaître-Tolman model can be characterized by three arbitrary functions M, f, and a the total gravitational mass, the total energy per unit mass, and the bang time function respectively. In order to extract the metric functions from observational data (i.e write the arbitrary functions as a set of differential equations (DEs) in terms of the data), let us assume that the redshift z, the diameter distance $\hat{R}(z)$ and the redshift space density $\mu n(z)$ are already given on the PNC. The derivation below follows [34–36] but includes non-zero Λ.

If the PNC path is $\hat{t}$, the diameter distance evaluated on the PNC is $R(\hat{t}(r), r) \equiv \hat{R}$. Now, the freedom in r allows us to make the coordinate choice $d\hat{t}/dr = 1$, on this single PNC only, therefore

$$\frac{\widehat{R'}(\hat{t}(r), r)}{\sqrt{1+f}} = 1\,, \tag{2.41}$$

and integrating Eq. (2.22) with (2.41) for the incoming radial null geodesics gives

$$\hat{t}(r) = t_0 - r\,. \tag{2.42}$$

Using this coordinate choice, the density and the generalised Friedmann equation (2.16) and (2.14) can be evaluated on the PNC, giving

$$\frac{\kappa\hat{\rho}\hat{R}^2}{2} = \frac{M'}{\sqrt{1+f}}\,, \tag{2.43}$$

$$\widehat{\dot{R}} = \pm\sqrt{\frac{2M}{\hat{R}} + f + \frac{\Lambda}{3}\hat{R}^2}\,. \tag{2.44}$$

The gauge equation can be obtained by taking the total derivative of $\hat{R}$ on the PNC,

$$\frac{d\hat{R}}{dr} = \widehat{R'} + \hat{\dot{R}}\,\frac{d\hat{t}}{dr}\,. \tag{2.45}$$

The first and the second term on the right hand side of Eq (2.45) can be replaced by using Eq (2.41), Eq (2.44) and the derivative of Eq (2.42), therefore, the gauge equation can be simplified to:

$$\frac{d\hat{R}}{dr} - \sqrt{1+f} = -\widehat{\dot{R}} = \mp\sqrt{\frac{2M}{\hat{R}} + f + \frac{\Lambda}{3}\hat{R}^2}\,. \tag{2.46}$$

Writing $W = W(r) = \sqrt{1+f}$ and squaring and rearranging Eq (2.46) we get

$$W = \sqrt{1+f} = \frac{1}{2}\left[\left(\frac{d\hat{R}}{dr}\right) + \frac{\left(1 - \frac{2M}{\hat{R}} - \frac{\Lambda}{3}\hat{R}^2\right)}{\frac{d\hat{R}}{dr}}\right]. \tag{2.47}$$

The active gravitational mass M can be described in the form of a first order linear inhomogeneous DE, obtained by substituting Eq (2.43) into Eq (2.47) to eliminate W

$$\frac{dM}{dr} + \left(\frac{\kappa\hat{\rho}\hat{R}/2}{\frac{d\hat{R}}{dr}}\right)M = \left(\frac{\kappa\hat{\rho}\hat{R}^2/4}{\frac{d\hat{R}}{dr}}\right)\left[\left(\frac{d\hat{R}}{dr}\right)^2 + 1 - \frac{\Lambda}{3}\hat{R}^2\right]. \tag{2.48}$$

Finally, to find the $\hat{r}(z)$ relation the right hand side of Eq. (2.29) must be expressed in terms of $\hat{R}$ and $\hat{n}(z)$. Differentiating Eq. (2.14) with respect to r at constant t and then substituting the gauge choice in Eq. (2.41) and evaluating on the PNC, we obtain

$$\frac{\widehat{\dot{R}'}}{W} = \frac{1}{\hat{\dot{R}}}\left[\frac{M'}{\hat{R}W} - \frac{M}{\hat{R}^2} + \frac{\Lambda\hat{R}}{3} + W'\right] . \tag{2.49}$$

Now, from Eq. (2.47) the derivative of W down the PNC can be found,

$$W' = \left[\frac{1}{2}\left(\frac{d^2\hat{R}}{dr^2}\right) - \frac{M'}{\hat{R}\,\frac{d\hat{R}}{dr}} + \frac{M}{\hat{R}^2} - \frac{\Lambda\hat{R}}{3} - \frac{\left(1 - \frac{2M}{\hat{R}} - \frac{\Lambda}{3}\hat{R}^2\right)\frac{d^2\hat{R}}{dr^2}}{2\left(\frac{d\hat{R}}{dr}\right)^2}\right] , \tag{2.50}$$

then, since

$$1 - \frac{2M}{\hat{R}} - \frac{\Lambda}{3}\hat{R}^2 = W^2 - \hat{\dot{R}}^2 ,$$

replacing M', $\hat{\dot{R}}$ and W' by Eq. (2.43), (2.46), and (2.50) then, Eq. (2.49) can be written as:

$$\frac{\widehat{\dot{R}'}}{W} = -\left(\frac{\kappa\hat{\rho}\hat{R}}{2} + \frac{d^2\hat{R}}{dr^2}\right) \Big/ \left(\frac{d\hat{R}}{dr}\right) . \tag{2.51}$$

Now using Eq. (2.51) the redshift formula (2.29) can be written as:

$$\frac{dz}{dr} = -\frac{(1+z)\left(\frac{\kappa\hat{\rho}\hat{R}}{2} + \frac{d^2\hat{R}}{dr^2}\right)}{\left(\frac{d\hat{R}}{dr}\right)} . \tag{2.52}$$

Since the coordinate r is not an observable, we transform all r derivative to z derivatives, by defining

$$\frac{dr}{dz} = \varphi , \tag{2.53}$$

$$\frac{d\hat{R}}{dz} = \frac{d\hat{R}}{dr}\varphi , \qquad \frac{d^2\hat{R}}{dz^2} = \frac{d^2\hat{R}}{dr^2}\varphi^2 + \frac{d\hat{R}}{dr}\frac{d\varphi}{dz} . \tag{2.54}$$

Using Eq. (2.43) in to Eq. (2.40) to eliminate $\hat{\rho}$ we find

$$\frac{dM}{dz} = \frac{\kappa\,\mu n\,W}{2} , \tag{2.55}$$

while inserting Eq. (2.53) into Eq. (2.47) converts $W(r)$ to $W(z)$

$$W = \frac{1}{2\varphi}\left(\frac{d\hat{R}}{dz}\right) + \frac{\left(1 - \frac{2M}{\hat{R}} - \frac{\Lambda}{3}\hat{R}^2\right)\varphi}{2\left(\frac{d\hat{R}}{dz}\right)} . \tag{2.56}$$

Inserting Eq. (2.53) and (2.54) into Eq. (2.52) the DE for φ is

$$\frac{d\varphi}{dz} = \varphi \left(\frac{1}{(1+z)} + \frac{\frac{\kappa\rho\varphi^2\hat{R}}{2} + \frac{d^2\hat{R}}{dz^2}}{\left(\frac{d\hat{R}}{dz}\right)} \right) . \tag{2.57}$$

Eqs. (2.53), (2.55), (2.56), and (2.57) represent the DEs that describe the Lemaître-Tolman model, and they can be solved numerically to calculate the values of $\varphi(z)$, $r(z)$, $M(z)$ and $W(z)$. Finally the "Big Bang" time $a(z)$ can be obtained by combining Eq. (2.19), (2.20) and (2.21) on the PNC.

Alternatively writing Eq. (2.52) as

$$\frac{d\hat{R}}{dr}\frac{dz}{dr} + \frac{d^2\hat{R}}{dr^2}(1+z) = -\frac{\kappa\hat{\rho}\hat{R}}{2}(1+z) . \tag{2.58}$$

and integrating with respect to r we get

$$\int_0^z \frac{d}{dr}\left[\frac{d\overline{z}}{dr}\frac{d\hat{R}}{d\overline{z}}(1+\overline{z})\right] dr = -\int_0^z \frac{\kappa\hat{\rho}}{2}(\overline{z})\hat{R}(\overline{z})(1+\overline{z})\frac{dr}{d\overline{z}}d\overline{z}. \tag{2.59}$$

Using the origin conditions $[(dz/dr)(d\hat{R}/dz)]_0 = [(d\hat{R}/dr)]_0 = 1$, $z(0) = 0$ (see §2.4) , and (2.40) the above can be rearranged to obtain:

$$\frac{dz}{dr} = \left[\frac{d\hat{R}}{dz}(1+z)\right]^{-1}\left\{1 - \frac{\kappa}{2}\int_0^z \frac{\mu(\overline{z})n(\overline{z})}{\hat{R}(\overline{z})}(1+\overline{z})\,d\overline{z}\right\} . \tag{2.60}$$

Finally the $r(z)$ relation is:

$$r(z) = \int_0^z \left[\frac{\mathrm{d}\hat{R}}{\mathrm{d}\tilde{z}}(1+\tilde{z})\right]\left\{1 - \frac{1}{2}\kappa\int_0^{\tilde{z}} \frac{\mu(\overline{z})n(\overline{z})}{\hat{R}(\overline{z})}(1+\overline{z})\,\mathrm{d}\overline{z}\right\}^{-1} \mathrm{d}\tilde{z} . \tag{2.61}$$

This equation can be solved numerically. Once $r(z)$ is known the inverse, $z(r)$ can be calculated, and from Eq. (2.48) $M(r)$ can be found. Then using Eq. (2.47) or (2.56) $W(r)$ can be calculated.

2.4 The behaviour at the Origin

In this subsection, our interest is to study the behaviour of the DEs at the origin or "centre". The origin occurs at the centre of spherical coordinates when $r = 0$, $R(t, 0) = 0$ and $\dot{R}(t, 0) = 0$ for all time t. Therefore, on the PNC Eq. (2.45) and (2.46) become

$$\left.\frac{d\hat{R}}{dr}\right|_{r=0} = \widehat{R'} = \left.\frac{d\hat{R}}{dr}\right|_{r=0} = \sqrt{1+f} = 1\,, \tag{2.62}$$

So $\hat{R} \sim r$ near the center $r = 0$. Therefore Eq. (2.16) implies

$$M' \approx \frac{\kappa}{2}\hat{\rho}_0 r^2\,, \qquad M \approx \frac{\kappa}{6}\hat{\rho}_0 r^3\,. \tag{2.63}$$

From Eq. (2.47), and using a Taylor expansion for $\hat{R}$ up to second order in r, we then have

$$f \approx \left(\left(\frac{d^2\hat{R}}{dr^2}\right)_o^2 - \frac{\kappa}{3}\hat{\rho}_o - \frac{\Lambda}{3}\right) r^2\,. \tag{2.64}$$

At $r = 0$ the term $d^2\hat{R}/dr^2$ is finite, thus from Eq. (2.63) and (2.64) we observe that $f \to 0$, $M \to 0$ and $M \propto f^{3/2}$.

2.5 The Apparent Horizon

Since the apparent horizon is the locus where the PNC reaches its maximum areal radius, we write $z = z_{max}$, $\hat{R} = \hat{R}_{max}$, and $d\hat{R}/dr|_{max} = 0$ here. Consider (2.46) and substitute from Eq. (2.14) and Eq. (2.22). This gives

$$\sqrt{\frac{2M}{\hat{R}} + f + \frac{\Lambda}{3}\hat{R}^2} = \sqrt{1+f}\,, \tag{2.65}$$

which implies

$$3\hat{R}_{max} = \Lambda \hat{R}^3_{max} + 6M_{max}\,. \tag{2.66}$$

If $\Lambda = 0$, then the relation becomes

$$\hat{R}_{max} = 2M_{max}\,. \tag{2.67}$$

We note that the relation between the maximum in the diameter distance and the gravitational mass at that locus is independent of any inhomogeneity between the observer and the source at this distance [37]. This maximum produces a difficulty in the numerical integration. In other words the DEs become singular at the maximum when $R = \hat{R}_{max}$. Since $3\hat{R}_{max} = \Lambda\hat{R}^3_{max} + 6M_{max}$ at the maximum then Eqs. (2.47), (2.48) and (2.57) contain zero over zero at that locus. On other hand the Eq. (2.60) shows

$$\frac{\kappa}{2}\int_{o}^{z_m}\frac{\mu n}{\hat{R}}(1+\overline{z})d\overline{z}=1$$

at $\hat{R}=\hat{R}_{max}$, since we do not expect $dz/dr=\infty$. So for Eq. (2.60) and (2.61) we also have $0/0$ at $\hat{R}_{max}$. Therefore, the numerical method breaks down at that locus. This problem can be avoided by doing a series expansion for $\hat{R}(z)$, $\hat{\mu}n(z)$, $\phi(z)$, $M(z)$ and $W(z)$ near the maximum of $\hat{R}$, which was studied in detail by Lu & Hellaby [35, 61] and McClure & Hellaby [36].

Chapter 3

Solving the Observer Metric

Cosmology is all about understanding the observed universe, and since Einstein's field equations are central to that endeavour, the primary problem is to determine the cosmic geometry, i.e. the metric, from observations of its matter content. A full understanding of the dynamics of the universe cannot be separated from understanding its geometry. Historically, it was very difficult to determine cosmological data with any precision, and the assumption of a homogeneous universe, which allowed a simple metric form, was entirely sufficient and indeed very fruitful. Consequently the problem was reduced to one of finding the best-fit parameter set, rather than determining the metric. Recent decades have seen a considerable improvement in the quality and quantity of cosmological data, mapping much more accurately the matter distribution and the structures that exist. Therefore, relaxing the assumption of homogeneity has become an important task.

This chapter provides a complementary approach to the observational metric. In the observational cosmology papers [17–23, 25–29], the approach to solving the problem has focused on using the observational data to analytically determine the metric functions. Here, with an eventual numerical scheme in mind, the presented approach will emphasise firstly the full formal solution of the field equations for the observer metric, particularly noting the 4 arbitrary functions that emerge in the process and how the evolution is determined by them, and secondly the algorithm for determining the metric from observational data, especially showing how the arbitrary functions are fixed by the data. The first provides a better understanding of the geometry and dynamics of the model, and the second shows the relationship between the data and the particular characteristics of the solution metric. The explicit transformation between the OC and LT forms of this metric is given in the appendix.

Although spherical symmetry about the observer is a strong assumption, it should be regarded as a first step — a useful and important one — towards the more general case. A proper understanding of this simpler case is essential for working with the more general forms of the observer metric. This chapter presents original work, published in collaboration with Charles Hellaby [65].

3.1 Spherically Symmetric Observer Metric

We choose coordinates $x^\mu = (w, y, \theta, \phi)$, and we assume (a) spherical symmetry about the origin, (b) the observer is at the origin, and (c) the (θ, ϕ) surfaces are orthogonal to the (w, y) surfaces. We work in geometric units. With these, the metric is

$$ds^2 = -A^2\,dw^2 + 2AB\,dw\,dy + C^2\left(d\theta^2 + sin^2\theta\,d\phi^2\right) . \tag{3.1}$$

Here $A = A(w, y)$ and $B = B(w, y)$ while $C = C(w, y)$ is an areal radius, and the lack of a dy^2 term ensures that the constant w surfaces are null,

$$dw = 0 = d\Omega \qquad \rightarrow \qquad ds^2 = 0 . \tag{3.2}$$

We further assume that the matter is a zero-pressure perfect fluid, comoving with the y coordinate,

$$T^{\mu\nu} = \rho u^\mu u^\nu , \tag{3.3}$$

where the constant y, θ, ϕ curves have timelike tangent vectors such that

$$u^\mu = \frac{1}{A}\,\delta^\mu_w , \qquad u_\mu = (-A, B, 0, 0) , \qquad u^\mu u_\mu = -1 . \tag{3.4}$$

Here ρ is the proper density relative to observers on the comoving worldlines, u^μ.

We note that the null tangent vector k_μ within the constant w surfaces,

$$k_\mu = \delta^w_\mu , \qquad k^\mu = \left(0, \frac{1}{AB}, 0, 0\right) , \qquad k^\mu k_\mu = 0 , \tag{3.5}$$

must be geodesic since it represents radial light rays, $k^\mu \nabla_\mu k^\nu = 0$, and in fact, for any $K(y)$, $k_\mu = \delta^w_\mu K(y)$ is geodesic. Similarly, the dust particles should follow geodesics, and $u^\mu \nabla_\mu u^\nu = 0$ leads to the two equations

$$u^w \partial_w u^w = -\left(\frac{A_w}{A} + \frac{B_w}{B} + \frac{A_y}{B}\right)(u^w)^2 \tag{3.6}$$

$$0 = \frac{A}{B}\left(\frac{B_w}{B} + \frac{A_y}{B}\right)(u^w)^2 , \tag{3.7}$$

where $A_w = \partial A/\partial w$, $A_y = \partial A/\partial y$, etc. The second of these imposes a restriction on the metric functions,

$$B_w = -A_y \ , \tag{3.8}$$

which reduces the first to $(\partial_w u^w)/u^w = -(\partial_w A)/A$, in agreement with the normalisation condition Eq. (3.4).

Near the central worldline $y = 0$ say, there is a spherical origin, where $C(w, 0) = 0$ for all w. The origin conditions for this metric, giving the limiting behaviours of A, B and C near $C = 0$, have been presented in several of the observational cosmology papers [17–23, 25–29].

3.1.1 Solving the EFEs

The Einstein field equations (EFEs) $G^{\mu\nu} = \kappa T^{\mu\nu} - \Lambda g^{\mu\nu}$ for this metric are

$$G^{ww} = \frac{2}{A^2B^2}\left(\frac{A_yC_y}{AC} + \frac{B_yC_y}{BC} - \frac{C_{yy}}{C}\right) = \frac{\kappa\rho}{A^2} \tag{3.9}$$

$$G^{wy} = \frac{2}{A^2B^2}\left(\frac{C_{wy}}{C} + \frac{C_wC_y}{C^2} + \frac{AC_y^2}{2BC^2} + \frac{A_yC_y}{BC} - \frac{AB}{2C^2}\right) = -\frac{\Lambda}{AB} \tag{3.10}$$

$$\begin{aligned} G^{yy} = \frac{2}{A^2B^2}\Bigg(&\frac{A_wC_w}{AC} + \frac{B_wC_w}{BC} - \frac{C_{ww}}{C} + \frac{A_yC_w}{BC} + \frac{AB_wC_y}{B^2C} + \frac{AC_wC_y}{BC^2} \\ &+ \frac{AA_yC_y}{B^2C} + \frac{A^2C_y^2}{2B^2C^2} - \frac{A^2}{2C^2}\Bigg) = -\frac{\Lambda}{B^2} \end{aligned} \tag{3.11}$$

$$\begin{aligned} G^{\theta\theta} = \frac{1}{ABC^2}\Bigg(&\frac{2C_{wy}}{C} + \frac{A_{wy}}{A} + \frac{B_{wy}}{B} - \frac{A_wA_y}{A^2} - \frac{B_wB_y}{B^2} + \frac{AC_{yy}}{BC} + \frac{A_{yy}}{B} \\ &- \frac{A_yB_y}{B^2} + \frac{A_yC_y}{BC} - \frac{AB_yC_y}{B^2C}\Bigg) = -\frac{\Lambda}{C^2} \ , \end{aligned} \tag{3.12}$$

and the conservation equations, $\nabla_\nu T^{\mu\nu} = 0$ are

$$\nabla_\nu T^{w\nu} = \frac{\rho_w}{A^2} + \frac{\rho}{A^2}\left(\frac{2C_w}{C} + \frac{2B_w}{B} + \frac{A_y}{B}\right) = 0 \tag{3.13}$$

$$\nabla_\nu T^{y\nu} = \frac{\rho(B_w + A_y)}{AB^2} = 0 \ . \tag{3.14}$$

Not surprisingly, we can obtain Eq. (3.8) directly from Eq. (3.14), since we do not expect the density to be zero.

From Eq. (3.10) & Eq. (3.11) above we obtain

$$\frac{A^2BC}{2}G^{wy} - \frac{AB^2C}{2}G^{yy} = \frac{C_{wy}}{B} - \frac{B_wC_y}{B^2} + \frac{C_{ww}}{A} - \frac{A_wC_w}{A^2} = 0 \ , \tag{3.15}$$

where two terms cancelled because of Eq. (3.8). This can be written as

$$\frac{\partial}{\partial w}\left(\frac{C_w}{A} + \frac{C_y}{B}\right) = 0 \ , \tag{3.16}$$

which solves to give

$$\frac{C_w}{A} + \frac{C_y}{B} = W(y) \ , \tag{3.17}$$

where $W(y)$ is an undetermined function of integration.

Next, from Eq. (3.17) we have

$$\begin{aligned} C_y &= B\left(W - \frac{C_w}{A}\right) \ , \\ C_{wy} &= B_w\left(W - \frac{C_w}{A}\right) - \frac{B}{A^2}\left(AC_{ww} - A_wC_w\right) \ , \end{aligned} \tag{3.18}$$

which combine with the G^{yy} equation (3.11) to give

$$-B^2C^2C_w\,G^{yy} = \frac{C_w^3}{A^2} + \frac{2CC_wC_{ww}}{A^2} - \frac{2CC_w^2A_w}{A^3} - C_w(W^2-1) = C^2C_w\Lambda \ , \tag{3.19}$$

where Eq. (3.8) was used again. The solution here is

$$\frac{\partial}{\partial w}\left(\frac{CC_w^2}{A^2} - C(W^2-1) - \frac{C^3\Lambda}{3}\right) = 0 \ , \tag{3.20}$$

which implies $$\frac{CC_w^2}{A^2} - C(W^2-1) - \frac{C^3\Lambda}{3} = 2M(y) \ , \tag{3.21}$$

where $M(y)$ is a second undetermined function of integration.

Equations Eq. (3.9) and Eq. (3.11) give the same as the y derivative of Eq. (3.21):

$$\begin{aligned} \kappa\rho BW &= (G^{ww} + \Lambda g^{ww})A^2BW - (G^{wy} + \Lambda g^{wy})ABC_y \\ &= -\frac{2C_wC_y^2}{ABC^2} - \frac{C_y^3}{B^2C^2} + \frac{C_y}{C^2} - C_y\Lambda + \frac{2B_yC_wC_y}{AB^2C} - \frac{2C_wC_{yy}}{ABC} \\ &\quad + \frac{2A_yC_wC_y}{A^2BC} + \frac{2B_yC_y^2}{B^3C} - \frac{2C_yC_{wy}}{ABC} - \frac{2C_yC_{yy}}{B^2C} \\ &= \frac{2M_y}{C^2} \ , \end{aligned} \tag{3.22}$$

where Eq. (3.17) and its y derivative were used. Therefore the density is given by

$$\kappa\rho = \frac{2M_y}{C^2 BW} \ , \tag{3.23}$$

which clearly satisfies Eq. (3.13). Similarly, the Kretschmann scalar is

$$K = R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} = \frac{48M^2}{C^6} + \frac{8\Lambda^2}{3} - \frac{32MM_y}{C^5BW} + \frac{12M_y^2}{C^4B^2W^2} + \frac{8\Lambda M_y}{3C^2BW} \ . \tag{3.24}$$

The solution Eq. (3.21) can be re-written as an evolution equation for C:

$$\frac{C_w}{A} = \pm\sqrt{\frac{2M}{C} + f + \frac{\Lambda C^2}{3}} \ , \tag{3.25}$$

$$\text{where} \qquad f(y) = W^2 - 1 \qquad \leftrightarrow \qquad W = \sqrt{1+f} \ , \tag{3.26}$$

and the sign depends on whether C is increasing or decreasing with time. In addition, equations Eq. (3.17) and Eq. (3.25) give

$$\frac{C_y}{B} = \sqrt{1+f} \mp \sqrt{\frac{2M}{C} + f + \frac{\Lambda C^2}{3}} \ . \tag{3.27}$$

Eq. (3.25) is clearly allied to the LT evolution equation Eq. (A.2), except that it contains two unknown functions, C and A, so it cannot be solved as is. Wherever a function is transformed between coordinates, we write e.g. $C(w, y) = C(t, r)$, meaning the two forms have the same numerical value at any given event, but they do not have the same functional dependence on their arguments. To solve Eq. (3.25), we define the proper time t along the worldlines of constant y by

$$t = \int_{\text{const}\, y} A\, dw \qquad \rightarrow \qquad \frac{\partial}{\partial w} = A\frac{\partial}{\partial t} \ , \tag{3.28}$$

which converts Eq. (3.25) to

$$\int \frac{dC}{\pm\sqrt{\frac{2M}{C} + f + \frac{\Lambda C^2}{3}}} = \int dt = t - a(y) \ . \tag{3.29}$$

In principle this gives us $t(C, y)$ or $C(t, y)$, and introduces $a(y)$, the initial t value at each y, as a third free function of integration. The solutions to Eq. (3.29) are identically those of the LT metric. The $\Lambda = 0$ solutions for each of the cases $f > 0$, $f = 0$ and $f < 0$ are well known, and are often given parametrically, $\{C(\eta, y), t(\eta, y)\}$. However, we don't yet have a transformation between t and w, since by Eq. (3.28)

$$t = \int_{\text{const}\, y} A\, dw \qquad \leftrightarrow \qquad A = t_w \ , \tag{3.30}$$

we must know A to calculate w and vice versa. Now from Eq. (3.8) we find

$$B_w = -A_y = -t_{wy} \quad \rightarrow \quad B = -t_y + \beta(y) \ . \tag{3.31}$$

The function $\beta(y)$ reflects a freedom in the definition Eq. (3.28) of t,

$$t \ \rightarrow \ t + \alpha(y) \ , \qquad \beta \ \rightarrow \ \beta - \alpha_y \ , \tag{3.32}$$

which we shall remove. We next define

$$r = r(w, y) = y \ , \quad \rightarrow \quad r_w = 0 \ , \ r_y = 1 \ , \tag{3.33}$$

and in the next few equations we use r when it is paired with t, as in $C_r = \partial_r C(t, r)$, but y when paired with w, as in $C_y = \partial_y C(w, y)$. By requiring that our t coordinate be orthogonal

to r, viz:

$$0 = g^{\mu\nu} \left(\partial_\mu t\right) \left(\partial_\nu r\right) = \frac{t_w \, r_y + t_y \, r_w}{AB} + \frac{t_y \, r_y}{B^2} = \frac{A + 0}{AB} + \frac{(\beta - B)}{B^2} \quad \rightarrow \quad \beta = 0 \ , \tag{3.34}$$

we reduce the freedom in t to a constant translation, i.e. α is a constant, but henceforth we shall drop it from our equations. Without this orthogonality condition, $\beta \neq 0$ and several subsequent equations contain large extra terms. Evidently, then, B is the negative of the rate of variation of proper time with respect to y down the past null cone. Even though we know $t(C, y)$, we can't calculate t_y unless we know how to hold w constant. Now the transformation between $C(w, y)$ and $C(t, r)$ allows us to write

$$C_y = C_t t_y + C_r r_y = -C_t B + C_r \ , \tag{3.35}$$

where $C_t \equiv \partial C/\partial t$ and $C_r \equiv \partial C/\partial r$. Combining Eq. (3.35) with Eq. (3.27) leads to

$$B = \frac{C_r}{\sqrt{1 + f}} \ , \tag{3.36}$$

and since $C(t, r)$ is known, this gives us $B(t, r)$. Using Eq. (3.31), Eq. (3.34) and Eq. (3.36) we obtain the differential equation

$$t_y = \frac{-C_r}{\sqrt{1 + f}} \ . \tag{3.37}$$

This equation specifies how much t changes for a given y change, when w is constant, so it may be integrated down the null cones, i.e. along constant w, from the origin outwards, giving $t(w, y)$. The boundary conditions, fixed say at the origin $y = 0$, give us a 4th undetermined function, $\gamma(w) = t(w, 0)$. Actually, this fixes the variation of w with respect to t, as t is fixed by integrating Eq. (3.29). An obvious choice is $w = t|_o = \gamma$. Having solved Eq. (3.37), we can then convert $C(t,r)$ to $C(w,y)$ using

$$C(t,r) \;\text{ plus }\; t(w,y) \quad \rightarrow \quad C(w,y) = C(t(w,y),y) \;, \tag{3.38}$$

and we finally determine $A(w,y)$ from Eq. (3.30), and $B(w,y)$ from Eq. (3.31), or possibly Eq. (3.36). The algorithm for calculating the model evolution is detailed in Section 3.3.6.

Having completed the solution, we see that this metric has 4 arbitrary functions — $f(y)$, $M(y)$, $a(y)$ and $\gamma(w)$ — of which $\gamma(w)$ represents a freedom to rescale w that is most naturally set to $\gamma = w$. The physical meanings of f, M and a are exactly as in the LT model; they represent two physical relationships plus a freedom to rescale y.

3.2 Observable quantities

Using the same ideas as in § 2.2, the cosmological observable quantities can obtained in the same way. Let w_0 label the past null cone (PNC) of present day observations by the central observer at $(t,r) = (t_0,0) \equiv (w,y) = (w_0,0)$, and let the evaluation of any quantity $Q(w,y)$ on this PNC be denoted $\hat{Q} = [Q]_\wedge = Q(w_0,y)$. We assume that emitters follow comoving worldlines y_e, and the observer is at the central worldline, $y_o = 0$. Let the evaluation of a quantity at the observer and the emitter be denoted $Q_o(w) = Q(w,0)$ and $Q_e(w) = Q(w,y_e)$ respectively. However we will often drop the subscript e.

The redshift of comoving sources on that null cone is given by the ratio of the light oscillation periods T measured at the observer, o, and the emitter, e,

$$(1+z) = \frac{T_o}{T_e} = \frac{\hat{A}_o\, dw}{\hat{A}_e\, dw} \quad \rightarrow \quad \hat{A} = \frac{\hat{A}_o}{(1+z)} \;, \tag{3.39}$$

and we can put $\hat{A}_o = 1$ since $\hat{A}_o = A(w_0,0) = \partial_w \gamma|_{w_0} = 1$ is the natural choice.

The definitions of the diameter and the luminosity distances are the same as in the Lemaître-Tolman model, (see chapter 2 for the explicit definitions of these quantities). Hence we have

$$d_D = \hat{C}_e \;, \qquad \text{and} \qquad d_L = (1+z)^2 \hat{C}_e \;. \tag{3.40}$$

In redshift space, (z, θ, ϕ), let $n(z)$ be the density of sources, that is the number per steradians per unit redshift interval[1]. Suppose that there are dN sources in solid angle $d\omega = \sin\theta \, d\theta \, d\phi$ between redshift z and $z + dz$, and that $\mu(z)$ is the mean mass per source[2], then the mass in that volume element of redshift space is, as before,

$$d\mathcal{M} = \mu \, dN = \mu \, n \, d\omega \, dz \ . \tag{3.41}$$

The proper 3-volume enclosing these sources at the time of emission, as measured by comoving observers u^μ, is spanned by

$$dx_1^\mu = \delta_y^\mu \, dy \ , \qquad dx_2^\mu = \delta_\theta^\mu \, d\theta \ , \qquad dx_3^\mu = \delta_\phi^\mu \, d\phi \ , \tag{3.42}$$

and evaluates to

$$\begin{aligned} d^3v &= \eta_{\mu\nu\lambda\rho} \, u^\mu \, dx_1^\nu \, dx_2^\lambda \, dx_3^\rho = \sqrt{|g|} \ \epsilon_{0123} \, u^0 \, dx_1^1 \, dx_2^2 \, dx_3^3 \\ &= \sqrt{|-A^2 B^2 C^4 \sin^2\theta|} \, \frac{1}{A} \, dy \, d\theta \, d\phi \\ &= BC^2 \sin\theta \, d\theta \, d\phi \, dy = BC^2 \, d\omega \, dy \ , \end{aligned} \tag{3.43}$$

so that the mass in this fluid element is

$$d\mathcal{M} = \rho \, B \, C^2 \, d\omega \, dy \ . \tag{3.44}$$

Clearly we have the following relationship between n and $\hat{\rho}$,

$$\mu \, n = \hat{\rho} \, \hat{B} \, \hat{C}^2 \, \frac{\mathrm{d}y}{\mathrm{d}z} \ . \tag{3.45}$$

The apparent horizon is where C is maximum on any given constant w cone,

$$C_y = 0 \ . \tag{3.46}$$

Now if the metric (3.1) is to be regular, and the density Eq. (3.23) and Kretschmann scalar Eq. (3.24) finite at a generic point, then B must be non-zero. For incoming light rays, Eq. (3.46) would need the upper sign in Eq. (3.27) — that is, the local matter-shells are expanding, $\dot{C} > 0$ — and

$$\sqrt{\frac{2M}{C} + f + \frac{\Lambda C^2}{3}} = \sqrt{1+f} \qquad \rightarrow \qquad 6M + \Lambda C^3 - 3C = 0 \ , \tag{3.47}$$

[1] Thus this n is different from the n used in the OC programme, which is number density on a constant time slice, see Section 3.4.

[2] For a treatment with a variety of source types, see [66].

along this locus. When $\Lambda = 0$ this simplifies to $C = 2M$. As shown in [35–37], this locus has considerable observational significance. See also [28].

3.3 Determining the Solution from Observational Data

Given the above observational data on the past null cone, that is $\hat{A}(z)$, $\hat{C}(z)$, and $\mu n(z)$, the solution process must determine the arbitrary functions f, M, a and γ. Knowing these enables all spacetime quantities to be calculated and evolved, via the results of Section 3.1.1. We envisage a numerical solution, so the emphasis here is on laying out a solution algorithm, rather than on formal integrals and functional dependence.

3.3.1 Gauge choices

The observational data must determine the physical properties of the model, but cannot restrict the coordinate freedoms. Therefore we will have to make some gauge choices in order to effect the solution. Firstly, we set $w = t$ along the central worldline, which implies

$$\gamma(w) = w \qquad \rightarrow \qquad A(w, 0) = t_w(w, 0) = 1 \qquad \rightarrow \qquad \hat{A}_o = 1 \qquad \rightarrow \qquad \hat{A} = \frac{1}{(1+z)} \; . \tag{3.48}$$

Secondly, we need to set the freedom in the y coordinate. We consider two options below. The 'LT' option specifies

$$\hat{t}_y = -1 \qquad \rightarrow \qquad \hat{B} = 1 \; , \tag{3.49}$$

as in many LT approaches. The OC papers choose $A(w_0, y) = B(w_0, y)$ on the PNC, so in the 'OC' option we choose

$$\hat{B} = \hat{A} = \frac{1}{(1+z)} \qquad \rightarrow \qquad \hat{t}_y = \frac{-1}{(1+z)} \; . \tag{3.50}$$

3.3.2 DE for y(z)

The coordinate y is of course not observable, but we have to determine it first. We define

$$\varphi = \frac{\mathrm{d}y}{\mathrm{d}z} \; , \tag{3.51}$$

and along the PNC we re-write our equations in terms of z derivatives rather than y derivatives; for any quantity $Q(w, y)$,

$$Q_y = \frac{Q_z}{\varphi} \quad \text{and} \quad Q_{yy} = \frac{Q_{zz}}{\varphi^2} - \frac{Q_z \varphi_z}{\varphi^3} \ . \tag{3.52}$$

Evaluating Eq. (3.9) on the PNC, and using Eq. (3.45), we find

$$\frac{\hat{A}_z \hat{C}_z}{\hat{A}\hat{C}} + \frac{\hat{B}_z \hat{C}_z}{\hat{B}\hat{C}} - \frac{\hat{C}_{zz}}{\hat{C}} + \frac{\hat{C}_z \varphi_z}{\hat{C}\varphi} - \frac{\kappa\mu n \hat{B}\varphi}{2\hat{C}^2} = 0 \ , \tag{3.53}$$

which, upon substituting for $\hat{A}_z/\hat{A}$ from Eq. (3.39), leads to

$$\varphi_z = \varphi \left(\frac{1}{(1+z)} - \frac{\hat{B}_z}{\hat{B}} + \frac{\hat{C}_{zz}}{\hat{C}_z} + \frac{\kappa\mu n \hat{B}\varphi}{2\hat{C}\hat{C}_z} \right) \ , \tag{3.54}$$

where all derivatives are now total derivatives along the PNC.

At this point we must fix the gauge in order to freeze out the coordinate freedom. The two options given above each convert Eq. (3.54) to an ODE for $\varphi(y)$ completely in terms of observables:

$$\text{OC:} \qquad (\varphi_1)_z = \varphi_1 \left(\frac{2}{(1+z)} + \frac{\hat{C}_{zz}}{\hat{C}_z} + \frac{\kappa\mu n \varphi_1}{2(1+z)\hat{C}\hat{C}_z} \right) \tag{3.55}$$

$$\text{LT:} \qquad (\varphi_2)_z = \varphi_2 \left(\frac{1}{(1+z)} + \frac{\hat{C}_{zz}}{\hat{C}_z} + \frac{\kappa\mu n \varphi_2}{2\hat{C}\hat{C}_z} \right) \ . \tag{3.56}$$

Integrating Eq. (3.55) or Eq. (3.56) followed by Eq. (3.51) yields $\varphi_i(z)$ and

$$y_i(z) = \int_0^z \varphi_i(z)\, dz \ . \tag{3.57}$$

This allows us to convert between functions of z and functions of y on the PNC.

3.3.3 DE for M(z) & W(z)

From Eq. (3.23) on the PNC and Eq. (3.54) we obtain

$$M_z = \frac{\kappa\mu n W}{2} \ , \tag{3.58}$$

where W is found by putting Eq. (3.27) on the PNC,

$$W = \frac{\hat{B}\varphi}{2\hat{C}_z} \left(1 - \frac{2M}{\hat{C}} - \frac{\Lambda \hat{C}^2}{3} \right) + \frac{\hat{C}_z}{2\hat{B}\varphi} \ . \tag{3.59}$$

The two gauge choices give

$$\text{OC:} \qquad W = \frac{\varphi_1}{2\hat{C}_z(1+z)}\left(1 - \frac{2M}{\hat{C}} - \frac{\Lambda\hat{C}^2}{3}\right) + \frac{\hat{C}_z(1+z)}{2\varphi_1} \tag{3.60}$$

$$\text{LT:} \qquad W = \frac{\varphi_2}{2\hat{C}_z}\left(1 - \frac{2M}{\hat{C}} - \frac{\Lambda\hat{C}^2}{3}\right) + \frac{\hat{C}_z}{2\varphi_2} \; . \tag{3.61}$$

Together Eq. (3.58) and Eq. (3.60) or Eq. (3.61) constitute an ODE for $M(z)$ that also generates $W(z)$. Note that Eq. (3.60) requires φ_1 from Eq. (3.55) and Eq. (3.61) requires φ_2 from Eq. (3.56). Technically, this is a first order linear inhomogeneous ODE for M, so the formal solution is well known. In practice, the two integrals involved would both have to be done numerically, so it is less work to solve the ODE directly in parallel with Eq. (3.54) and Eq. (3.51), using say a Runge-Kutta method.

3.3.4 Obtaining a(z)

From Eq. (3.31) and Eq. (3.34) on the PNC we get

$$\hat{t}_z = -\varphi\hat{B} \; , \tag{3.62}$$

which, in the OC & LT gauges, simplifies to the ODEs

$$\text{OC:} \qquad \hat{t}_z = \frac{-\varphi_1}{(1+z)} \tag{3.63}$$

$$\text{LT:} \qquad \hat{t}_z = -\varphi_2 \; , \tag{3.64}$$

thus giving the worldline proper time, $\hat{t}(z)$ or $\hat{t}(y)$, on the PNC. The appropriate φ_i must be used in each equation. From Eq. (3.29) on the PNC, we write

$$\int_0^{\hat{C}} \frac{dC}{\pm\sqrt{\frac{2M}{C} + f + \frac{\Lambda C^2}{3}}} = \tau \; , \tag{3.65}$$

where f is given in Eq. (3.26) and τ is the proper time from the bang to the PNC along the matter worldlines, and after performing the integral at each z, we calculate

$$a(z) = \hat{t}(z) - \tau(z) \; . \tag{3.66}$$

We now have M, $W = \sqrt{1+f}$, and a, and we've also used up the freedom to rescale y by directly or indirectly fixing t_y, and $\gamma(w)$ is fixed by $\hat{A}_o = t_w(w_o, 0) = 1$.

3.3.5 Evolving off the PNC

In principle, no amount of data on the PNC is sufficient to determine the future evolution of any part of the spacetime, because new information can arrive along succeeding incoming light rays. But since we have already assumed a dust equation of state, in order to get the arbitrary functions, it is not unreasonable to expect the worldlines continue their dust evolution into the future. This same assumption is tacitly made when fitting a Robertson-Walker model to observational data.

Away from the PNC, z is no longer a useful variable, and we should rather use y. Also, the gauge choices do not give B off the PNC, so gauge-specific equations such as Eq. (3.55) or Eq. (3.64) are not applicable. One may determine the full evolution of C using the algorithm below, based on the solution of section 3.1.1. In addition, Eq. (3.9) together with Eq. (3.23) provides a cross-check on the calculated propagation of the metric components.

The remaining question is whether, given that we have initial data for $\hat{C}$, $\hat{B}$ & $\hat{A}$ on the PNC, there is a better way to evolve C than integrating over the entire (t, r) domain twice, first calculating $C(t, r)$ and then finding $t(w, y)$ and converting to $C(w, y)$. Can we integrate directly with respect to w, giving $C(w, y)$ straight off? This would be especially important if there were detectable time evolution in cosmological observables. The key difficulty is that we do not know any of A, B or C away from the PNC and the central worldline, and though we have direct evolution equations for B_w and C_w, there isn't one for A_w.

Once the arbitrary functions W, M and a are known, the observational data plus the gauge choices give us all of A, B & C on an initial constant w null cone, w_0. For clarity of argument, consider Euler integration. Evolution equations for C and B follow from Eq. (3.25) and Eq. (3.8),

$$C_{i+1} = C_i + (C_w)_i \, dw \ , \qquad C_w = AV \ ,$$

$$V = \pm\sqrt{\frac{2M}{C} + W^2 - 1 + \frac{\Lambda C^2}{3}} \ , \tag{3.67}$$

$$B_{i+1} = B_i + (B_w)_i \, dw \ , \qquad B_w = -A_y \ , \tag{3.68}$$

but the difficulty is finding an evolution equation for A. For example, Eq. (3.17), in the form

$$A_{i+1} = \frac{(C_w)_{i+1}}{W - (C_y)_{i+1}/B_{i+1}} = \frac{A_{i+1}V_{i+1}}{W - (C_y)_{i+1}/B_{i+1}} \ , \tag{3.69}$$

does not help because A_{i+1} cancels out. The yy EFE contains A_w, but as soon as we substitute $C_{ww} = A_w V + AV_w$, then A_w vanishes from the equation. The best we can do is to put $B_w = -A_y$, $C_w = AV$ and $C_{ww} = A_w V + AV_w$ in the wy EFE (3.10), to obtain an expression for A_y/A that is free of w derivatives; but this requires an integration along constant w, and is at least as much work as solving Eq. (3.37). Similarly the $\theta\theta$ EFE gives an expression for $\partial_w(A_y/A)$.

Combining Eq. (3.25) and Eq. (3.27) with Eq. (3.8) eliminates A & B, but leads to a second order non-linear PDE for C that depends only on M, f (or W), and their y derivatives. This would also require a double numerical integration over the (w, y) space. We have not been able to cast it in a simpler form, and we do not regard it as a better alternative, so we don't write it out here.

Evidently there is no direct integration off the PNC along the constant y worldlines, though it should be possible to program the numerical integration as a single sweep across the spacetime.

3.3.6 The Algorithm

The procedure for obtaining the observer metric from observational data may be presented as a two-part algorithm, the first for obtaining the undetermined functions from the data, and the second for calculating the model evolution from the functions. The arbitrary functions M, $W = \sqrt{1+f}$ and a are obtained as follows:

- Assume the following observational data for a large number of sources on the PNC:
  — redshift z,
  — apparent luminosity ℓ and absolute luminosity L,
  (and/or angular diameter δ and true diameter D),
  — number density of sources in redshift space n, and mass per source μ.

  From these calculate diameter distance $\hat{C} = d_D(z)$ using Eq. (3.40), and redshift space mass density $\mu n(z)$.

- Make a gauge choice, as in §3.3.1, which fixes $\hat{B}$ and $\hat{t}_y$.

- Integrate down the PNC one of equations Eq. (3.54), (3.55) or (3.56), as appropriate to the gauge choice, which gives $\varphi(z)$, then integrate $\varphi(z)$ as in Eq. (3.57) to produce $y(z) \to z(y)$.

- Integrate Eq. (3.58) with the appropriate choice of Eq. (3.59), (3.60) or (3.61) down the PNC to calculate $M(z)$ and $W(z) \to M(y)$ & $W(y)$.

- Integrate the relevant choice of Eq. (3.62), (3.63) or (3.64) to give the time on the PNC $\hat{t}(z)$; integrate Eq. (3.65) along each constant y worldline, producing $\tau(z)$ the proper time from bang to null cone; then calculate the bang time $a(z)$ from Eq. (3.66) thus giving $\hat{t}(y)$ & $a(y)$.

Having found the 3 arbitrary functions, the evolution of the model is determined as follows:

- Use equation Eq. (3.29) and integrate up and down each matter worldline to evaluate $t(C, r) \to C(t, r)$ everywhere $(r = y)$. Initial conditions are provided on the PNC by $\hat{C}(z(y))$ and $\hat{t}(y)$. In practice, this could be done in the same step as the τ integration above.
- Choose the gauge function $\gamma(w) = t(w, 0)$ to fix w all along the central worldline.
- Knowing $C(t, r)$, calculate C_r everywhere; and hence find $B(t, r)$ everywhere from Eq. (3.36).
- From each w on the central worldline, integrate equation Eq. (3.37) to trace the (t, y) locus of its past null cone, allocating each point the same w, thus obtaining $w(t, y) \to t(w, y)$.
- Calculate $C(w, t) = C(t(w, y), y)$ and $B(w, t) = B(t(w, y), y)$ everywhere, as shown in Eq. (3.38).
- Differentiate $t(w, y)$ to find $A(w, y)$ according to Eq. (3.30).

The above steps are written for clarity rather than numerical efficiency. In coding it, certain steps may be combined. As explained in [35, 36], the neighbourhoods of the origin, the bang, parabolic worldlines, and the maximum in d_D require special numerical treatment.

3.4 Relationship to Other Work

This solution must obviously be a version of the Lemaître-Tolman metric, so it would be useful to see the transformation. We give this in appendix A.

We here compare the present paper with earlier work, particularly noting any differences, and we give the conversion between the different notations that have been used.

The notation for coordinates, (w, y, θ, ϕ); and metric functions A, B, C is common to all the papers, as are those for the redshift z and the primary PNC w_0. In the OC papers, the w and y partial derivatives are written $\dot{C} = \partial_w C = C_w$, $C' = \partial_y C = C_y$, etc.

Concerning [19], we note there is a factor of κ missing on the right of their equations (13), or it has been absorbed into the definitions of μ, μ_0 and p. We will assume the latter. They use a completeness factor F — the fraction of sources that are actually counted. Whereas survey completeness is a significant concern, in our paper we have assumed it has been corrected for. (The effect of systematic errors was investigated in [36].) Combining our Eq. (3.13) and Eq. (3.14) gives

$$\frac{\rho_w}{\rho} = -\left(\frac{2C_w}{C} + \frac{B_w}{B}\right) \qquad \rightarrow \qquad \rho = \frac{\overline{\rho}(y)}{C^2 B} \, , \qquad \overline{\rho} = \frac{2M_y}{\kappa W} \, , \tag{3.70}$$

which is (21a) in [19]. As noted in [23], Eq (30) of [19] does not hold, so their Eqs (34)-(45) are incorrect.

In [23], they effectively obtain orthogonality of the constant t and y surfaces in their (13) by comparing the matter flow lines in the LT and OC metrics. Their variable $N_*(y)$ is never interpreted — it is 8π times the total gravitational mass M within shell y, divided by the mean galaxy mass. Since M contains both the integrated rest mass and the curvature, N_* is not proportional to the number of galaxies (except near the centre). Item 4 in the corrigenda is misleading; to get that solution, put $u = F - v$ into $v^2 + 2uv = 1 - mN_*/C$, multiply through by C, rearrange, differentiate with respect to y, and use the derivative of (27), giving

$$(2CvF)' + mFN' = (Cv^2 + C)' \, , \tag{3.71}$$

which can be evaluated on the PNC and integrated to give F. Note the prime in their paper (and in this paragraph) is $\partial/\partial y$ while w is held constant. This must be remembered when taking the prime derivatives on the right of their (29), where functions are expressed in terms of t, y, and parameter Γ. The prime derivative of Γ is rather a long expression. In fact, we recommend integrating the negative of the left hand side of (29), to get the null cone path $t(y)$, and then determining T rather than T'. Thus the derivative of the parametric solution, which includes calculating Γ', is not needed. There seem to be some oddly placed factors of 4π: if their (17) and the equation below their (10) are correct, then their ρ is 4π times the density. Also their (17) and (27) imply their mN_* is our M, whereas their (25) implies it is our $2M$. We will assume the latter, as their equations involving ρ do not play a significant role. Otherwise that paper (with corrigenda) is basically correct, though a little circuitous. However the propagation of the solution off the observer's PNC in observer coordinates is not discussed.

In [26], μ and μ_0 also contain absorbed factors of κ. The function of integration $l(y)$ in their (47) corresponds to $\beta(y)$ in our Eq. (3.31), but, without the orthogonality condition, they haven't set it to zero. In steps 2 and 3 of their integration procedure, they argue that $A(w, y)$ must have the same functional form as $A(w_0, y)$, though they acknowledge that this does not mean simple replacement of w_0 by w. In their FLRW example, they use the central conditions plus homogeneity to obtain $1 \rightarrow w/w_0$ in the formula for A. Unfortunately, we cannot see how this can be implemented in general, away from the origin, when we are not assuming homogeneity. There are an unlimited number of functions $A(w, y)$ that become the given $A(w_o, y)$ when $w = w_o$.

TABLE 3.1: This table summarises the correspondence between the different notations used in this thesis and certain OC papers

Here	In [19]	In [23]	In [26]
$\hat{C}$	r_0	d_A	r_0
μ	M	$4\pi m$	m
ρ	$\frac{\mu}{\kappa}$	$\frac{\rho}{4\pi}$	$\frac{\mu}{\kappa}$
$\frac{\rho}{\mu}$	n	n	n
$\frac{\kappa\mu n}{\hat{C}^2}$	$M_0 F$		$M_0 J$
n	$\frac{F M_0 r_0^2}{\kappa M}$	$\frac{N'(\mathrm{d}y/\mathrm{d}z)}{4\pi}$	$\frac{J M_0 r_0^2}{\kappa m}$
$\kappa\overline{\rho} = \frac{2M_y}{W}$	μ_0		μ_0
M	$-\omega_0$	$\frac{mN_*}{2}$	$-\omega_0$
$-\frac{M}{C^3}$	ω		ω
$\frac{8\pi M}{\mu}$		N_*	
W	W	F	W
f		$-kf^2$	
a		$-T$	
β			l
γ	$A(w,0)$		$A(w,0)$

In this chapter we have presented the spherically symmetric observer coordinates and metric. Then we showed how one can determine the observer coordinates from observations of redshift, diameter distance, and number count density, which provide initial data on the PNC. We focused on clarifying procedures, especially the evolution of the metric off the PNC.

Chapter 4

Generalisation of the Cosmic Mass to the Lemaître model

As early as 1933, Lemaître [31] considered the general diagonal metric, imposing spherically symmetry and comoving matter[1]. He obtained the conservation equations, and from them obtained an expression for the mass. He reduced the field equations to a system of first order differential equations (DEs), constraining the initial configuration and fixing the evolution. In doing this he imposed a comoving, diagonal matter tensor, but allowed the radial and tangential pressures to be different and the cosmological constant to be non-zero. He then went on to consider a large variety of very interesting cases and applications — see [67]

In 1964, Podurets [68] wrote out the field equations for a spherically symmetric "star" consisting of a comoving perfect fluid (with isotropic pressure) and zero cosmological constant. He solved them down to a slightly simpler set of evolution and constraint DEs, gave a brief thermodynamic interpretation of the equation relating pressure to the time derivative of mass, and presented only a sketch of how a numerical procedure would work. In order to gain a basic interpretation of the equations, he showed how the Euler, continuity and Poisson equations are obtained in the Newtonian limit.

That same year, Misner and Sharp [49] considered the same kind of star, pointing out it cannot include heat conduction or a flux of radiation. They discussed the thermodynamics in some detail, and from these, they obtained the the same evolution and constraint DEs — the relativistic Euler and continuity equations. They defined a particle number and showed it is conserved. They imposed an origin condition at zero radius, and a comoving boundary condition where the metric joins to a vacuum exterior.

[1] Lemaître's metric form is actually the most general spherically symmetric spacetime.

Cahill and McVittie's [69] derivation of the spacetime is essentially the same as the above two, except they don't assume a diagonal matter tensor. They provide several justifications for the choice of mass function, which we discuss below. They go on to consider the perfect fluid, 'negative mass shells'[2], and some specific metrics.

The notion of the mass has applications not only to astrophysical models of stars, but it is also important in cosmology . As it was pointed out [37], the apparent horizon — the locus where the diameter distance is maximum — has important observational consequences, and enables us to determine the cosmic mass at that radius. This is due to a simple relationship between the maximum in the diameter distance, the cosmological constant, and the mass within a sphere of that size, that is unique to that distance. This relationship allows us to check for systematic errors in the observational data, and partially correct them [36].

Given the significance of the apparent horizon result, it important to check how general the relationship is. After presenting the Lemaître metric as a calculational tool, this chapter extends the apparent horizon-cosmic mass result to the case of non-zero pressure. It also considers how the definition of mass is affected by the introduction of non zero pressure and cosmological constant.

4.1 The Lemaître Metric

We consider an inhomogeneous, spherically symmetric spacetime, filled with a perfect fluid, in which the coordinates $x^\mu = (t,\, r,\, \theta,\, \phi)$ are comoving with the matter flow. The metric may be written as

$$ds^2 = -e^{2\sigma}\, dt^2 + e^{\lambda}\, dr^2 + R^2\, (d\theta^2 + \sin^2\theta\, d\phi^2)\ , \tag{4.1}$$

where $\sigma = \sigma(t, r)$, $\lambda = \lambda(t, r)$ are functions to be determined, and $R = R(r, t)$ corresponds to the areal radius. The energy momentum tensor is given by:

$$T^{\mu\nu} = (\rho + p)\, u^\mu\, u^\nu + g^{\mu\nu} p\ , \tag{4.2}$$

where $\rho = \rho(t, r)$ is the mass-energy density of the perfect fluid, $p = p(t, r)$ is the matter pressure, and $u^\mu = (e^{-\sigma}, 0, 0, 0)$ is the fluid four-velocity. This is less general than the full Lemaître metric as we assume isotropic pressure.

[2]Since they assume ρ & p are positive, these shells would appear to be the region beyond a maximum of R in the spatial sections of constant t, where both R and M are decreasing towards a second origin.

4.1.1 The Field Equations

The EFEs, $G^{\mu\nu} = \kappa T^{\mu\nu} - g^{\mu\nu}\Lambda$ can be reduced to the following set of equations:

$$e^{2\sigma} G^{tt} = -\left(\frac{2R''}{R} + \frac{R'^2}{R^2} - \frac{R'}{R}\lambda'\right) e^{-\lambda} + \left(\frac{\dot{R}^2}{R^2} + \frac{\dot{R}}{R}\dot{\lambda}\right) e^{-2\sigma} + \frac{1}{R^2} = \kappa\rho + \Lambda \ , \tag{4.3}$$

$$e^{\lambda} G^{tr} = \left(\frac{2\dot{R}'}{R} - \frac{2\dot{R}}{R}\sigma' - \frac{R'}{R}\dot{\lambda}\right) e^{-2\sigma} = 0 \ , \tag{4.4}$$

$$e^{\lambda} G^{rr} = \left(\frac{R'^2}{R^2} + \frac{2R'}{R}\sigma'\right) e^{-\lambda} - \left(\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{2\dot{R}}{R}\dot{\sigma}\right) e^{-2\sigma} - \frac{1}{R^2} = \kappa p - \Lambda \ , \tag{4.5}$$

$$\begin{aligned} R^2 G^{\theta\theta} = &\left(\frac{R''}{R} + \frac{R'}{R}\sigma' + \sigma'' + \sigma'^2 - \frac{R'}{2R}\lambda' - \frac{1}{2}\sigma'\lambda'\right) e^{-\lambda} \\ &+ \left(\frac{\dot{R}}{R}\dot{\sigma} - \frac{\ddot{R}}{R} - \frac{1}{2}\ddot{\lambda} + \frac{1}{2}\dot{\lambda}\dot{\sigma} - \frac{\dot{R}}{2R}\dot{\lambda} - \frac{1}{4}\dot{\lambda}^2\right) e^{-2\sigma} = \kappa p - \Lambda \ , \end{aligned} \tag{4.6}$$

where the dot means a derivative with respect to t, and prime means a derivative with respect to r. We use geometric units, $G = 1 = c$, so that $\kappa = 8\pi$. The conservation equations $\nabla_\mu T^{\mu\nu} = 0$ are:

$$\frac{2e^{2\sigma}}{(\rho+p)} \nabla_\mu T^{t\mu} = \dot{\lambda} + \frac{2\dot{\rho}}{(\rho+p)} + \frac{4\dot{R}}{R} = 0 \tag{4.7}$$

$$\frac{e^{\lambda}}{(\rho+p)} \nabla_\mu T^{r\mu} = \sigma' + \frac{p'}{p+\rho} = 0 \ . \tag{4.8}$$

To solve these equations, we multiply Eq (4.3) by R^2R' and use Eq (4.4) to eliminate the term that contains $\dot{\lambda}$, which produces

$$\frac{\partial}{\partial r}\left[R + R\dot{R}^2 e^{-2\sigma} - RR'^2 e^{-\lambda} - \frac{1}{3}\Lambda R^3\right] = \kappa\rho R^2 R' \ . \tag{4.9}$$

Multiplying Eq (4.5) by $R^2\dot{R}$ and using Eq (4.4) to eliminate the term that contains σ', shows Eq (4.5) can be rewritten as

$$\frac{\partial}{\partial t}\left[R + R\dot{R}^2 e^{-2\sigma} - RR'^2 e^{-\lambda} - \frac{1}{3}\Lambda R^3\right] = -\kappa p R^2 \dot{R} \ . \tag{4.10}$$

The term in square brackets is related to the total mass-energy of the system, M, interior to a comoving shell of constant r, and is normally defined by

$$\frac{2M}{R} = \dot{R}^2 e^{-2\sigma} - R'^2 e^{-\lambda} + 1 - \frac{1}{3}\Lambda R^2 \ . \tag{4.11}$$

The justifications for this will be discussed in section (4.2). With this definition, Eqs (4.9) and (4.10) can by rewritten as:

$$\kappa\rho = \frac{2M'}{R^2 R'} \; , \tag{4.12}$$

$$\kappa p = -\frac{2\dot{M}}{R^2 \dot{R}} \; . \tag{4.13}$$

Eq (4.11) may be rearranged as an evolution equation for the model

$$\dot{R} = \pm e^{\sigma} \sqrt{\frac{2M}{R} + f + \frac{\Lambda R^2}{3}} \; , \tag{4.14}$$

where

$$f(t, r) = R'^2 e^{-\lambda} - 1 \; , \tag{4.15}$$

acts as the curvature term, or twice the total energy of the particles at r (analogous to f in the LT model). However, the solution for $R(t, r)$ cannot be directly obtained from this, because of the unknown functions λ, σ and M.

The metric variables g_{tt} and g_{rr} can be obtained be integrating Eqs (4.8) and (4.7) as follows:

$$\sigma = \sigma_0(t) - \overset{\text{const } t}{\int_{r_0}^{r}} \frac{p' \,\mathrm{d}r}{(\rho + p)} = \sigma_0 - \overset{\text{const } t}{\int_{\rho_0}^{\rho}} \frac{(\partial p / \partial \rho)}{(\rho + p(\rho))} \,\mathrm{d}\rho \; , \tag{4.16}$$

and

$$\lambda = \lambda_0(r) - 2 \overset{\text{const } r}{\int_{\rho_0}^{\rho}} \frac{\mathrm{d}\rho}{(\rho + p(\rho))} - 4 \ln\left(\frac{R}{R_0}\right) \; , \tag{4.17}$$

where $\sigma_0(t)$ and $\lambda_0(r)$ are arbitrary functions of integration. Typically, we would choose $r_0 = 0$ to be the origin, where $R(t, 0) = 0$. Having solved (4.3)-(4.5) and (4.7)-(4.8), the

θ-θ field equation (4.6) is now satisfied, since

$$\begin{aligned} R^2\, G^{\theta\theta} = & \left(R \left(\frac{\sigma'}{4R'} - \frac{\dot\lambda}{8\dot R} \right) - \frac{1}{2} \right) e^{2\sigma}\, G^{tt} - \left(\frac{R e^{2\sigma}}{4\dot R} \right) \partial_t G^{tt} \\ & + \left(R \left(\frac{\sigma'}{4R'} - \frac{\dot\lambda}{8\dot R} \right) + \frac{1}{2} \right) e^{\lambda}\, G^{rr} + \left(\frac{R e^{\lambda}}{4R'} \right) \partial_r G^{rr} \\ & + \left(\frac{e^{2\sigma}}{4\dot R} \left(\frac{R\lambda'}{2} - R\sigma' - R' \right) + \frac{e^{\lambda}}{4R'} \left(\frac{R\dot\lambda}{2} - R\dot\sigma + \dot R \right) \right) G^{tr} \\ & + \left(\frac{R e^{\lambda}}{4R'} \right) \partial_t G^{tr} - \left(\frac{R e^{2\sigma}}{4\dot R} \right) \partial_r G^{tr} \;, \end{aligned} \tag{4.18}$$

and to get the right hand side of (4.6) we must also add

$$0 = \frac{\kappa R e^{2\sigma}}{4\dot R}\, \nabla_\nu T^{t\nu} - \frac{\kappa R e^{\lambda}}{4R'}\, \nabla_\nu T^{r\nu} \;. \tag{4.19}$$

4.1.2 Constructing the Lemaître model

To actually generate the Lemaître in the general case requires a numerical integration of the DEs. In order to do that, we first need to specify the free functions, and then integrate the DEs along constant t or r paths. Below is a brief summary of the procedure.

- Choose an initial time slice t_0, and make a gauge choice $R(t_0, r) = R_0(r)$, for example $R_0 = r$. This means R_0' is also known.
- Specify $\rho_0(r) = \rho(t_0, r)$ on that initial time slice.
- Select a specific equation of state, $p = p(\rho)$, thus giving $p_0 = p(t_0, r)$, on the initial surface.
- Integrate Eq (4.12) along constant t

  $$M_0 = \int_{r_0}^{r} {}^{\text{const } t} \; \frac{\kappa \rho_0 R_0^2 R_0'}{2}\, \mathrm{d}r \tag{4.20}$$

  to calculate $M_0 = M(t_0, r)$ everywhere on the initial time slice. Alternatively, one may specify $M_0(r)$ and determine $\rho_0(r)$ from it.
- Choose $\lambda(t_0, r)$, thereby fixing the free function $\lambda_0(r)$. Note that by (4.15) this is equivalent to choosing the geometry $f_0(r)$ on the initial surface, so $\lambda_0(R) = 0$, $e_0^{\lambda} = 1$ is not necessarily a good choice. One may instead prefer to choose $f_0(r)$ and then calculate $\lambda_0(r)$ from $e^{\lambda_0} = R_0'^2/(1 + f_0)$.

- Choose $\sigma(t, r_0)$ along the central worldline $r_0 = 0$, which we can take to be identically $\sigma_0(t)$. This relates the time coordinate to the central observer's proper time.

- Integrate (4.16) along $t = t_0$ to get $\sigma(t_0, r)$.

We have now determined all the functions on the initial time surface, which means we can obtain their r derivatives too. We next wish to evolve the metric forwards in time, along the worldlines of constant r. This can be done as follows:

- Equation (4.14) gives $\dot{R}$ everywhere on our initial time slice.

- The $\dot{M}$ DE can be obtained from Eq (4.13) as

$$\dot{M} = \frac{-\kappa p \, \dot{R} R^2}{2} \; . \tag{4.21}$$

- Eliminating σ' between Eqs (4.8) and (4.4), and then substituting for $\dot{\lambda}$ from (4.7), produces the following DE for $\dot{\rho}$

$$\dot{\rho} = -p' \, \frac{\dot{R}}{R'} - (\rho + p) \left[\frac{\dot{R}'}{R'} + \frac{2\dot{R}}{R} \right] \; . \tag{4.22}$$

- Having chosen the equation of state[3] $p = p(\rho)$, an equation for $\dot{p}$ follows,

$$\dot{p} = \frac{dp}{d\rho} \, \dot{\rho} \; . \tag{4.23}$$

- Eq (4.7) provides a DE for $\dot{\lambda}$, which may be used as is or combined with (4.22) to give

$$\dot{\lambda} = \frac{2}{R'} \left(\frac{p' \dot{R}}{\rho + p} + \dot{R}' \right) \; . \tag{4.24}$$

- Having these 5 coupled DEs, and the initial values on $t = t_0$, we can create a numerical procedure to integrate the DEs in parallel, thus giving $R(t, r)$, $M(t, r)$, $\rho(t, r)$, $p(t, r)$ and $\lambda(t, r)$ everywhere. Eq (4.12) can be used as a cross-check on the results. Note that the spatial derivatives $\dot{R}'$, p' and M' will be needed at each step.

- Finally, $\sigma(t, r)$ is obtained from Eq (4.16) by integrating along each slice of constant t.

[3] More general equations of state would require a slightly different procedure.

Notice that we have chosen 4 functions, $R_0(r)$, $\rho_0(r)$, $\lambda_0(r)$ and $\sigma_0(t)$, as well as the equation of state $p(\rho)$.

4.1.3 Special cases

The Lemaître metric contains the Lemaître-Tolman and FLRW metrics as special cases.

The spherically symmetric inhomogeneous dust cosmology of Lemaître-Tolman is obtained when the pressure is zero. Setting $p = 0$ in (4.8) gives $\sigma = \sigma_0(t)$, and we are free to set $\sigma = 1$ so that t becomes the comoving proper time, i.e. the natural choice of time coordinate. Eq (4.13) gives

$$\dot{M} = 0 \quad \rightarrow \quad M = M(r) = M_0(r) \; . \tag{4.25}$$

Similarly, (4.7) and (4.17) simplify to

$$\lambda = \lambda_0(r) - 2\ln\left(\frac{\rho}{\rho_0}\right) - 4\ln\left(\frac{R}{R_0}\right) \tag{4.26}$$

$$\rightarrow \quad e^{\lambda/2} = \frac{e^{\lambda_0/2}\rho_0 R_0^2}{\rho R^2} = \frac{e^{\lambda_0/2} R' M_0'}{M' R_0'} = \frac{R'}{\sqrt{1+f_0}} \tag{4.27}$$

where we used (4.12) and then (4.15), so clearly $f = f_0(r) = f(r)$ too. Therefore the evolution equation reduces to the familiar

$$\dot{R}^2 = \frac{2M}{R} + f + \frac{\Lambda R^2}{3} \; . \tag{4.28}$$

On the other hand, the standard cosmological model, the homogeneous RW metric, is obtained by setting to zero the spatial variation of any physical invariants. Again $p' = 0$ allows us to choose $\sigma = 1$, and requiring the constant t 3-spaces to have a constant curvature form requires $g_{ij} = S^2(t)\tilde{g}_{ij}(r)$, $i, j = 1, 2, 3$, with the canonical r coordinate choice giving

$$ds^2 = -dt^2 + S(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \; . \tag{4.29}$$

4.2 On the definition of mass

In this section we will consider the justification for naming M the gravitational mass felt at comoving radius r. First we review the arguments given by some earlier authors, especially Cahill and McVittie, then we consider a geodesic deviation approach.

Lemaître [31] does not really justify calling M the mass, other than an implied comparison with Newtonian equations. Podurets [68] merely says "it is not difficult to see that" (4.11) gives "the total mass of matter contained in the interval $(0, r)$, the mass being defined in terms of the gravitational field created on the boundary", and then calls it a definition of mass. Misner and Sharp [49] write (4.12) in the form (with $\Lambda = 0$)

$$M = \int_{V(r)} \rho \left(1 + \dot{R}^2 e^{-2\sigma} - \frac{2M}{R}\right)^{1/2} \mathrm{d}^3 V \tag{4.30}$$

pointing out that the gravitational mass M contains contributions from the kinetic energy and the gravitational potential energy, as well as the matter density ρ. (We note that the contributions $U^2 = \dot{R}^2 e^{-2\sigma}$ and $2M/R$ are twice the two energy values. We also note that the term M appears in both sides, which cause a difficulty in interpreting this equation)

On the other hand Cahill & McVittie [69], assuming zero Λ, define the mass in terms of a Riemann component

$$M_{CM}(t, r) = \frac{R}{2} R^{\phi}{}_{\theta\phi\theta} = \frac{R}{2}\left(1 + \frac{\dot{R}^2}{e^{2\sigma}} - \frac{R'^2}{e^{\lambda}}\right) . \tag{4.31}$$

which involves only first metric derivatives, and is independent of any rescaling of the t & r coordinates. Their justifications are: (i) if joined to a Schwarzschild exterior, M must equal the exterior mass; (ii) the Bianchi identities lead to generalisations of (4.12) & (4.13)

$$2M' = \kappa R^2 (T^t_t\, R' - T^t_r\, \dot{R}) , \qquad 2\dot{M} = \kappa R^2 (T^r_r\, \dot{R} - T^r_t\, R') \tag{4.32}$$

but the roles of the various terms in the interpretation are not discussed; (iii) the proper acceleration of R on a comoving worldline is

$$u^{\mu} \nabla_{\mu} (u^{\nu} \nabla_{\nu} R) = \frac{\kappa}{2} R T^r_r - \frac{M}{R^2} + R' e^{-\lambda} \lambda' \tag{4.33}$$

which contains a Newtonian-like gravitational force term $-M/R^2$, as well as the pressure acting on the comoving shell, and a term that's hard to interpret; (iv) they show that the mass flow vector

$$J^{\alpha} = \frac{\sin\theta}{4\pi\sqrt{|g_{\mu\nu}|}} (M', -\dot{M}, 0, 0) , \qquad J^{\alpha}{}_{;\alpha} = 0 , \tag{4.34}$$

is conserved.[4] (For the case of the Reissner-Nordstrom metric, with mass and charge parameters m and q, they find that

$$M = m - \frac{2\pi q^2}{R} = m - 2\pi \int_R^\infty E^2 R^2 \, \mathrm{d}R \; , \tag{4.35}$$

where E is the electric field. The two masses agree at $R = \infty$, but the effective gravitational mass M decreases as the charge is approached, because the energy density in the E field that is outside radius R is not included in M. As is well known, this energy density diverges close to the charge, so M can go negative.)

Aspects of the above arguments are clear and easy to follow, and for the $\Lambda = 0 = p$ case they are convincing. It is not quite so easy to be certain we've properly understood the various relativistic corrections when $p \neq 0$. A strong argument is that (4.12) is the same as in the LT model, and it applies at all times. In other words, on every constant time 3-space, the gravitational mass is the integral of the density with a curvature (or gravity) correction, d^3V being the proper 3-volume element:

$$M = \int_{V(r)} \rho \sqrt{1+f} \, \mathrm{d}^3 V \; . \tag{4.36}$$

A similar treatment of (4.13) might seem to involve more complicated contributions from the gravity (curvature). If we consider all the matter within the comoving sphere of constant r, then the work done by the pressure on the matter outside the boundary as it expands is $\mathrm{d}W = p \, A \, \mathrm{d}L$, where A is the area of the boundary and $\mathrm{d}L$ its displacement. Now if, following traditional Newtonian thinking, $\mathrm{d}L$ is taken to be the physical distance the boundary has moved, then this is

$$\mathrm{d}W = p \, A \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^r e^{\lambda/2} \, \mathrm{d}r \right) \mathrm{d}t \tag{4.37}$$

$$= \left(-\frac{2\dot{M}}{\kappa R^2 \dot{R}} \right) (4\pi R^2) \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^r \frac{R'}{\sqrt{1+f}} \, \mathrm{d}r \right) \mathrm{d}t \tag{4.38}$$

$$= -\frac{\dot{M}}{\dot{R}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^r \frac{R'}{\sqrt{1+f}} \, \mathrm{d}r \right) \mathrm{d}t \; . \tag{4.39}$$

which differs from the loss of mass-energy in the interior. If instead we consider how much the boundary has expanded[5], it gives the familiar Newtonian result:[6] $\mathrm{d}L = \dot{R} \, \mathrm{d}t$

[4] Their (3.5) and (4.3) also seem to use a comma for a covariant derivative.

[5] This is not obviously the correct thing to do, since it has more to do with increase in surface area than radial distance moved.

[6] This thinking does not extend to (4.36), which is unavoidably an integral over a volume, whereas (4.40) is really about the boundary only.

leads to

$$\mathrm{d}W = -\dot{M}\,\mathrm{d}t \; . \tag{4.40}$$

Another problem arises once the cosmological constant is introduced, because Cahill & McVittie's definition (4.31) then involves more than just the mass,

$$\frac{R}{2}\,R^{\phi}{}_{\theta\phi\theta} = M + \frac{\Lambda R^3}{6} \; . \tag{4.41}$$

Consequently, in the following we investigate whether new invariant expressions can provide definitions of M, Λ and other quantities independently.

4.2.1 Geodesic deviation Equation

Another method that can be used to try and justify M as the total gravitational mass-energy is the geodesic deviation equation (GDE). The GDE measures the relative acceleration between two nearby geodesics, which is itself a measure of the tidal effects of gravity and therefore is related to the matter present. For a congruence of geodesics,

$$\frac{\delta^2 \xi^\mu}{\delta \tau^2} = -R^{\mu}{}_{\nu\rho\lambda}\,U^\nu\,\xi^\rho\,U^\lambda \; , \tag{4.42}$$

where $R^{\mu}{}_{\nu\rho\lambda}$ is the Riemann tensor, U^ν is the geodesic tangent vector $\mathrm{d}x^\nu/\mathrm{d}\tau$, τ is proper distance or time along the geodesics, and ξ^λ is the geodesic deviation vector. Contracting this equation with ξ_λ produces the scalar

$$\mathcal{A}(\xi^\mu, U^\nu) = \xi_\mu\,\frac{\delta^2 \xi^\mu}{\delta \tau^2} = -R_{\mu\nu\rho\lambda}\,\xi^\mu\,U^\nu\,\xi^\rho\,U^\lambda \; , \tag{4.43}$$

If ξ^λ has unit magnitude, then $\mathcal{A}$ is the size of the relative acceleration of the chosen geodesics, and hence it is a measure of the strength of the tidal effects. Now the Riemann tensor determines the tidal effects due to local as well as distant matter. If one wishes to remove the effect of local matter, then one may consider the quantities

$$-\,C^{\mu}{}_{\nu\rho\lambda}\,U^\nu\,\xi^\rho\,U^\lambda \; , \tag{4.44}$$

and

$$\mathcal{B}(\xi^\mu, U^\nu) = -C_{\mu\nu\rho\lambda}\,\xi^\mu\,U^\nu\,\xi^\rho\,U^\lambda \; , \tag{4.45}$$

where $C^{\mu}{}_{\nu\rho\lambda}$ is the Weyl tensor. Since we only wish to "feel" the tides in a small neighbourhood, and we don't need to propagate the geodesics, we are free to choose U^{ν} and ξ^{ν} at will.[7]

In order to make these measures meaningful, we construct a set of canonical vectors using definitions that are invariant for any spherical metric. In spherical symmetry, the "areal radius" or "curvature coordinate" is the metric component that multiplies the unit 2-sphere, $\mathrm{d}\Omega^2$. In (4.1) it is R. Then the unit timelike vector that follows constant R is,

$$\begin{aligned} u^{\mu}u_{\mu} = -1 \;, \qquad & u^{\mu}\nabla_{\mu}R = 0 \qquad \rightarrow \\ u^{\mu} = & \frac{e^{-\sigma-\lambda/2}}{\sqrt{e^{-\lambda}R'^2 - e^{-2\sigma}\dot{R}^2}}\left(R', -\dot{R}, 0, 0\right) \;. \end{aligned} \tag{4.46}$$

Using this, we define a canonical unit spacelike vector v^{μ} in the radial direction that is orthogonal to u^{μ} and satisfies

$$\begin{aligned} v^{\mu}v_{\mu} = 1 \;, \qquad & v^{\mu}u_{\mu} = 0 = v^{\theta} = v^{\phi} \qquad \rightarrow \\ v^{\mu} = & \frac{1}{\sqrt{e^{-\lambda}R'^2 - e^{-2\sigma}\dot{R}^2}}\left(-\dot{R}e^{-2\sigma}, R'e^{-\lambda}, 0, 0\right) \;. \end{aligned} \tag{4.47}$$

It is evident that the spacelike or timelike character of u^{ν} & v^{ν} will flip if $(e^{-\lambda}R'^2 - e^{2\sigma}\dot{R}^2)$ changes sign. We then define unit vectors in the θ and ϕ directions that are also orthogonal to u^{ν} and v^{ν}, and tangent to the constant R spheres,

$$\begin{aligned} w^{\mu}w_{\mu} = 1 \;, \qquad & u^{\mu}w_{\mu} = v^{\mu}w_{\mu} = 0 \;, \\ w^{\mu}\nabla_{\mu}R = 0 \qquad \rightarrow \qquad & w^{\mu} = \left(0, 0, \frac{1}{R}, 0\right) \;, \end{aligned} \tag{4.48}$$

and

$$\begin{aligned} z^{\mu}z_{\mu} = 1 \;, \qquad & u^{\mu}z_{\mu} = v^{\mu}z_{\mu} = 0 \;, \\ z^{\mu}\nabla_{\mu}R = 0 \qquad \rightarrow \qquad & z^{\mu} = \left(0, 0, 0, \frac{1}{R\sin\theta}\right) \;. \end{aligned} \tag{4.49}$$

Lastly, the radial incoming null vector k^{μ} is given by

$$k^{\mu}k_{\mu} = 0 = k^{\theta} = k^{\phi} \qquad \rightarrow \qquad k^{\mu} = k\left(e^{\lambda/2}, -e^{\sigma}, 0, 0\right) \;, \tag{4.50}$$

[7] More general measures of the curvature may be defined via the parallel transport equation, i.e. using the quantities $W_{\mu}\, R^{\mu}{}_{\nu\lambda\rho}\, X^{\nu}\, Y^{\lambda}\, Z^{\rho}$, but for the spacetimes under consideration, all the Riemann components with more than 2 different indices are zero.

where k is undetermined by the above conditions. In addition, one could use the unit eigenvectors of the Einstein tensor $(G_{\mu\nu} - \ell g_{\mu\nu})\, V^\mu = 0$ which are $e^{-\sigma}\delta^\mu_t$, $e^{-\lambda/2}\delta^\mu_r$, $R^{-1}\delta^\mu_\theta$, $(R\sin\theta)^{-1}\delta^\mu_\phi$, but it turns out these do not extend the range of independent scalars we can define.

The scalars $\mathcal{A}$ & $\mathcal{B}$, when combined with any canonical vectors, are determined solely by the metric, and therefore represent a property of the spacetime. Below we use the notation

$$AH = e^{-\lambda}R'^2 - e^{2\sigma}\dot{R}^2 = 1 - \frac{2M}{R} - \frac{\Lambda R^2}{3} \; , \tag{4.51}$$

$$AHm = e^{-\lambda/2}R' - e^{\sigma}\dot{R} \; , \qquad AHp = e^{-\lambda/2}R' + e^{\sigma}\dot{R} \; . \tag{4.52}$$

Evaluating the scalars $\mathcal{A}$ and $\mathcal{B}$ using the various combinations of the above vectors gives:

$$\mathcal{A}(u^\mu, v^\nu) = \frac{2M}{R^3} + \frac{\Lambda}{3} - \frac{\kappa}{2}(\rho + p) \tag{4.53}$$

$$\mathcal{B}(u^\mu, v^\nu) = \frac{2M}{R^3} - \frac{\kappa\rho}{3} \tag{4.54}$$

$$\mathcal{A}(u^\mu, w^\nu) = \mathcal{A}(u^\mu, z^\nu) = \frac{\kappa\rho}{2} - \frac{\kappa(\rho + p)R'^2 e^{-\lambda}}{2(AH)} - \frac{M}{R^3} + \frac{\Lambda}{3} \tag{4.55}$$

$$\mathcal{B}(u^\mu, w^\nu) = \mathcal{B}(u^\mu, z^\nu) = \frac{\kappa\rho}{6} - \frac{M}{R^3} \tag{4.56}$$

$$\mathcal{A}(u^\mu, k^\nu) = \mathcal{A}(v^\mu, k^\nu) = \frac{k^2 e^{\lambda+2\sigma}(AHm)}{(AHp)}\left[\frac{2M}{R^3} + \frac{\Lambda}{3} - \frac{\kappa}{2}(\rho + p)\right] \tag{4.57}$$

$$\mathcal{B}(u^\mu, k^\nu) = \mathcal{B}(v^\mu, k^\nu) = \frac{k^2 e^{\lambda+2\sigma}(AHm)}{(AHp)}\left[\frac{2M}{R^3} - \frac{\kappa\rho}{3}\right] \tag{4.58}$$

$$\mathcal{A}(v^\mu, w^\nu) = \mathcal{A}(v^\mu, z^\nu) = \frac{\kappa p}{2} - \frac{\kappa(\rho + p)R'^2 e^{-\lambda}}{2(AH)} + \frac{M}{R^3} - \frac{\Lambda}{3} \tag{4.59}$$

$$\mathcal{B}(v^\mu, w^\nu) = \mathcal{B}(v^\mu, z^\nu) = \frac{M}{R^3} - \frac{\kappa\rho}{6} \tag{4.60}$$

$$\mathcal{A}(w^\mu, z^\nu) = -\frac{2M}{R^3} - \frac{\Lambda}{3} \tag{4.61}$$

$$\mathcal{B}(w^\mu, z^\nu) = \frac{\kappa\rho}{3} - \frac{2M}{R^3} \tag{4.62}$$

$$\mathcal{A}(w^\mu, k^\nu) = \mathcal{A}(z^\mu, k^\nu) = -k^2 e^{\lambda+2\sigma}\frac{\kappa}{2}(\rho + p) \tag{4.63}$$

$$\mathcal{B}(w^\mu, k^\nu) = \mathcal{B}(z^\mu, k^\nu) = 0 \tag{4.64}$$

We notice that all the $\mathcal{B}$s are multiples of $\kappa\rho/3 - 2M/R^3$, while the $\mathcal{A}$s are more diverse. Combining the right hand sides of the above equations, we can define the following

$$\mathcal{A}(v^\mu, w^\nu) - \mathcal{A}(u^\mu, w^\nu) = \frac{2M}{R^3} - \frac{2\Lambda}{3} - \frac{\kappa(\rho - p)}{2} \tag{4.65}$$

$$-\mathcal{A}(w^\mu, z^\nu) = \frac{2M}{R^3} + \frac{\Lambda}{3} \tag{4.66}$$

$$\mathcal{A}(v^\mu, w^\nu) - \mathcal{A}(u^\mu, w^\nu) - \mathcal{A}(u^\mu, v^\nu) = \kappa p - \Lambda \tag{4.67}$$

$$-2\Big\{\mathcal{A}(u^\mu, v^\nu) + \mathcal{A}(w^\mu, z^\nu)\Big\} = \kappa(\rho + p) \tag{4.68}$$

$$\frac{1}{3}\Big\{\mathcal{A}(v^\mu, w^\nu) - \mathcal{A}(u^\mu, w^\nu) - \mathcal{A}(u^\mu, v^\nu)\Big\} - \mathcal{A}(w^\mu, z^\nu) = \frac{2M}{R^3} + \frac{\kappa p}{3} \tag{4.69}$$

$$-\mathcal{A}(u^\mu, v^\nu) - 2\mathcal{A}(w^\mu, z^\nu) - \mathcal{A}(v^\mu, w^\nu) + \mathcal{A}(u^\mu, w^\nu) = \kappa\rho + \Lambda \tag{4.70}$$

$$\frac{1}{3}\Big\{\mathcal{A}(v^\mu, w^\nu) - \mathcal{A}(u^\mu, w^\nu) + \mathcal{A}(u^\mu, v^\nu) - \mathcal{A}(w^\mu, z^\nu)\Big\} = \mathcal{B}(w^\mu, z^\nu) = \frac{2M}{R^3} - \frac{\kappa\rho}{3}\,. \tag{4.71}$$

However, there are no combinations that give the value of just M/R^3, or $\kappa\rho$, or κp or Λ. In the case of zero Λ, though, Eqs (4.66), (4.67), (4.70) do give M/R^3, κp and $\kappa\rho$. Similarly, in the case of zero pressure, Eqs (4.67), (4.68), (4.69) do give Λ, $\kappa\rho$ and M/R^3.

It appears from the results above, then, that we cannot separate the mass, the cosmological constant, the density and the pressure from each other, and so we cannot create a unique definition of mass based on geometric invariants of the metric in the general case. In contrast, if $p = 0$ or $\Lambda = 0$ (the LT and Misner-Sharp-Podurets cases), then the remainig quantities are uniquely defined from these scalars. This has important implications. The results of Cahill and McVittie do not generalise to the case of non-zero Λ, and the justification for calling the M of (4.11) the gravitational mass is weaker than was thought. It is based solely on comparing (4.12) and (4.13) with a few special cases.

4.3 The Past Null Cone and the Apparent Horizon Relation

We consider the past null cone (PNC) of the observation event $t = t_o$, $r = r_o = 0$. For any quantity $Q(t, r)$, its value on our PNC will be indicated with a hat: $\hat{Q} = \hat{Q}(r) = Q(\hat{t}(r), r)$, or for expressions, a square bracket with subscript "$\wedge$" will be used. For the metric (4.1), the path of an incoming radial light ray is given by

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}r} = -\frac{e^{\lambda/2}}{e^{\sigma}}\,, \tag{4.72}$$

and the solution that reaches (t_o, r_o) is $t = \hat{t}(r)$.

The redshift of comoving sources on that null cone is given by the ratio of the light oscillation periods T measured at the observer, o, and the emitter, e,

$$(1+z) = \frac{T_o}{T_e} \ . \tag{4.73}$$

For the two successive wavefronts passing through the events B and A or D and C respectively on worldlines of constant r (see § (2.2) Fig (2.1)), the change in the light oscillation period T over a distance dr is given by

$$\mathrm{d}T = \left.\frac{\mathrm{d}\hat{t}}{\mathrm{d}r}\right|_C \mathrm{d}r - \left.\frac{\mathrm{d}\hat{t}}{\mathrm{d}r}\right|_A \mathrm{d}r \ , \tag{4.74}$$

The first term on the right of Eq. (4.74) can be written as a Taylor expansion about A, so that

$$\mathrm{d}T = \left\{ \left.\frac{\mathrm{d}\hat{t}}{\mathrm{d}r}\right|_A + \left.\frac{\partial}{\partial t}\left(\frac{\mathrm{d}\hat{t}}{\mathrm{d}r}\right)\right|_A T \right\} \mathrm{d}r - \left.\frac{\mathrm{d}\hat{t}}{\mathrm{d}r}\right|_A \mathrm{d}r \ , \tag{4.75}$$

$$\frac{\mathrm{d}T}{T} = \left.\frac{\partial}{\partial t}\left(\frac{\mathrm{d}\hat{t}}{\mathrm{d}r}\right)\right|_A \mathrm{d}r \ . \tag{4.76}$$

Therefore Eq (4.74) can be integrated down the PNC to give

$$-\ln\left(\frac{T_o}{T_e}\right) = \int_0^{r_e} \frac{\partial}{\partial t}\left(\frac{\mathrm{d}\hat{t}}{\mathrm{d}r}\right) \mathrm{d}r \ , \tag{4.77}$$

so that, for the Lemaître metric, Eqs (4.72) and (4.73), give the redshift as

$$\ln(1+z) = \int_0^{r_e} \frac{\partial}{\partial t}\left(\frac{e^{\lambda/2}}{e^{\sigma}}\right) \mathrm{d}r \ . \tag{4.78}$$

Recall that with any cosmological source there are primary observable quantities such as redshift z, angular diameter δ, apparent luminosity ℓ, and number density in redshift space n, and a ssociated with each of δ, ℓ, and n is a source property, true diameter D, absolute luminosity L, and mass per source μ, which may evolve with time, and therefore vary with redshift. These combine to give the luminosity distance d_L, the diameter distance d_D and the mass density in redshift space, μn.

The expressions for the diameter and the luminosity distances do not change in the Lemaître and Lemaître-Tolman models, i,e

$$d_D = \hat{R} \ , \qquad d_L = (1+z)^2 \hat{R} \ . \tag{4.79}$$

Similarly, the mass of the sources between z and $z + dz$ in solid angle $d\omega$ is unchanged:

$$d\mathcal{M} = \mu\, dN = \mu\, n\, d\omega\, dz \ . \tag{4.80}$$

The 3-d volume d^3V of proper space that encloses these sources at the time of emission t_e, as measured by comoving observers u^μ, is given by $d^3V = e^{\lambda/2} R^2\, d\omega\, dr$, so that the mass in the fluid element is

$$d\mathcal{M} = \rho\, e^{\lambda/2} R^2\, d\omega\, dr \ . \tag{4.81}$$

Therefore the relationship between n and $\hat{\rho}$ is given by

$$\mu\, n = \left[\rho\, e^{\lambda/2}\, R^2\, \frac{\mathrm{d}r}{\mathrm{d}z}\right]_\wedge \ . \tag{4.82}$$

The apparent horizon is the locus where the observer's PNC reaches it's maximum areal radius (diameter distance), which we denote $z = z_{max}$, $\hat{R} = \hat{R}_{max}$, and obviously we have $[\mathrm{d}R/\mathrm{d}r]_{max} = 0$. The total derivative of $\hat{R}$ along the ray is

$$\frac{\mathrm{d}\hat{R}}{\mathrm{d}r} = \left[R' + \dot{R}\, \frac{\mathrm{d}\hat{t}}{\mathrm{d}r}\right]_\wedge \tag{4.83}$$

and using Eqs (4.14) and (4.72) to replace $\dot{R}$ & $\mathrm{d}\hat{t}/\mathrm{d}r$ shows

$$\frac{\mathrm{d}\hat{R}}{\mathrm{d}r} = e^{\lambda/2}\left[R' e^{-\lambda/2} - \sqrt{\frac{2M}{R} + R'^2 e^{-\lambda} - 1 + \frac{\Lambda R^2}{3}}\,\right]_\wedge = 0 \tag{4.84}$$

$$\longrightarrow \qquad 6M_{max} - 3\hat{R}_{max} + \Lambda \hat{R}^3_{max} = 0 \tag{4.85}$$

at $\hat{R}_{max}$. This is exactly the same as the result in [37], showing that the non-zero pressure does not affect the apparent horizon condition. Note too that M and Λ appear in exactly the same inseparable combination as in (4.66), and (4.85) can be written $\hat{R}^2_{max} = -1/\mathcal{A}(w^\mu, z^\nu)|_{max}$.

We solved the EFEs for spherical symmetric for the inhomogeneous Lemaître model with non-zero pressure, and used it to discuss the definition of gravitational mass, and generalise the cosmic mass result.

Chapter 5

Conclusion

In this thesis some aspects of an inhomogeneous approach to cosmology have been discussed and studied. In the standard approach to cosmology, it is often assumed that the Universe is homogeneous. However, since the Universe is full of inhomogeneities like galaxies, galaxy clusters, voids, groupings of galaxies resembling “ filaments” and “walls” there is a need to develop tools that will enable us to take into account inhomogeneities when analysing observations. A contribution to such a framework was the main aim of this thesis. Here a brief summary is presented.

Chapter 2 discussed an application of spherically symmetric inhomogeneous pressure free Lemaitre–Tolman models. It also reviewed definitions of observable quantities and an algorithm that shows how to define a model based on observations.

In Chapter 3 an alternative formalism was presented, one which is based on observer coordinates. This formalism is based on the observer past null cone and it provides a natural framework for the analysis of cosmological observations. The spherically symmetric case was carefully discussed, and two approaches to solving the spherical observer metric were presented.

These were firstly a formal solution of the EFEs in terms of arbitrary functions, and secondly a procedure for determining the arbitrary functions and the metric evolution from observational data. The main aim was to lay out a solution algorithm that could be coded for numerical implementation, which was presented in a detailed and step by step manner. It was emphasised that, if we are given observational data, then they fix the arbitrary functions, and we are only free to make the gauge choices that pin down the coordinate freedoms. On the other hand, if we choose all the arbitrary functions, then the observational relations are already fixed, and there is no room to fit observations.

In the formal sense, the approach presented in Chapter 3 has a lot in common with previous approaches up to (3.31), but thereafter, up to the final solution (3.38), it is new. In the solution from observations, the method presented is similar to others in Sections 3.3.2 and to some extent 3.3.4 where $y(z)$ and $a(z)$ are found, but differs in Sections 3.3.3 and 3.3.5 where $M(z)$ and $W(z)$ are found and the evolution off the PNC is discussed. In particular, the demonstration that the evolution from one constant w null cone to the next necessarily involves an integration down the null cone at each step, is new. The orthogonality condition leading to $\beta = 0$ is important for the solution from scratch, but up to now has only been obtained indirectly by comparing with the LT metric.

The available data on galaxy observations only extends to a finite redshift z. This is well suited to the algorithms presented in §3.3.6, which involve integrations along the null cones outwards from the centre, and integrations along the constant y worldlines. The LT functions are fully determined within the range of reliable data. Even if quite large scale inhomogeneity is discovered, it is still possible the cosmos approaches homogeneity on even larger scales.

In Chapter 4 the Lemaître metric was considered. The Lemaître model describes a spherically symmetric perfect fluid distribution in comoving coordinates, and its reduces to a system of DEs. In §4.1.2 the solution in the form of an explicit algorithm for numerical calculation was presented. This algorithm clearly lists those functions that need to be specified on an initial surface, and on a chosen worldline.

Further in Chapter 4 the concept of the total gravitational mass within a given comoving sphere was reviewed. The main aim was to present a definition of the mass that is independent of the case of both pressure and cosmological constant being non-zero. In seeking for alternative definitions or justifications, a number of invariants of a spherical metric has been considered. The approach was based on the geodesic deviation equation and a set of canonical vector fields. It was found that the physical variables M, ρ, p and Λ always appear in combinations of two or more, and it is not possible to separate them using the methods considered, which were based on the properties of the metric. Since M is not just the integral of the density, but contains a contribution from the curvature, it was shown that there is always some ambiguity in the definition of the effective gravitational mass. In other words, the concept of mass in general spherical symmetry is not as easily defined as was thought.

In §4.3 the past null cone of a central observer was considered, in praticular how the areal radius (diameter distance) varies along it. The apparent horizon relation found in [37], that is the relationship between the maximum in the diameter distance $\hat{R}_{max}$, Λ, and the mass M within that sphere M_{max} — the “cosmic mass” — has been shown to hold also in the Lemaître model. This relationship, previously only shown for the zero pressure case, holds only at the apparent horizon, and is independent of any intervening inhomogeneity. It provides a way of measuring the cosmic mass on gigaparsec scales [37].

The framework studied in this thesis has a great potential for further generalisation. It is known [70] that the Lemaître-Tolman model can fit any given observational functions for diameter distance and number counts versus redshift. Thus, an extension of this programme to the Lemaître model will offer more flexibility, and thus the possibility of fitting other kind of astronomical data. The algorithm presented in Chapter 4 is a first step towards such a goal.

Also the observer coordinates, since they offer a natural choice of analysis of observations may prove to be of great value in the near future, not only for studying the observational data, but also as recently suggested [71] for the study of averaging. Since this thesis presented an observerational approach, and developed it a bit further, it is hoped the findings of this thesis will be helpful of use in this branch of cosmology.

Appendix A

The Transformation of LT to Null-Comoving Coordinates

The LT (Lemaître-Tolman) metric is [31, 32]

$$ds^2 = -dt^2 + \frac{(R')^2}{1+f}\,dr^2 + R^2\,d\Omega^2 \ , \tag{A.1}$$

where $R = R(t, r)$ and $R' = \partial R/\partial r$. It depends on 3 arbitrary functions, $f = f(r)$, $M = M(r)$ and $a = a(r)$. The matter is comoving and has zero pressure. The evolution equation is

$$\dot{R}^2 = \frac{2M}{R} + f + \frac{\Lambda R^2}{3} \ , \tag{A.2}$$

where $\dot{R} = \partial R/\partial r$, and the density is given by

$$\kappa\rho = \frac{2M'}{R^2 R'} \ . \tag{A.3}$$

The arbitrary functions each have a physical meaning; f gives the deviation of the constant t 3-spaces from flatness, and also gives twice the energy per unit mass of the dust particles; M gives the gravitational mass within the comoving shells of constant r, and a gives the local time of the big bang on each constant r worldline.

We propose the transformation

$$\begin{aligned} & t = t(w, y) \ , \qquad r = y \\ \rightarrow \qquad & J = \frac{\partial(t, r)}{\partial(w, y)} = \begin{pmatrix} t_w & t_y \\ r_w & r_y \end{pmatrix} = \begin{pmatrix} t_w & t_y \\ 0 & 1 \end{pmatrix} \ , \end{aligned} \tag{A.4}$$

which retains y as a comoving coordinate, so the metric becomes

$$ds^2 = -(t_w\,dw + t_y\,dy)^2 + \frac{(R')^2}{1+f}\,(r_w\,dw + r_y\,dy)^2 + R^2\,d\Omega^2 \tag{A.5}$$

$$= -t_w^2\,dw^2 - 2\,t_w\,t_y\,dw\,dy + \left(-t_y^2 + \frac{(R')^2}{1+f}\right)dy^2 + R^2\,d\Omega^2\ . \tag{A.6}$$

We want w to be a coming null coordinate, i.e. $dw = 0 = d\theta = d\phi$ must give $ds = 0$, which leads to

$$g_{yy} = 0 \qquad \rightarrow \qquad t_y = \frac{-R'}{\sqrt{1+f}} \tag{A.7}$$

$$\rightarrow \qquad ds^2 = -t_w^2\,dw^2 + 2\,t_w\,\frac{R'}{\sqrt{1+f}}\,dw\,dy + R^2\,d\Omega^2\ , \tag{A.8}$$

where the sign choice is because, on the past null cone (PNC), t must decrease as $r = y$ increases. Eq (A.7) is a PDE for $t(w, y)$, and its solution will introduce a function of w, $\gamma(w)$.

Applying this transformation to the LT evolution equation (A.2)

$$R_w = \dot{R}\,t_w + R'\,r_w = \dot{R}\,t_w\ , \tag{A.9}$$

we find the new evolution equation in the new coordinates,

$$R_w = \pm t_w\sqrt{\frac{2M}{R} + f + \frac{\Lambda R^2}{3}}\ . \tag{A.10}$$

Similarly, by transforming R_y we obtain

$$R_y = \dot{R}\,t_y + R'\,r_y = -\dot{R}\left(\frac{R'}{\sqrt{1+f}}\right) + R' \tag{A.11}$$

$$\rightarrow \qquad R' = R_y\left(\frac{\sqrt{1+f}}{\sqrt{1+f} \mp \sqrt{\frac{2M}{R} + f + \frac{\Lambda R^2}{3}}}\right)\ , \tag{A.12}$$

where (A.10) was used. The metric now becomes

$$ds^2 = -t_w^2\,dw^2 + \frac{2\,R_y\,t_w}{\left(\sqrt{1+f} \mp \sqrt{\frac{2M}{R} + f + \frac{\Lambda R^2}{3}}\right)}\,dw\,dy + R^2\,d\Omega^2\ . \tag{A.13}$$

Up to this point $t(w, y)$ contains an undetermined function of w, obtained when integrating (A.7). We now specify that, at the origin, $y = r = 0$, w is the observer's proper time, $w = t$, i.e.

$$t(w,0) = w \ , \qquad w(t,0) = t \ , \qquad t_w(w,0) = 1 \ ,$$
$$w_t(t,0) = 1 \ . \tag{A.14}$$

and this will fix the function of integration introduced in solving (A.7).

Comparing (3.1) and (A.13), it is clear that

$$A = t_w \ ,$$
$$B = -t_y = \frac{R_y}{\left(\sqrt{1+f} \ \mp \sqrt{\frac{2M}{R} + f + \frac{\Lambda R^2}{3}} \ \right)} \ . \tag{A.15}$$

in agreement with (3.30) and (3.27).

The matter tensor transforms to

$$\tilde{T}^{ab} = T^{cd} \, (J^{-1})^a_c \, (J^{-1})^b_d = \begin{pmatrix} \rho/t_w^2 & 0 \\ 0 & 0 \end{pmatrix} \ ,$$
$$\tilde{T}_{ab} = T_{cd} \, J^c_a \, J^d_b = \begin{pmatrix} \rho \, t_w^2 & \rho \, t_w \, t_y \\ \rho \, t_w \, t_y & \rho \, t_y^2 \end{pmatrix} \ . \tag{A.16}$$

Appendix B

The Lemaître Model with a Barotropic Equation of State

For the case of the commonly used equation of state, $p = w\rho$, with w the equation of the state parameter, the Lemaître model has a more explicit solution. Eq (4.16) and (4.17) can be integrated as follows

$$\sigma - \sigma_0(t) = -\int_{r_0}^{r} \overset{\text{const } t}{} \frac{w\rho' \, \mathrm{d}r}{\rho(1+w)} = \frac{-w}{(1+w)} \ln\left(\frac{\rho}{\rho_{t0}}\right) \tag{B.1}$$

where $\rho_{t0} = \rho(t, r_0)$, and

$$\frac{\lambda}{2} - \frac{\lambda_0(r)}{2} = -\int_{t_0}^{t} \overset{\text{const } r}{} \frac{\rho' \, \mathrm{d}r}{\rho(1+w)} - 2\ln\left(\frac{R}{R_{0r}}\right) = \frac{-1}{(1+w)} \ln\left(\frac{\rho\, R^2}{\rho_{0r}\, R_{0r}^2}\right) \tag{B.2}$$

where $R_{or} = R(t_0, r)$ etc, so that

$$e^{\sigma - \sigma_0} = \left(\frac{\rho}{\rho_{t0}}\right)^{-w/(1+w)} \quad , \qquad e^{(\lambda - \lambda_0)/2} = \left(\frac{\rho\, R^2}{\rho_{0r}\, R_{0r}^2}\right)^{-1/(1+w)} \quad . \tag{B.3}$$

Making the gauge choice $R_{0r} = r$, and assuming $\rho(t, r)$ is known, Eqs (4.12) and (4.13) can be integrated along constant r or t to give

$$M(t_i, r) = \frac{\kappa}{6} \int_{t=t_i} \rho(t_i, r)\, r^2 \, dr \ , \tag{B.4}$$

$$M(t, r_i) = \frac{-\kappa w}{6} \int_{r=r_i} \rho(t, r_i)\, R^2(t, r_i)\, \dot{R}(t, r_i)\, dt \ . \tag{B.5}$$

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